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Fronts propagating with signal dependent speed in limited diffusion and related Hamilton–Jacobi formulations

Susana Serna [a,*], Antonio Marquina [b]

[a] Departament de Matematiques, Universitat Autonoma de Barcelona, 08193 Bellaterra, Spain
[b] Departament de Matematica Aplicada, Universitat de Valencia, 46100 Burjassot, Spain

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ABSTRACT

We consider a class of limited diffusion equations and explore the formation of diffusion fronts as the result of a combination of diffusive and hyperbolic transport. We analyze a new class of Hamilton–Jacobi equations arising from the convective part of general Fokker–Planck equations ruled by a non-negative diffusion coefficient that depends on the unknown and on the gradient of the unknown. We explore the main features of the solution of the Hamilton–Jacobi equations that contain shocks and propose a suitable numerical scheme that approximates the solution in a consistent way with respect to the solution of the associated Fokker–Planck equation. We analyze three model problems covering different scenarios. One is the relativistic heat equation model where the speed of propagation of fronts is constant. A second one is a standard porous media model where the speed of propagation of fronts is a function of the density, is unbounded and can exceed any fixed value. We propose a third one which is a porous media model whose speed of propagating fronts depends on the density media and is limited. The three model problems satisfy a general Darcy law. We perform a set of numerical experiments under different piecewise smooth initial data with compact support and compare the behavior of the three different model problems.

1. Introduction

Many complex physical systems like plasmas, porous media, transport in statistical mechanics, geometric flows among others are described by a class of (anisotropic) diffusion equations that model transport by diffusion of a physical magnitude in a continuum medium [1–4,7,17–19,22,23]. Fokker–Planck formulation represents these models whose solutions contain diffusion fronts propagating with finite speed. Because of this feature these equations are known as limited diffusion equations.

Limited diffusion equations can be understood as an extension of Fourier theory of heat conduction ruled by the heat equation $u_t = \nu \Delta u$ where $\nu > 0$ is the coefficient of heat diffusion in a specific media [10]. The solution to this model for a delta initial signal is a Gaussian distribution defined over all $\mathcal{R}^n$ for any time t. The main drawback of this model relies on the fact that transport of heat conduction occurs at infinite speed making the model not appropriate for the description of many dissipative processes in thermodynamics. This issue has been widely discussed in the literature and different attempts to overcome it have been presented [6,12,13,18,22].

We can understand the inadequacy of Fourier theory from the theory of relativity. Relativistic astrophysical flows are based on the assumption that light propagates with finite speed in vacuum, c, where c is an upper bound of light speed

* Corresponding author.
E-mail addresses: serna@mat.uab.es (S. Serna), marquina@uv.es (A. Marquina).

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