Qualitative behaviour analysis of feedback-controlled Buck-Boost power converters thru three different techniques

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SUMMARY

This work is based on the comparison of three techniques for analyzing the qualitative behaviour of nonlinear dynamic systems, including the study of their finite and infinite equilibrium points. The qualitative techniques used are: the direct method of Lyapunov, The theorems of Dickson and Perko for second order quadratic differential systems and the linearization around finite equilibrium points. These techniques provide information about the local or global stability of nonlinear systems. The state feedback controlled Buck-Boost power converter will be used as a case of study.

KEY WORDS: Nonlinear system, bounded system, qualitative analysis of dynamical systems, Lyapunov method, Buck-Boost power converter

1. INTRODUCTION

It is known that some of the inherent qualitative characteristics of dynamic systems have been specified through rigorous analytic techniques. However in the specific case of nonlinear systems, there may be examples where there are not explicit solutions for the differential equations that describe their dynamics, and further, there are systems that exhibit multiple equilibrium points, limit cycles, bifurcations, among other features. Under these circumstances, the qualitative analysis of differential equations is a viable alternative to learn about the dynamic behaviour of these systems.

In this sense, the converse theorems are key tools in the stability analysis of dynamic systems. Some classical references on the subject are the work of [8] and [6]. More recent references are the work of [7] and [5]. The references mentioned have been developed as a result of the work of [9]; where the local and global equilibrium points in linear systems and in some nonlinear systems are studied. A concise reference to the concepts of the theory of Lyapunov is the text of [11].

In general, local results do not provide a comprehensive explanation of the behaviour of nonlinear systems. Therefore it is necessary to use other tools for the study of systems of second order quadratic differential equations, as the one considered in this paper. For this purpose, two references that analyze the behaviour of these differential equations are used: the first is [1] aimed at sorting through the use of inequalities the different behaviours of bounded quadratic systems, and the second is the work of [2] which, through qualitative analysis of these dynamic systems, seeks to classify them in terms of an atlas represented in phase portraits. Both references are summarized in the textbook [3].

An application to stability analysis through the qualitative techniques referenced above is presented in the work of [12], where the behaviour of the Boost power converter is discussed. This article presents a qualitative analysis of a nonlinear closed-loop system, specifically the Buck-Boost power converter with a state vector feedback. It is a non-linear, quadratic, second order differential system that has no explicit solution. It is intended to study the behaviour of the trajectories between finite and infinite equilibrium points, with respect to changes in system and controller parameters.

2. BUCK-BOOST POWER CONVERTER

The main characteristic of the Buck-Boost circuit design is that it can operate as a step up or as a step down voltage converter, that is, its output voltage may be lower or higher than the power supply. Fig 1 illustrates its circuit diagram.

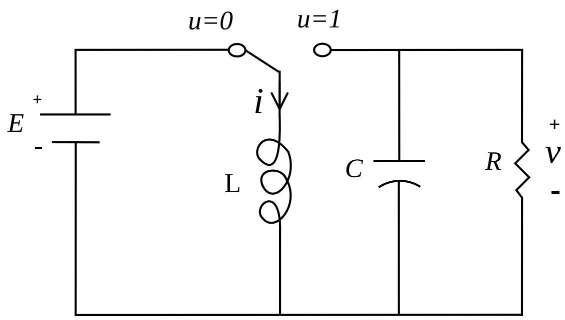

Figure 1. Illustration of the Buck-Boost circuit design.

The paper by [10] shows how the average behaviour of the circuit of Fig. 1 may be represented by a continuous time model, applying the laws of Kirchhoff and Ohm. This is:

$$\begin{aligned} L\frac{di}{dt} &= (1-u)\,v + (u)\,E, \\ C\frac{dv}{dt} &= -\,(1-u)\,i - \frac{v}{R}, \end{aligned} \tag{1}$$

where i is the inductor current, v the voltage on the capacitor, R the resistance of the load, L the inductance of the coil, C the capacitance, E the power supply and u the DC control input, which is defined in the range $[0,1]$. In order to facilitate the calculations, let τ and Q be defined as $\tau = \dfrac{t}{\sqrt{LC}}$ and $Q = R\sqrt{\dfrac{C}{L}}$, and let the linear transformation given in equation (2) be applied to system (1)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{1}{E}\dfrac{\sqrt{C}}{\sqrt{L}} & 0 \\ 0 & \dfrac{1}{E} \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix}. \tag{2}$$

The normalized system (1) may be represented as

$$\begin{aligned} \frac{dx}{d\tau} &= \dot{x} = (1-u)\,y + u, \\ \frac{dy}{d\tau} &= \dot{y} = -\,(1-u)\,x - \frac{y}{Q}, \end{aligned} \tag{3}$$

where the normalized variable $x(\tau) = x$ is the coil current, $y(\tau) = y$ is the capacitor voltage, Q is the charge and $u \in [0,1]$ is the control input. The equilibrium points of the open loop system (3) are given as

$$\bar{x} = \frac{\bar{y}\,(\bar{y}-1)}{Q}, \quad \bar{u} = \frac{\bar{y}}{\bar{y}-1}, \tag{4}$$

wherein the desired value of the output voltage of the capacitor V_d is equilibrium value of the system, that is, $\bar{y} = V_d < 0$.

By moving system (3) to the origin, which is achieved through the change of coordinates defined by $e_1 = x - \bar{x}$, $e_2 = y - \bar{y}$, $e_u = u - \bar{u}$, the following exact error dynamics is obtained

$$\begin{aligned} \dot{e}_1 &= (1 - e_u - \bar{u})\,e_2 + \bar{y}\,(1 - e_u - \bar{u}) + e_u + \bar{u}, \\ \dot{e}_2 &= (e_u + \bar{u} - 1)\,e_1 - \frac{e_2}{Q} - \bar{x}\,(1 - e_u - \bar{u}) - \frac{\bar{y}}{Q}. \end{aligned} \tag{5}$$

Before making the analysis of equilibrium points, the feedback control loop using the state vector with a gain $k = [\alpha \quad \beta]$ will be considered. Thus, the equation of the control law is defined by

$$e_u = -k \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = -\left(\alpha e_1 + \beta e_2\right). \tag{6}$$

The dynamics of the closed loop system is obtained replacing the controller (6) in (5) and the equilibrium values of $\bar{x}$ and $\bar{u}$ given in (4). That is

$$\begin{aligned} \dot{e}_1 &= \alpha\,(\bar{y}-1)\,e_1 + \frac{\left(\beta(\bar{y}-1)^2 - 1\right)}{\bar{y}-1} e_2 + \alpha e_1 e_2 + \beta e_2^2, \\ \dot{e}_2 &= \frac{\left(Q - \alpha\bar{y}(\bar{y}-1)^2\right)}{(\bar{y}-1)\,Q} e_1 - \frac{\beta\bar{y}\,(\bar{y}-1) + 1}{Q} e_2 - \beta e_1 e_2 - \alpha e_1^2. \end{aligned} \tag{7}$$

Equating to zero the right sides of (7), solving for e_1 in the first equation and substituting it into the second equation, the equilibrium equation based on $\bar{e}_2$ is obtained

$$p\,(\bar{e}_2) = \left(\bar{e}_2^2 + \left(\frac{\beta}{\alpha}Q + 3\bar{y} - 2\right)\bar{e}_2 + \frac{Q\left(\beta(\bar{y}-1)^2 - 1\right) + \alpha(\bar{y}-1)^2\,(2\bar{y}-1)}{\alpha\,(\bar{y}-1)}\right)\bar{e}_2 = 0. \tag{8}$$

Since (8) is a polynomial of degree three, the Cardano method to characterize its roots will be used. The discriminant of Cardano Δ_C is as follows

$$\begin{aligned} \Delta_C = \quad &\left(Q\left(\beta(\bar{y}-1)^2 - 1\right) + \alpha(\bar{y}-1)^2\,(2\bar{y}-1)\right)^2 \\ &\left(Q^2\,(\bar{y}-1)\,\beta^2 + 2\alpha Q\,(\bar{y}-1)\,\bar{y}\beta \; + \alpha^2\,(\bar{y}-1)\,\bar{y}^2 + 4Q\alpha\right). \end{aligned} \tag{9}$$

- If $\Delta_C > 0$, there will be a real root and two complex non-real roots.
- If $\Delta_C = 0$, there will be a double real root and a single real root.
- If $\Delta_C < 0$, there will be three real roots.

This work will analyze only the case when there is one real equilibrium point; for which Proposition 1 establishes the range of values for the parameters.

Proposition 1. *The existence of a single real equilibrium point is defined by the conditions*

$$\alpha > 0 \quad \text{and} \quad -\alpha\frac{\bar{y}}{Q} - 2\sqrt{-\frac{\alpha}{Q(\bar{y}-1)}} < \beta < -\alpha\frac{\bar{y}}{Q} + 2\sqrt{-\frac{\alpha}{Q(\bar{y}-1)}}.$$

Proof
Restrictions for the parameters are: $Q > 0$, $\bar{y} < 0$, $\alpha, \beta \in (-\infty, \infty)$. The equation of Cardano states that if $\Delta_C > 0$ then there will be a single real equilibrium point. Thus, from (9) the following set of inequality solutions is obtained

- $\alpha > 0$,
- $\beta_1 < \beta < \beta_2$, where $\beta_{1,2} = -\alpha\dfrac{\bar{y}}{Q} \mp 2\sqrt{-\dfrac{\alpha}{Q(\bar{y}-1)}}$.

□

This range is defined as $R_{\Delta_C} = \{(\alpha,\beta) \in \mathbb{R} | \alpha > 0 \wedge \beta_1 < \beta < \beta_2\}$.

3. LYAPUNOV STABILITY ANALYSIS

In order to apply the concept of stability in the sense of Lyapunov, under the conditions provided in Proposition 1, it is necessary to have a single real equilibrium point located at the origin $(0, 0)$.

Theorem 1. *System (7) is globally stable if there is a unique equilibrium point.*

Proof
Let the positive definite Lyapunov function candidate $V(e)$ be defined as

$$V(e) = \frac{1}{2}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

According to the direct method of Lyapunov, if the derivative of the Lyapunov function candidate evaluated in the trajectories of the dynamical system is negative definite, then the system will display a globally stable behaviour. The derivative $\dot{V}(e)$ is defined as

$$\dot{V}(e) = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix}. \tag{10}$$

Substituting equation (7) in (10) yields an error dependent equation, which has the form $\dot{V}(e) = e^T M e + e^T K e$, where M is a symmetric matrix and K is an skew symmetric matrix defined as follow

$$M = \begin{bmatrix} \alpha(\bar{y}-1) & \dfrac{\beta(\bar{y}-1)^2 - 1}{\bar{y}-1} \\ \dfrac{Q - \alpha\bar{y}(\bar{y}-1)^2}{(\bar{y}-1)\,Q} & -\dfrac{\beta\bar{y}\,(\bar{y}-1)+1}{Q} \end{bmatrix}, \quad K = \begin{bmatrix} 0 & \alpha e_1 + \beta e_2 \\ -\alpha e_1 - \beta e_2 & 0 \end{bmatrix}.$$

Since $\dot{V}(e)$ is the sum of two quadratic forms and $e^T K e = 0$, by decomposing the matrix M in a symmetric matrix plus an skew symmetric matrix, i.e. $M = M_s + M_a$, the derivative of the Lyapunov function candidate turns into $\dot{V}(e) = e^T M_s e + e^T M_a e$, where

$$M_s = \begin{bmatrix} \alpha\,(\bar{y}-1) & \dfrac{1}{2}\dfrac{(\bar{y}-1)\,(Q\beta - \alpha\bar{y})}{Q} \\ \dfrac{1}{2}\dfrac{(\bar{y}-1)\,(Q\beta - \alpha\bar{y})}{Q} & \dfrac{\bar{y}\,(1-\bar{y})\,\beta - 1}{Q} \end{bmatrix},$$

and M_a is the skew symmetric matrix of M_s. Since the term $e^T M_a e = 0$, it suffices analyzing the function $\dot{V}(e) = e^T M_s e$.

Rewriting $\dot{V}(e)$, the derivative of the Lyapunov function candidate takes the form $\dot{V}(e) = -e^T(-M_s)e$, namely

$$\dot{V}(e) = -\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^T \begin{bmatrix} -\alpha\,(\bar{y}-1) & -\dfrac{1}{2}\dfrac{(\bar{y}-1)\,(Q\beta - \alpha\bar{y})}{Q} \\ -\dfrac{1}{2}\dfrac{(\bar{y}-1)\,(Q\beta - \alpha\bar{y})}{Q} & -\dfrac{\bar{y}\,(1-\bar{y})\,\beta - 1}{Q} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where the sign of $\dot{V}(e)$ depends on the sign of the matrix $-M_s$.

According to Sylvester criterion it suffices that $-M_s > 0$, so that $\dot{V}(e) < 0$. In order to satisfy that $-M_s > 0$, the minors of the matrix $-M_s$ must have positive determinants. These determinants are defined as:

- The determinant of the first minor is $\Delta_1 = -\alpha(\bar{y} - 1)$.
- The determinant of the matrix $-M_s$ is given as

$$\Delta_2 = \beta^2 (\bar{y} - 1) Q^2 + 2\alpha \left(\beta\bar{y}^2 - \beta\bar{y} + 2\right) Q + \alpha^2\bar{y}^2 (\bar{y} - 1).$$

To ensure compliance of $\Delta_1 > 0$, it is necessary that $-\alpha(\bar{y} - 1) > 0$, which is true if and only if $\alpha > 0$, because by definition $\bar{y} < 0$. For $\Delta_2 > 0$, it is necessary to write it in terms of a quadratic polynomial defined as

$$p(\beta) = \beta^2 \left(Q^2(\bar{y} - 1)\right) + \beta \left(2\alpha Q\bar{y}(\bar{y} - 1)\right) + \left(4\alpha Q + \alpha^2\bar{y}^2(\bar{y} - 1)\right) > 0, \tag{11}$$

since the term that accompanies β^2 is negative definite ($Q^2(\bar{y} - 1) < 0$), then the polynomial $p(\beta)$ is positive definite within the interval of its solutions

$$-\alpha\frac{\bar{y}}{Q} - 2\sqrt{-\frac{\alpha}{Q(\bar{y} - 1)}} < \beta < -\alpha\frac{\bar{y}}{Q} + 2\sqrt{-\frac{\alpha}{Q(\bar{y} - 1)}}.$$

Note that since of $Q > 0$, $\alpha > 0$ and $\bar{y} < 0$, then the determinant of (11) satisfies $\Delta_{p(\beta)} = -\dfrac{\alpha}{Q(\bar{y} - 1)} > 0$; implying that $\beta_{1,2} \in \mathbb{R}$ and there will always exist an interval (β_1, β_2) in which $p(\beta) > 0$ and therefore, the interval where $\dot{V}(e) < 0$ is defined by the set R_L, given by

$$R_L = \{(\alpha, \beta) \in \mathbb{R} | \alpha > 0 \wedge \beta_1 < \beta < \beta_2\}. \tag{12}$$

□

From the above analysis it is shown that $\dot{V}(e) < 0$ on R_L defined by (12). Also, $R_L = R_{\Delta_C}$ when there is a unique point of equilibrium and therefore this equilibrium point is globally stable.

Corollary 1. *In the boundary conditions for the system* (7) *stability in the sense of Lyapunov does not apply.*

Proof
If the discriminant of Cardano Δ_C is analyzed using equation (9), it may be appreciated that if $\beta = \beta_1$ or $\beta = \beta_2$ then $\Delta_C = 0$, which implies that there are two equilibrium points, and therefore the concept of global stability in the sense of Lyapunov cannot be applied. □

4. ANALYSIS VIA THEOREMS OF DICKSON AND PERKO

Theorems of [2], see Appendix A, allow qualitative analysis of quadratic second order systems. Theorem 5 is formulated to analyze Bounded Quadratic Systems (BQS), while Theorem 6 facilitates studying the qualitative behaviour of systems with a unique real equilibrium point ($BQS1$).

4.1. Bounded Quadratic Systems (BQS)

According to Theorem 5 of Appendix A, after applying a linear transformation, system (7) must be affine and equivalent to one of the following systems (23), (24) or (25).

Theorem 2. *The quadratic system described by* (7) *is bounded.*

Proof
Consider system (7) and the linear matrix transformation defined by $e = \theta z$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \tag{13}$$

where θ_{11}, θ_{12}, θ_{21} and θ_{22} are constants. Substituting (13) into (7) the following system of differential equations is obtained

$$\begin{aligned} \dot{z}_1 &= \Theta_1 z_1^2 + \Theta_2 z_1 z_2 + \Theta_3 z_2^2 + \Theta_4 z_1 + \Theta_5 z_2, \\ \dot{z}_2 &= \Theta_6 z_1^2 + \Theta_7 z_1 z_2 + \Theta_8 z_2^2 + \Theta_9 z_1 + \Theta_{10} z_2. \end{aligned} \tag{14}$$

Where Θ_1,...,Θ_{10} are parameters which depend on θ_{11}, θ_{12}, θ_{21}, θ_{22}, Q, $\bar{y}$, α and β. Equation (14) may be rewritten in the form of the system of equations (25). This is

$$\begin{aligned} \dot{z}_1 &= a_{11} z_1 + a_{12} z_2 + {z_2}^2, \\ \dot{z}_2 &= a_{21} z_1 + a_{22} z_2 - z_1 z_2 + c{z_2}^2. \end{aligned} \tag{15}$$

In order to accomplish this transformation, it is necessary to select $\Theta_1 = 0$, $\Theta_2 = 0$, $\Theta_3 = 1$, $\Theta_6 = 0$, $\Theta_7 = -1$ and to solve the system of algebraic equations which result in

$$\theta_{11} = \frac{\beta}{\alpha^2 + \beta^2}, \quad \theta_{12} = \theta_{21} = \frac{-\alpha}{\alpha^2 + \beta^2}, \quad \theta_{22} = \frac{-\beta}{\alpha^2 + \beta^2}. \tag{16}$$

By replacing the coefficients (16) in (14), the parameters of system (15) are

$$\begin{aligned} a_{11} &= \Theta_4 = -\frac{\alpha^2}{(\alpha^2 + \beta^2)\,Q}, \\ a_{12} &= \Theta_5 = \frac{(\alpha^2 + \beta^2)\left(\beta(\bar{y}-1)^2 - 1\right)Q + \alpha^3(\bar{y}-1)^2\bar{y}}{(\alpha^2 + \beta^2)\,(1-\bar{y})\,Q} + \frac{\alpha\beta\,(\bar{y}-1)\,(1 + \beta\,(\bar{y}-1)\,\bar{y})}{(\alpha^2 + \beta^2)\,(1-\bar{y})\,Q}, \\ a_{21} &= \Theta_9 = -\frac{\left(Q\left(\alpha^2 + \beta^2\right) + \alpha\beta\,(\bar{y}-1)\right)}{(\alpha^2 + \beta^2)\,(\bar{y}-1)\,Q}, \\ a_{22} &= \Theta_{10} = -\left(\frac{\beta^2}{(\alpha^2 + \beta^2)\,Q} + \alpha\,(\bar{y}-1) - \frac{\beta\,(\bar{y}-1)\,\bar{y}}{Q}\right), \\ c &= \Theta_8 = 0. \end{aligned} \tag{17}$$

Since $Q > 0$ then $a_{11} < 0$ and therefore, according to Theorem 5, systems (15) and (7) are bounded. □

It should be noted that Theorem 2 ensures that system (7) is bounded for any configuration of finite and infinite equilibrium points. Fig. 2 shows the equilibrium points at infinity in a saddle–node configuration, where the circle corresponds to the neighborhood of infinite.

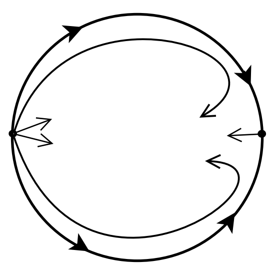

Figure 2. Phase Portraits of a Bounded Quadratic Systems (*BQS*).

4.2. *Bounded quadratic systems with a unique real equilibrium point (BQS1)*

Theorem 6 of the appendix allows analyzing the different qualitative behaviours when there is a unique finite equilibrium point in a system of the type (25); and does so by means of phase diagrams, including both the finite equilibrium point as those at infinity. According to Theorem 6 there are four configurations, of which (a) and (b) cannot be used because system (7) is affine to a (25) type system, and these are mutually exclusive. Thus, system (7) may be of type (c) or (d). It will be shown that the only feasible configuration for (7) is (c).

Theorem 3. *Given the following parameters restrictions: $Q > 0$, $\bar{y} < 0$, $\alpha > 0$ and $-\alpha\frac{\bar{y}}{Q} - 2\sqrt{-\frac{\alpha}{Q(\bar{y}-1)}} < \beta < -\alpha\frac{\bar{y}}{Q} + 2\sqrt{-\frac{\alpha}{Q(\bar{y}-1)}}$, then system (7) has the configuration type (c) of Fig. 3.*

Proof
According to Theorem 6(c) of the appendix, system (25) will have a unique equilibrium point if the following conditions are satisfied:

$$(i)\quad a_{11} < 0, \qquad (ii)\quad (a_{12} - a_{21} + ca_{11})^2 < 4(a_{11}a_{22} - a_{21}a_{12}), \qquad (iii)\; a_{11} + a_{22} \leq 0. \quad (18)$$

In Theorem 2 is proved that condition (18)(i) is satisfied. For the second condition, substituting parameters (17) in (18)(ii) results

$$f = \beta^2 + 2\frac{\alpha}{Q}\bar{y}\beta + \left(\frac{\alpha}{Q}\bar{y}\right)^2 + 4\frac{\alpha}{Q\,(\bar{y}-1)} < 0. \quad (19)$$

Equation (19) is valid in the interval $\overline{(\beta_1, \beta_2)}$ where $\beta_{1,2} = -\alpha\frac{\bar{y}}{Q} \mp 2\sqrt{-\frac{\alpha}{Q(\bar{y}-1)}}$, i.e.,

$$-\alpha\frac{\bar{y}}{Q} - 2\sqrt{-\frac{\alpha}{Q(\bar{y}-1)}} < \beta < -\alpha\frac{\bar{y}}{Q} + 2\sqrt{-\frac{\alpha}{Q(\bar{y}-1)}}. \quad (20)$$

The discriminant of f is $\Delta_f = -\frac{\alpha}{Q(\bar{y}-1)}$ and $\beta_{1,2} \in \mathbb{R}$ if and only if $\alpha > 0$.

For the third condition, parameters (17) are substituted into (18)(iii) which results in $\beta \geq \frac{Q(\bar{y}-1)\alpha - 1}{(\bar{y}-1)\bar{y}}$. This defines an interval $\overline{[\beta_a, \infty)}$ where $\beta_a = \frac{Q(\bar{y}-1)\alpha - 1}{(\bar{y}-1)\bar{y}}$.

Note that $\overline{(\beta_1, \beta_2)} \cap \overline{[\beta_a, \infty)} = \overline{(\beta_1, \beta_2)}$, so the range in which (18)(iii) is fulfilled is (20), with $\alpha > 0$, which shows that (7) is a type (c) system. According to Theorem 5, (7) will be a type (d) system if it satisfies the following conditions:

$$(i)\quad a_{11} < 0, \qquad (ii)\quad (a_{12} - a_{21} + ca_{11})^2 < 4(a_{11}a_{22} - a_{21}a_{12}), \qquad (iii)\; a_{11} + a_{22} > 0. \quad (21)$$

Analogously, Theorem 2 proves that (21)(i) is valid; and the above analysis, in the interval $\overline{(\beta_1, \beta_2)}$, shows that (21)$(ii)$ =(18)(ii). For the third condition, parameters (17) are substituted into (21)(iii) and $\beta < \frac{Q(\bar{y}-1)\alpha - 1}{(\bar{y}-1)\bar{y}}$ is obtained, which defines an interval $\overline{(-\infty, \beta_b)}$, where $\beta_b = \frac{Q(\bar{y}-1)\alpha - 1}{(\bar{y}-1)\bar{y}}$. The intervals are such that $\overline{(\beta_1, \beta_2)} \cap \overline{(-\infty, \beta_b)} = \emptyset$ and therefore, since condition (21)(iii) is not satisfied, (7) cannot be a type (d) system.

In summary, the set R_L given in (12) defines the range within which system (7) is *BQS1*. □

Corollary 2. *Since (7) is a BQS1 type (c) system, it is globally stable.*

Proof
Since (7) is *BQS1* type (c), then there is a unique real equilibrium point to which all trajectories converge; that is, the system is globally stable. □

Corollary 3. *Since* (7) *is BQS1 type* (c) *system, then it has no limit cycles.*

Proof
Since (7) is *BQS1* type (c) and it cannot be represented as type (d), then there is no limit cycle or periodic solution. □

5. QUALITATIVE ANALYSIS AROUND FINITE EQUILIBRIUM POINTS

System (7) will be analyzed locally with respect to its finite equilibrium points using the linearization method that is described in [4], section 1.5.

The procedure consists of two steps; first the finite equilibrium point is obtained. Next, the Jacobian matrix associated with (8) at this equilibrium point is evaluated. Thus the linearized version of the original nonlinear system is obtained. (8) and (9) define the equilibrium equation and the determinant of Cardano Δ_C, respectively. The Jacobian matrix A is given as

$$A = \begin{pmatrix} \alpha\,(\bar{y}-1) & \dfrac{\beta(\bar{y}-1)^2-1}{(\bar{y}-1)} \\ \dfrac{Q-\alpha(\bar{y}-1)^2\bar{y}}{Q\,(\bar{y}-1)} & -\dfrac{\beta\,(\bar{y}-1)\,\bar{y}+1}{Q} \end{pmatrix}.$$

On the other hand, the eigenvalues of A are defined as

$$\lambda_{1,2} = -\frac{1}{2Q} + \alpha\frac{(\bar{y}-1)}{2} - \beta\frac{(\bar{y}-1)}{2Q}\bar{y} \pm \frac{\sqrt{4Q\left((\bar{y}-1)^2(2\bar{y}-1)\,\alpha + Q\left((\bar{y}-1)^2\beta-1\right)\right) + (\bar{y}-1)^2(1+(\bar{y}-1)\,(\bar{y}\beta-Q\alpha))^2}}{2Q\,(\bar{y}-1)}. \tag{22}$$

The local behaviour at the origin of coordinates may be interpreted using the following theorem.

Theorem 4. *The origin of coordinates is an attractor.*

Proof
The origin of the linearized system is an attractor, if the system possesses a unique equilibrium point and its eigenvalues are negative. This is accomplished with the following restrictions:

(i) $\Delta_C > 0$,

(ii) $\lambda_1\lambda_2 = \dfrac{Q-(\bar{y}-1)^2(2\bar{y}-1)\,\alpha - Q(\bar{y}-1)^2\beta}{Q(\bar{y}-1)^2} > 0$,

(iii) $\lambda_1+\lambda_2 = \dfrac{(\bar{y}-1)\,(Q\alpha-\bar{y}\beta)-1}{Q} < 0.$

The above restrictions are satisfied for the following conditions on the parameter

$$\alpha > 0, \quad -\alpha\frac{\bar{y}}{Q} - 2\sqrt{-\frac{\alpha}{Q(\bar{y}-1)}} < \beta < -\alpha\frac{\bar{y}}{Q} + 2\sqrt{-\frac{\alpha}{Q(\bar{y}-1)}}.$$

These conditions are the same that define the set R_L given in (12).

In order to verify if the origin is a repeller, restriction (i) in addition to the following restrictions are used

$(iv)\ \ \lambda_1\lambda_2 = \dfrac{Q - (\bar{y}-1)^2\,(2\bar{y}-1)\,\alpha - Q(\bar{y}-1)^2\beta}{Q(\bar{y}-1)^2} > 0,$

$(v)\ \ \lambda_1 + \lambda_2 = \dfrac{(\bar{y}-1)\,(Q\alpha - \bar{y}\beta) - 1}{Q} > 0.$

It is easy to verify that there are not values of parameters that would make the origin to behave as a repeller.

To check if the origin of coordinates is a saddle, the restriction (i) and the following restriction are taken into account

$(vii)\ \ \lambda_1\lambda_2 = \dfrac{Q - (\bar{y}-1)^2\,(2\bar{y}-1)\,\alpha - Q(\bar{y}-1)^2\beta}{Q(\bar{y}-1)^2} < 0.$

There are not values of parameters that would make the origin to behave as a saddle.

To verify if the origin of coordinates is a center, the restriction (i) and the following restrictions are considered

$(viii)\ \ \mathrm{Re}\{\lambda_1, \lambda_2\} = 0,$

$(ix)\ \ \ \mathrm{Im}\{\lambda_1, \lambda_2\} \neq 0.$

From $(viii)$ and (22) results $\mathrm{Re}\{\lambda_1, \lambda_2\} = 0 \Rightarrow -\dfrac{1}{2Q} + \alpha\dfrac{(\bar{y}-1)}{2} - \beta\dfrac{(\bar{y}-1)}{2Q}\bar{y} = 0$. Also, from (ix) and (22) results $\mathrm{Im}\{\lambda_1, \lambda_2\} \neq 0 \Rightarrow 4Q\left((\bar{y}-1)^2\,(2\bar{y}-1)\,\alpha + Q\left((\bar{y}-1)^2\beta - 1\right)\right) + (\bar{y}-1)^2(1 + (\bar{y}-1)\,(\bar{y}\beta - Q\alpha))^2 < 0.$

There are not values of parameters that would make the origin to behave as a center. □

Theorem 4 may be used to prove that the origin of coordinates, which is the unique real equilibrium point, can only be an attractor; and it is in the R_L set defined in (12). Also, the system is locally stable.

6. ANALYSIS OF RESULTS

There have been used three techniques to study the qualitative behaviour of a second order nonlinear dynamic system. These techniques have corresponded to the direct method of Lyapunov, theorems of Dickson and Perko and the approximate linearization of nonlinear systems. In all three cases, the analysis has led to the same set where the parameters of the system were defined.

The results of the analysis performed with each of the techniques have been summarized in Table I.

- The direct method of Lyapunov may be used to prove that when there is only one real equilibrium point, system (7) is globally stable. But it does not provide any information regarding the boundedness (*BQS*) of system (7) for any combination of parameter values.
- Theorems 5 and 6 of Dickson and Perko may be used to prove that system (7) is bounded (*BQS*), regardless of the values of its parameters. In addition, to prove that when there is a unique real equilibrium point (*BQS1*), system (7) is globally stable and there are no limit cycles in its trajectories.
- Approximate Linearization allows local analysis and provides no information on the overall behaviour of system (7) or its boundedness (*BQS*). When there is only one real equilibrium point, it may be used to prove that the finite equilibrium point is an attractor, and therefore system (7) is locally stable.
- If the boundedness feature (*BQS*) of system (7) obtained thru Theorems of Dickson and Perko, together with the attractor behaviour of the single real equilibrium point, which is derived using the approximate linearization method is used, it can be concluded that system (7) is *BQS1*.

Table I. Qualitative analysis results of system (7) with different techniques.

Qualitative Analysis Technique	**Bounded for any value of the parameters (*BQS*)**	**Bounded with a unique real equilibrium point (*BQS1*)**
Lyapunov	Not shown if bounded	Globally Stable
Dickson–Perko	Bounded	Globally Stable
Linearization	Not shown if bounded	Locally Stable
Dickson–Perko and Linearization	Bounded	Globally Stable

7. CONCLUSIONS

In this work an analysis of the behaviour of the trajectories around the equilibrium points of the Buck–Boost power converter with state vector feedback, using qualitative techniques for dynamic systems has been presented.

For this type of closed-loop systems, there is a bifurcation of equilibrium points; so there may be one, two or three finite equilibrium points. Regardless of the value of the system parameters, all trajectories converge to finite equilibrium points; that is, the system is bounded (*BQS*) for any configuration of finite equilibrium points and there are no limit cycles.

The direct method of Lyapunov may be used to prove the existence of conditions on the control parameters to ensure global stability of the system. Through theorems of Dickson and Perko a global qualitative behaviour of the system with a single equilibrium point is obtained, as well as they provide conditions on the control parameters that make all trajectories converge to the equilibrium point. Also, the theorems allow verifying the existence, or not, of limit cycles.

Linearization around the origin of coordinates, where it is located the equilibrium point, facilitates establishing conditions on the control parameters to ensure that it is a local attractor.

The direct method of Lyapunov and the theorems of Dickson and Perko permitted to obtain results about the overall behaviour of the system; while the approximate linearization only allowed giving local results. By using the results of boundedness, together with the absence of limit cycles for the *BQS1*, the local analysis may be considered as global for the case study considered.

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A. THEOREM OF DICKSON AND PERKO

Theorem 5. *Any* (BQS) *is affinely equivalent to*

$$\dot{x} = a_{11}x, \qquad \dot{y} = a_{21}x + a_{22}y + xy \tag{23}$$

with $a_{11} < 0$ *and* $a_{22} \le 0$*, or*

$$\dot{x} = a_{11}x + a_{12}y + y^2, \qquad \dot{y} = a_{22}y \tag{24}$$

with $a_{11} \le 0$*,* $a_{22} \le 0$ *and* $a_{11} + a_{22} < 0$*, or*

$$\dot{x} = a_{11}x + a_{12}y + y^2, \qquad \dot{y} = a_{21}x + a_{22}y - xy + cy^2 \tag{25}$$

with $|c| < 2$ *and either* (i) $a_{11} < 0$*;* (ii) $a_{11} = 0$ *y* $a_{21} = 0$*; or* (iii) $a_{11} = 0$*,* $a_{21} \neq 0$*,* $a_{12} + a_{21} = 0$ *and* $ca_{21} + a_{22} \le 0$*.*

Theorem 6. *The phase portrait of any (BQS1) is determined by one of the separatrix configurations in Figure 3. Furthermore, the phase portrait of a quadratic system is given by Figure 3.*

(a) iff the quadratic system is affinely equivalent to (23) with $a_{11} < 0$ *and* $a_{22} < 0$;
(b) iff the quadratic system is affinely equivalent to (24) with $a_{11} < 2a_{22} < 0$*;*
(c) iff the quadratic system is affinely equivalent to (24) with $2a_{22} \le a_{11} < 0$ *or (25) with* $|c| < 2$ *and either*
   (i) $a_{11} = a_{22} + a_{21} = 0$*,* $a_{21} \neq 0$ *and* $a_{22} < min(0, -ca_{21})$ *or* $a_{22} = 0 < -ca_{21}$*,*
   (ii) $a_{11} < 0$*,* $(a_{12} - a_{21} + ca_{11})^2 < 4(a_{11}a_{22} - a_{21}a_{12})$*, and* $a_{11} + a_{22} \le 0$*, or*
   (iii) $a_{11} < 0$ *y* $(a_{12} - a_{21} + ca_{11}) = (a_{11}a_{22} - a_{21}a_{12}) = 0$;
(d) iff the quadratic system is affinely equivalent to (25) with $|c| < 2$ *and either*
   (i) $a_{11} = a_{12} + a_{21} = 0$ *and* $0 < a_{22} < -ca_{21}$*, or*
   (ii) $a_{11} < 0, a_{11} + a_{22} > 0$*, and* $(a_{12} - a_{21} + ca_{11})^2 < 4(a_{11}a_{22} - a_{21}a_{12})$*.*

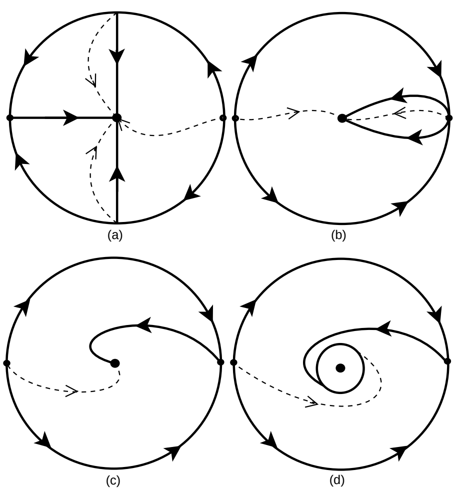

Figure 3. All possible phase portraits for (*BQS1*).