Singular Points and Periodic Orbits for a Vector Fields

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In the qualitative theory of planar differential equations knowing the number of limit cycles of a concrete family is, from the work of H. Poincaré, an open problem. In this memory, we mainly study the following two problems: How many limit cycles can be obtained from a critical point of monodromic type in a degenerate Hopf bifurcation? How many limit cycles can bifurcate from the level curves of the Hamiltonian $H = \frac{1}{2}(x^2 + y^2)$?

In Chapter 1 we study the center-focus problem, i. e. how to obtain necessary conditions to decide whether the origin is a center or a focus. A. M. Lyapunov proved that the return map can be used to study the stability of the origin by means of the sign of the first non-vanishing coefficient of the return map. This method introduces the Lyapunov constants. J. P. Françoise, in 1996, obtains an algorithm to compute this first non-vanishing coefficient of the return map for a system which is a perturbation of a linear center. In this chapter, we generalise this result when the perturbation is an analytic function of the parameter. This new method is very useful to study the center-focus problem for several families both for theoretical and for practical purposes.

In Chapter 2, we study the relation between both numbers of limit cycles presented by the two problems above mentioned. In particular, we compute how many limit cycles can bifurcate from the level curves of the Hamiltonian in the case of a Liénard equation. With some natural hypothesis, for systems with homogeneous nonlinearities, we prove that these two numbers coincide. Finally, we analyse the main difficulties to study the number of limit cycles that can born from the origin in a degenerate Hopf bifurcation, i.e. the ciclicity of the origin, and also the number of necessary Lyapunov constants to determine the centers of an equation.

Chapter 3 is dedicated to study the return map for the perturbed problem, and to use it to obtain the shape of the periodic orbits in terms of the parameter. As an example, we apply this method to the limit cycle of the van der Pol equation and, moreover, we show an example of double bifurcation in a Liénard equation. Finally, we obtain the return map for two piecewise analytic systems, which allows us to characterize all their centers.

In Chapter 4, we generalise a known result by Cherkas that characterizes the centers of a Liénard equation for the degenerate case. This result can be also applied to obtain all the centers for a polynomial case and for a subclass of a quasihomogeneous family, in contrast to the method described in Chapter 1 which we have found to be useful only to obtain a set of necessary center conditions.

In Chapter 5, we study the maximum order of degeneracy of the origin as a fine focus for a Liénard equation. From a recent paper of C. J. Christopher and N. G. Lloyd, the computation of this number can be reduced to compute the multiplicity at the origin of a polynomial map. Moreover, we can obtain new lower and upper bounds for this maximum order, and for some particular examples we show that this method is easier to use and faster than the methods based in the computation of Lyapunov constants. In this chapter, we generalise these results for the degenerate case.

Finally, in Chapter 6, we introduce the notion of index and multiplicity at zero of a map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and we study if there exists any relation between these two numbers. In particular, we show that the inequality given by D. Eisenbud and H. Levine is optimal for $n = 2$ but it is not for the case $n \geq 2$. Finally, we can compute the sum of indices of all zeros of a map from $\mathbb{R}^n$ to $\mathbb{R}^n$.