Lucyjane de Almeida Silva Menezes

On classical results for discontinuous and constrained differential systems.

Tese apresentada ao Programa de Pós–Graduação do Instituto de Matemática e Estatística da Universidade Federal de Goiás, como requisito parcial para obtenção do título de Doutor em Matemática.

Área de concentração: Sistemas Dinâmicos.

Orientador: Prof. João Carlos da Rocha Medrado
Co–Orientador: Prof. Jaume Llibre, Universitat Autònoma de Barcelona - UAB

Goiânia
2019

Abstract

Menezes, Lucyjane de Almeida Silva. **On classical results for discontinuous and constrained differential systems**. Goiânia, 2019. 85p. PhD. Thesis . Instituto de Matemática e Estatística, Universidade Federal de Goiás.

The present work concerns the study of classes of discontinuous differential systems addressing the following topics: global attractors, linearization, and codimension–one singularities for constrained differential systems. The Markus–Yamabe conjecture deals with global stability and it states that if a differentiable system $\dot{x} = f(x)$ has a singularity and the Jacobian matrix $Df(x)$ has everywhere eigenvalues with negative real part, then the singularity is a global attractor. We consider a piecewise smooth differential systems $Z = (X,Y)$ separated by one straight line Σ. Assuming that X and Y are linear vector field, $0 \in \Sigma$, $Y(0) = 0$, $X(0) \neq 0$, and the Jacobian matrices of the subsystems X and Y has everywhere eigenvalues with negative real part, we prove that Z can have one crossing limit cycle. That is about similar hypotheses to that of the Markus–Yamabe conjecture the origin is not necessarily a global attractor of Z. Consider Z defined in $\mathbb{R}^n$. In this work, we provide linear normal forms around generic singularities of Z. Let $A(x)$ be a $n \times n$ matrix valued function, $n \geq 2$, and $F(x)$ a vector field defined on $\mathbb{R}^n$. Assuming that A and F are smooth, a constrained system on $\mathbb{R}^n$ is a differential equations system of the form $A(x)\dot{x} = F(x)$, where $x \in \mathbb{R}^n$. The set $I_A = \{x \in \mathbb{R}^n : \det A(x) = 0\}$ is called the impasse hypersurface whose points are called impasse points. In this thesis we classify the one codimension singularities of the constrained systems defined on $\mathbb{R}^3$. Moreover we provide the respective normal forms in the one parameter space.

Keywords

Discontinuous systems, linearization, global attractor, limit cycles, constrained systems.

Contents

List of Figures **9**

Introduction **10**

1 Preliminary **15**
- 1.1 Smooth differential systems 16
- 1.2 Discontinuous differential systems 19
- 1.3 Vector fields on manifolds with boundary 22
- 1.4 Constrained systems 25
  - 1.4.1 Typical singularities of a constrained systems on $\mathbb{R}^3$ 26

2 On global attractors for discontinuous systems **28**
- 2.1 On limit cycles for Piecewise MY–systems 30
  - 2.1.1 Piecewise MY–systems of type FF 31
  - 2.1.2 Piecewise MY–systems of type FN 37
  - 2.1.3 The piecewise MY–system of type FiN 42

3 On linearization for discontinuous differential systems **47**
- 3.1 A Tubular Flow Theorem for pairs of vector fields 48
- 3.2 Σ–equivalence of hyperbolic pseudo–equilibrium 50
- 3.3 Σ–equivalence of fold–regular singularity 54
- 3.4 Weak equivalence for tangency points of order $k > 2$ 56

4 Codimension–one singularities of constrained systems on $\mathbb{R}^3$ **60**
- 4.1 Tangency impasse points 61
- 4.2 Equilibrium impasse points 63
  - 4.2.1 Nonresonant case 64
  - 4.2.2 Resonant case 67

Bibliography **83**

Introduction

Discontinuous differential systems appear in the history of dynamical systems as models of real phenomena presenting abrupt switching of behavior such as electronic relays and mechanic impact, see [1]. The theory of discontinuous systems has been in wide development and it can applicable in different areas of knowledge such as electrical and mechanic engineering and biology, see [28]. The qualitative analysis of discontinuous differential systems is an essential issue for the study and comprehension of such models. However, the qualitative analysis requires the establishment of new techniques and the adaptations of the classic tools for continuous differential systems, see [4], [16], and [29]. Moreover even in the planar case there are important unsolved questions for this class of differential system as to know the maximum number of limit cycles.

Let $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ be a C^r function, $r \geq 1$, for which 0 is a regular value. We define a *discontinuous differential system*, also named *Filippov system* as

$$Z(z) = \begin{cases} X(z), & \text{if} \quad f(z) > 0, \\ Y(z), & \text{if} \quad f(z) < 0, \end{cases} \tag{0-1}$$

where $z \in \mathbb{R}^n$ and $\Sigma = f^{-1}(0)$ is the *discontinuity manifold*.

According to Filippov's convention, the manifold Σ is decomposed in crossing, sliding, and escaping regions. Furthermore, we define the sliding vector field Z^s as the linear convex combination of X and Y tangent to Σ at p, that is

$$Z^s(p) = \frac{Yf(p)X(p) - Xf(p)Y(p)}{Yf(p) - Xf(p)}$$

where $Xf(p) = \langle \nabla f(p), X(p) \rangle$,

The singularities of Filippov system (0-1) are the points p such that: $p \notin \Sigma$ and $X(p) = 0$ or $Y(p) = 0$, p belonging to sliding or escape region such that $Z^s(p) = 0$, and $p \in \Sigma$ such that $Xf(p) = 0$ or $Yf(p) = 0$. Notice that in Filippov systems there exist singularities p which is not a stationary point of Z. We say that $p \in \Sigma$ is a tangency of order k of X if $Xf(p) = 0, \ldots, X^{k-1}f(p) = 0$, and $X^k f(p) \neq 0$. Furthermore, if $\{DXf(p), \ldots, DX^{k-1}f(p), Df(p)\}$ is linearly independent then the tangency point p is said to be generic.

We define a *crossing limit cycle* as a isolated periodic orbit formed by the union of regular orbits which cross the discontinuity only through crossing points. In this work a crossing limit cycle will be called simply *limit cycle*.

The present work concerns the study of different classes of discontinuous differential systems with emphasis on the problems as the Markus–Yamabe conjecture, Hartman–Grobman Theorem, maximum number of limit cycles, and one–codimension singularities for constrained systems.

On global attractors for discontinuous systems

In 1960 Markus and Yamabe stated that if $f(0) = 0$ and all eigenvalues of $Df(x)$ have negative real part then the differential system

$$\dot{x} = f(x), \tag{0-2}$$

where $f \in C^1(\mathbb{R}^n)$, is such that the origin is a global attractor. In their paper [19] the conjecture was proved under some strong additional hypotheses. This statement became known as the Markus–Yamabe conjecture.

Many authors have dedicated their work in proving the Markus–Yamabe conjecture. In 1988 Meisters and Olech proved this conjecture for polynomial vector fields in the plane, see [20]. Considering vector fields of class C^1 defined in $\mathbb{R}^2$, Gutierrez [15], Fleber [9] and Glutsyuk [12] for this order provided different proofs of the Markus–Yamabe conjecture in the years 1994 and 1995. However counterexamples have been constructed in higher dimensions. Bernat and Llibre [3], in 1996, presented a counterexample to the conjecture in dimension larger than 3. In 1997, Cima et al. [8] provided a counterexample for the case $n = 3$. More precisely, they proved that the Markus–Yamabe conjecture is false for polynomial vector fields in $\mathbb{R}^n$ with $n \geq 3$. More details can be found in [5, 11, 14]. In this work we consider a discontinuous differential system satisfying similar hypotheses to that of the Markus–Yamabe conjecture and prove that the origin is not necessarily a global attractor.

We define a *piecewise linear Markus–Yamabe differential system* as a discontinuous differential system (0-1) defined for $n = 2$ and $f(x,y) = x$, where X and Y are linear vector fields, the real part of the eigenvalues of $DX(z)$ and $DY(z)$ are negative, $Y(0) = 0$, and the singularity of X is virtual, i.e. it leaves in the half–plane $x < 0$. In this case the discontinuity set is the straight line $\Sigma = \{(x,y) \in \mathbb{R}^2; x = 0\}$. In order to simplify the notation we denote the piecewise Markus–Yamabe system as $Z = (X,Y)$ and call it simply as *piecewise MY–system*. The extension of the conjecture of Markus–Yamabe to piecewise MY–system claims: *The origin of any piecewise MY–system is a global attractor*. Our main goal is to prove that the extension of the Markus–Yamabe conjecture does not hold for piecewise MY–systems. For this we should characterize that these systems can have limit cycles. This characterization is done in the Theorem A.

On linearization for discontinuous differential systems

A classical result on linearization for continuous differential system is the Grobman–Hartman Theorem, [21]. It states that topologically the local behavior of the non linear system

$$\dot{x} = X(x) \tag{0-3}$$

near an hyperbolic singularity x_0 where $X(x_0) = 0$ is typically determined by the behavior of the linear system

$$\dot{x} = DX(x_0)x \tag{0-4}$$

near the origin. Assuming that the singularity x_0 has been translated to the origin, we have the following:

Grobman–Hartman Theorem. Let E be an open subset of $\mathbb{R}^n$ containing the origin let $X \in C1(E)$, and let Φ_t be the flow of the nonlinear system (0-3). Suppose that $X(0) = 0$ and that 0 is hyperbolic singularity of X, that is the matrix $A = DX(0)$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $x_0 \in U$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $x_0 \in U$ and $t \in I_0$

$$H \circ \Phi_t(x_0) = e^{At} H(x_0).$$

That is H maps trajectories of (0-3) near the origin onto trajectories of (0-4) near the origin and preserves the parameterization by time.
Our main scope on linearization for discontinuous differential systems is to establish results such as versions of the Grobman–Hartman Theorem in the discontinuous context.

Consider $Z = (X,Y)$ a discontinuous differential system of the form (0-1). The generic singularities of Z are

1. the hyperbolic singularities of X and Y in $\Sigma^+ = \{z \in \mathbb{R}^n : f(z) > 0\}$ and $\Sigma^- = \{z \in \mathbb{R}^n : f(z) < 0\}$, respectively.
2. hyperbolic singularities of the sliding vector field: $p \in \Sigma$ such that $Z^s(p) = 0$.
3. the tangency–regular points: $p \in \Sigma$ such that p is a generic tangency of X with order k, $1 < k \leq n$, and $Yf(p) \neq 0$.

In this work we assume that the origin is a generic singularity of Z and provide linear normal forms for Z around the origin. This is done in the Theorem B. Moreover we prove a version of the tubular flow theorem for pairs of vector fields, called theorem C, that is a key tool for proving the Theorem B for generic singularities what are hyperbolic singularities of the sliding vector field.

Codimension–one singularities for constrained system

Let $A = A(x)$ be a $n \times n$ matrix valued function, $n \geq 2$, and $F = F(x)$ a vector field defined on $\mathbb{R}^n$. Assuming that A and F are smooth, a *constrained system* on $\mathbb{R}^n$ is a differential equations system of the form

$$A(x)\dot{x} = F(x), \tag{0-5}$$

where $x \in \mathbb{R}^n$. The constrained system are characterized by the existence of *impasse hypersurface*

$$I_A = \{x \in \mathbb{R}^n : \delta_A = \det A(x) = 0\}$$

whose points are called *impasse points*. Notice that, outside of the impasse hypersurface the constrained system can be rewritten as

$$\dot{x} = A^{-1}(x)F(x) = \delta_A^{-1}A^*(x)F(x),$$

where A^* denotes the adjoint matrix of A. Then, for every system of the form (0-5) , we can define the vector field $\widetilde{X} = A^*(x)F(x)$ called the *regularization* of the constrained system. A constrained systems CS_1 at point p is said to be topologically equivalent to a constrained system CS_2 provided that there is a homeomorphism h from a neighborhood U of p into a neighborhood V of q which maps the impasse hypersurface I_{A_1} onto I_{A_2} and sends the arcs of orbits of the regularization $\widetilde{X}_1$ in $U \setminus I_{A_1}$ onto those of $\widetilde{X}_2$ in $V \setminus I_{A_2}$ preserving there constrained positive orientations. Such constrained orientation, in the case (0-5) is defined by $\widetilde{X}_1$ multiplied by the sign of δ_{A_1}.

Systems of the form (0-5) appear in electrical circuit theory and in the study of the inversion of smooth mappings, see [6], [7], and [26]. The numerical analysis of a class of constrained systems has been carried out by Riaza and Zufiria in [23].

An impasse point p of the constrained system (0-5) is said regular if δ_A is a regular function at p, that is $D(\delta_A(x))(p) \neq 0$. Moreover an regular impasse point p will be called non singular if the space $\ker A(p)$ is transversal to I_A and the vector $F(p)$ does not belong to the image of $A(p)$. Otherwise, if at least one at least of these conditions are violated, p is called a impasse singularity. In [31], Zhitomirskii provided the local normal forms for constrained systems on 2–manifolds. In [27], Sotomayor and Zhitomirskii classified the generic impasse singularities of constrained systems defined on $\mathbb{R}^n$, $n \geq 3$, and provided the respective normal forms. In [18], the authors provided a complete list of the one–parameter impasse bifurcations. A Peixoto's Theorem for constrained systems defined on $\mathbb{S}^2$ was proved by Buzzi, Medrado, and Silva in [4].

In this work we consider the constrained systems (0-5) defined on $\mathbb{R}^3$ and we classify the codimension–one singularities and provide the respective local normal forms in the one parameter space. This classification is done in Theorem D.

Organization of the thesis

In Chapter 1 we introduce the basic concepts and the necessary results on differential systems that will be fundamental tools for the development of this thesis.

Chapter 2 is devoted to study on Markus–Yamabe conjecture for discontinuous differential systems. In order to see that the Markus–Yamabe conjecture does not hold for discontinuous differential systems separated by one straight line, we should characterize which piecewise MY–systems can have limit cycles. This characterization is done in the Theorem A.

Chapter 3 concerns to the versions of the Grobman–Hartman Theorem around generic singularities and their respective normal forms are exhibited, this is done in the Theorem B. In the Section 3.1 we prove a version of the Tubular Flow Theorem for pairs of vector fields, called Theorem C. This result will be a key tool for proving the Theorem B in a neighborhood of the hyperbolic singularity of the sliding vector field in the Section 3.2. In Sections 3.3 and 3.4 we provide the linear normal forms of the discontinuous differential system in a neighborhood of a tangency–regular point with order 2 and greater than 2, respectively.

The last Chapter 4 we present a classification of the codimension–one singularities for constrained systems defined on $\mathbb{R}^3$ and the respective normal forms. These are the contents of the Theorem D. In the section 4.1 we classify the impasse singularities that are of type tangency impasse points. The section 4.2 concerns the classification of the impasse singularities that are of type equilibrium impasse points.

The figures of this work were made on maple and corel draw.