Connectivity of Julia sets
of transcendental meromorphic functions

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Introduction

In the year 1669, a young fellow of the Trinity College of the University of Cambridge presented a treatise on quadrature of simple curves and on resolution of equations. Concerning the latter, he wrote: "Because the whole difficulty lies in the resolution, I shall first illustrate the method I use in a numeral equation," and the procedure he described next became the germ of possibly the most powerful root-finding algorithm used today. The young fellow was Isaac Newton and the treatise was *De analysi per æquationes numero terminorum infinitas*, one of his most celebrated works.

Using the "numeral equation" $y^3 - 2y - 5 = 0$, Newton then illustrates his resolution method as follows: He proposes the number 2 as an initial guess of the solution which differs from it by less than a tenth part of itself. Calling p this *small* difference between 2 and the solution y, he writes $2 + p = y$ and substitutes this value in the equation, which gives a new equation to be solved: $p^3+6p^2+10p-1 = 0$. Since p is small, the higher order terms $p^3 + 6p^2$ are quite smaller relatively, therefore they can be neglected to give $10p - 1 = 0$, from where $p = 0.1$ may be taken as an initial guess for the solution of the second equation. Now, it is clear how the algorithm continues, since, writing $0.1 + q = p$ and substituting this value in the second equation, a third equation $q^3+6.3q^2+11.23q+0.061 = 0$ is obtained, and so on.

Using this method, Newton constructs a sequence of polynomials, plus a sequence of root approximations that converge to 0 and add up to the solution of the original equation.

A couple of decades later, Joseph Raphson discussed Newton's recurrence and improved the method by using the concept of derivative of a polynomial. It was in 1740 that Thomas Simpson described the algorithm as an iterative method for solving general nonlinear equations using fluxional calculus, essentially obtaining the well-known formula $x_{n+1} = x_n - f(x_n)/f'(x_n)$ for finding the roots of a function f. In the same publication, Simpson also gave the generalisation to systems of two equations and noted that the method can be used for solving optimisation problems. Today, the so-called *Newton's method* (or *Newton-Raphson method*) is probably the most common — and usually efficient — root-finding algorithm.

As in the previous example, Newton's method is frequently used to solve problems of real variable — either in dimension one or greater —, although the plane of complex numbers is often the natural environment provided that the functions to be dealt with do have a certain regularity. Already in 1879, Arthur Cayley applied Newton's method to complex polynomials and tried to identify the basins of attraction of its roots. Cayley did provide a neat solution for this problem in the case of quadratic polynomials, but the cubic case appeared to be far more difficult — and a few years later he finally gave only partial results. Today, it is enough to see the pictures of a few cubic polynomials' dynamical planes to understand why Cayley was never able to work out such a complex structure with the mathematical tools of 125 years ago.

Newton's method associated to a complex holomorphic function f is then defined by the dynamical system

$$N_f(z) = z - \frac{f(z)}{f'(z)}\,.$$

A natural question is what kind of properties we might be interested in or, put more generally, what kind of study we want to make of it. From the dynamical point of view, and given the purpose of any root-finding algorithm, a fundamental question is to understand the dynamics of N_f about its fixed points, as they correspond to the roots of the function f; in other words, we would like to understand the *basins of attraction* of N_f, the sets of points that converge to a root of f under the iteration of N_f.

Basins of attraction are actually just one type of *stable* component or component of the *Fatou set* $\mathcal{F}(f)$, the set of points $z \in \widehat{\mathbb{C}}$ for which $\{f^n\}_{n\geq 1}$ is defined and normal in a neighbourhood of z (recall $\widehat{\mathbb{C}}$ stands for the *Riemann sphere*, the compact Riemann surface $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$). The *Julia set* or *set of chaos* is its complement, $J(f) := \widehat{\mathbb{C}} \setminus \mathcal{F}(f)$. These two sets are named after the French mathematicians Pierre Fatou and Gaston Julia, whose work began the study of modern Complex Dynamics at the beginning of the 20th century.

At first, one could think that if the fixed points of N_f are exactly the roots of f, then Newton's method is a neat algorithm in the sense that it will always

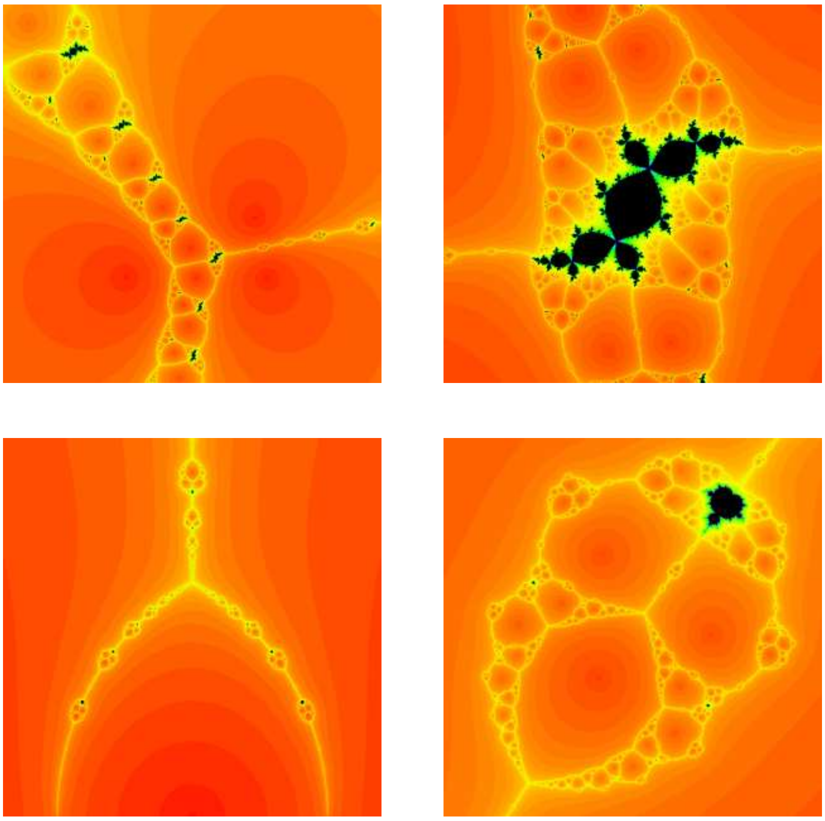

Figure 1.1: The two images above are the dynamical plane of $f_a(z)$ for $a = 0.913+0.424\,i$, and the images below are the parameter space of this family. The black regions on the right-hand pictures (magnifications of the other two) indicate the values of non-convergence. The parameter a has been chosen so that there exists an attracting periodic orbit of period 6.

converge to one of the roots. But notice that not every stable component is a basin of attraction; even not every attracting behaviour is suitable for our purposes: Basic examples like Newton's method applied to cubic polynomials of the form $f_a(z) = z(z-1)(z-a)$, for certain values of $a \in \mathbb{C}$, lead to open sets of initial values converging to attracting periodic cycles. Actually, also the set of such parameters $a \in \mathbb{C}$, for this family of functions, is an open set of the corresponding parameter space (see [15, 19]). Figure 1.1 shows both phenomena in coloured complex planes. Different colours represent different rates of convergence towards the roots of f_a, while black means either convergence somewhere else or non-convergence.

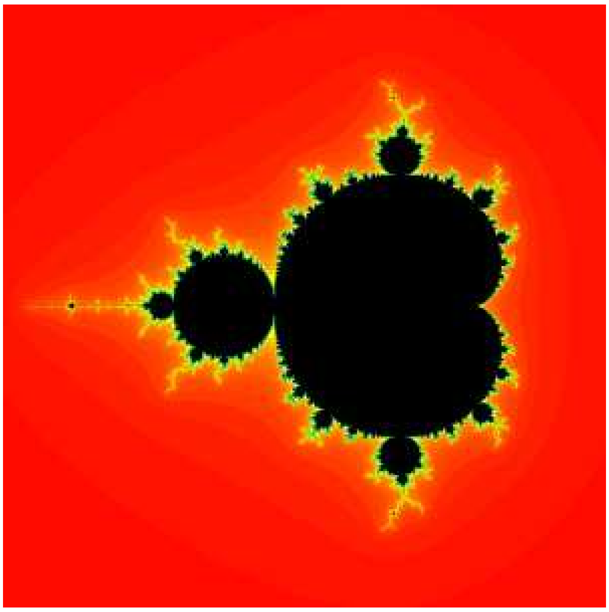

Figure 1.2: The Mandelbrot set.

These facts suggest a division between two directions of dynamical study: On the one hand, given a certain function f, we can try to understand the general behaviour of points under iterates of f, that is to say, the study of its stable and chaos sets — the *dynamical plane*. On the other hand, if we have a family of functions depending on one or several parameters, we might then be interested in knowing for which values of the parameter(s) a certain property occurs — the *parameter space*. A well-known example of this division is given by the family of quadratic polynomials $f_c(z) = z^2 + c$, $c \in \mathbb{C}$, for which the dichotomy between connected and totally disconnected Julia sets has been proved. In this case, the parameter space shows the *Mandelbrot set*, the locus of polynomials $f_c(z)$ with connected Julia set (see Figure 1.2).

The fixed points of N_f are the roots of the function f and the poles of f', since

$$N_f(z) = z \iff z - \frac{f(z)}{f'(z)} = z \iff \frac{f(z)}{f'(z)} = 0\,.$$

When the method is applied to a polynomial, infinity becomes a fixed point as well, whereas if N_f is transcendental, this point is an essential singularity. In Lemma 1.1 we will see when this case occurs.

As for their behaviour, if we compute Newton's method's derivative we have

$$N_f' = 1 - \frac{(f')^2 - f \cdot f''}{(f')^2} = \frac{f \cdot f''}{(f')^2},$$

which means that simple roots of f are superattracting fixed points of N_f. This is an extraordinary property from the point of view of root-finding algorithms, as it is equivalent to say that, in a neighbourhood of such points, N_f is conjugate to $z \mapsto z^k$, for some $k > 1$, for which local convergence is very fast.

Multiple roots of f are attracting fixed points of N_f, but no longer superattracting. In fact, their multiplier is $(m-1)/m$, where m is the multiplicity of the root, so in this case the rate of attraction is linear.

When Newton's method is applied to a polynomial P of degree d, the point at infinity has multiplier $N_P'(\infty) = d/(d-1)$, so it is repelling — in particular, weakly repelling.

Notice that the critical points of N_f are the simple roots of f, as well as its inflection points $\{z \in \widehat{\mathbb{C}} : f''(z) = 0\}$. Of course, every simple root of f is both a critical point and a fixed point of N_f, but inflection points of f become *free* critical points of N_f, which can lead to undesirable Fatou components (as mentioned earlier). From the root-finding point of view, some tools have been developed to cope with this kind of situations: Given a polynomial P, one can find explicitly a finite set of points such that, for every root of P, at least one of the points will converge to this root under N_P (see [29]).

Now let us focus our attention on the case in which f is transcendental. We have the following result (see [8]).

Lemma 1.1. *If a complex function f is transcendental, then so is N_f, except when f is of the form $f = R\,e^P$, with R rational and P a polynomial. In this case, N_f is a rational function.*

The dynamical system N_f for functions of the form $f = R\,e^P$ has also been investigated, especially when f is entire, i.e., of the form $f = Pe^Q$, where P and Q are polynomials. Mako Haruta [28] proved that, if $\deg Q \geq 3$, the area of the basins of attraction of the roots of f is finite. Figen Çilingir and Xavier Jarque [14] studied the area of the basins of attraction of the roots of f in the case $\deg Q = 1$, and Antonio Garijo and Jarque [26] extended the previous results in the cases $\deg Q = 1$ and $\deg Q = 2$. For yet another reference on the subject, see also [30].

It is worth saying that there exist a number of variations of Newton's method, which can improve its efficiency in some cases. One of the most usual versions is the *relaxed Newton's method*, which consists in the iteration of the map $N_{f,h} = \mathrm{id} - h \cdot f/f'$, where h is a fixed complex parameter. In general, for certain choices of rational functions R and parametres h, the method has additional attractors, which causes the algorithm not to work reliably. Nevertheless, it has been proved in [44] that, for almost all rational functions R, the additional attractors vanish if h is chosen sufficiently small.

A lot of literature concerning Newton's method's Julia and Fatou sets has been written, above all when applied to algebraic functions. Feliks Przytycki [35] showed that every root of a polynomial P has a simply connected immediate basin of attraction for N_P. Hans-Günter Meier [33] proved the connectedness of the Julia set of N_P when $\deg P = 3$, and later Tan Lei [43] generalised this result to higher degrees of P. In 1990, Mitsuhiro Shishikura [40] proved the result that actually sets the basis of the present work: For any non-constant polynomial P, the Julia set of N_P is connected (or, equivalently, all its Fatou components are simply connected). In fact, he obtained this result as a corollary of a much more general theorem for rational functions. We denote by a *weakly repelling fixed point* a fixed point which is either repelling or parabolic of multiplier 1 (see Subsection 2.1.1). It was proven by Fatou that every rational function has at least one weakly repelling fixed point (see Theorem 2.6).

Theorem 1.2 (Shishikura [40]). *If the Julia set of a rational function R is disconnected, then R has at least two weakly repelling fixed points.*

Let us see how this applies to Newton's method. If P is a polynomial, then N_P is a rational function whose fixed points are exactly the roots of the polynomial P, plus the point at infinity. The finite fixed points are all attracting, even superattracting if, as roots of P, they are simple. The point at infinity, instead, is always repelling. Hence, rational functions arising from Newton's methods of polynomials have exactly one weakly repelling fixed point and, in view of Theorem 1.2, their Julia set must be connected.

This Thesis, however, deals with Newton's method applied to transcendental maps. In the same direction, in 2002 Sebastian Mayer and Dierk Schleicher [32] extended Przytycki's theorem by showing that every root of a transcendental entire function f has a simply connected immediate basin of attraction for N_f. This work has been recently continued by Johannes Rückert and Schleicher in [38], where they study Newton maps in the complement of such Fatou components. Our long-term goal is to prove the natural transcendental versions of Shishikura's results — although this Thesis covers just part of it —, which can be conjectured as follows.

Conjecture 1.3. *If the Julia set of a transcendental meromorphic function f is disconnected, there exists at least one weakly repelling fixed point of f.*

It is important to notice that essential singularities are always in the Julia set of a transcendental meromorphic function f and therefore infinity can connect two unbounded connected components of $\mathcal{J}(f) \cap \mathbb{C}$ otherwise disconnected.

Observe that Fatou's theorem on weakly repelling fixed points only applies to rational maps. For transcendental maps, the essential singularity at infinity plays the role of the weakly repelling fixed point, and therefore no such point must necessarily be present for an arbitrary map. From this fact, and from the discussion above about Newton's method, it follows that transcendental meromorphic functions that come from applying Newton's method to transcendental entire functions happen to have no weakly repelling fixed points at all, so the next result is obtained forthwith.

Conjecture 1.4 (Corollary). *The Julia set of the Newton's method of a transcendental entire function is connected.*

Recall that the Julia set (closed) is the complement of the Fatou set (open). Hence, as it was already mentioned, the connectivity of the Julia set is equivalent to the simple connectivity of the Fatou set. Because of this fact, a possible proof of Conjecture 1.3 splits into several cases, according to different Fatou components (see Section 3.2). In this Thesis we will see three of such cases (see [23, 24]), which, together, give raise to the following result.

Main Theorem 1.5. *Let f be a transcendental meromorphic function with either a multiply-connected attractive basin, or a multiply-connected parabolic basin, or a multiply-connected Fatou component with simply-connected image. Then, there exists at least one weakly repelling fixed point of f.*

Notice how this theorem actually connects with the result of Mayer and Schleicher mentioned above.

In order to prove this theorem, we use mainly two tools: the method of *quasiconformal surgery* and a theorem of Xavier Buff on *virtually repelling fixed points*. On the one hand, quasiconformal surgery (see Section 2.4) is a powerful tool that allows to create holomorphic maps with some prescribed dynamics. One usually starts glueing together — or *cutting* and *sewing*, this is why this procedure is called 'surgery' — several functions having the required dynamics; in general, the map f obtained is not holomorphic. However, if certain conditions are satisfied, the Measurable Riemann Mapping Theorem, due to Charles Morrey, Bogdan Bojarski, Lars Ahlfors and Lipman Bers, can be applied to find a holomorphic map g, conjugate to the original function g. On the other hand, Buff's theorem states that, under certain local conditions, a map possesses a virtually repelling fixed point. These conditions are a generalization of the polynomial-like setup and the property of being a virtually repelling fixed point is only slightly stronger than that of weakly repelling. Hence in those cases where we can apply Buff's theorem, the result follows in a very direct way.

Structure of the Thesis. This Introduction puts the subject of the Thesis into historical context and gives a little state of the art about the study of the topology of the Fatou and Julia sets of the dynamical system generated by applying Newton's method to polynomials and transcendental entire functions. In particular, it gives Shishikura's main result and our 'transcendental' conjectures and Main

Theorem. Chapter 2 provides us with some background tools from various topics in Holomorphic Dynamics, to be used in the following chapters. These topics range from pure Dynamical Systems stuff, such as basics on iteration theory or the classification of Fatou components, to concepts coming from other fields, like quasiconformal surgery from Analysis or local connectivity from Topology. In these 'borrowed stuff' cases we will see how such concepts are adapted to Holomorphic Dynamics and become actual tools in our context. Sections 3, 4 and 5 contain our proof for our Main Theorem, separated by type of Fatou component. Thus, Section 3 is dedicated to the proof for the case of immediate attractive basins, Section 4 to parabolic basins and Section 5 to preperiodic Fatou components. Also, what actually opens Section 3 is a preamble with Shishikura's proof for the attractive rational case plus an introduction to the general transcendental case that tells how our main conjecture splits into the different Fatou-component cases. Finally, Section 6 rounds up our global case-by-case discussion with a collection of results and ideas about wandering domains, Herman rings and Baker domains, for completeness. The section concludes with some remarks about future projects and further work on the subject.