Facultat de Cincies
Departament de Matemtiques

Index of vector fields on manifolds and isochronicity for planar Hamiltonian differential systems

Jordi Villadelprat Yage

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Els Drs. Anna Cima Mollet i
Francesc Maosas Capellades
CERTIFIQUEM que la present memria ha estat realitzada per Jordi Villadelprat Yage sota la nostra direcci, al Departament de Matemtiques de la Universitat Autonma de Barcelona.

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Contents

Introduction

The contents of this memoir deal with Qualitative Theory of Differential Equations. It consists essentially of two parts that can be read independently. In the first part, composed by Chapters 1, 2 and 3, we consider vector fields on manifolds of arbitrary dimension and we discuss questions related to their index. In the second part, which is formed by Chapters 4, 5, 6 and 7, we study the isochronicity problem for planar Hamiltonian differential equations.

The first part of this work treats questions concerning the index of vector fields. Given a tangent vector field X on a compact manifold M, the index of X, that we denote by $\mathrm{Ind}(X)$, measures the variation of the direction of X around its critical points. The index of X provides qualitative information about the global behaviour of the flow generated by the differential equation $\dot{x} = X(x)$. For instance, in case that M is an even dimensional sphere we can assure the existence of at least one critical point since the index of X must be different from zero. The index can also be used to investigate if a critical point is asymptotically stable because the local index of such a critical point in a n-dimensional manifold is equal to $(-1)^n$.

The Poincar-Hopf Theorem is perhaps one of the most classical results dealing with the question of the index of a vector field. It can be applied in case that M is compact and X is nonvanishing on ∂M, the boundary of M. It shows that if X is never pointing outward at ∂M then

$$\mathrm{Ind}(X) = (-1)^n \mathcal{X}(M), \tag{1}$$

where $\mathcal{X}(M)$ and n denote respectively the Euler characteristic of M and its dimension. It is usually stated for vector fields having finitely many critical points, and in this case $\mathrm{Ind}(X)$ is the sum of the local indices of X at all its critical points.

A two-dimensional version of this theorem was proved by Poincar in 1885. The full theorem was proved by Hopf in 1926 after earlier partial results by Brouwer, Lefschetz and Hadamard.

The Poincar-Hopf Theorem is a very useful tool to prove results that relate the topology of M with the qualitative behaviour of the solutions of $\dot{x} = X(x)$. Many works about polynomial vector fields use it because a polynomial vector field on $\mathbb{R}^n$ can be extended analytically to a vector field defined on the n-dimensional sphere (which is a compact and boundaryless manifold). For example, Cima and Llibre [13] prove that any bounded polynomial vector field X on $\mathbb{R}^2$ satisfy that the sum of the local indices of X at all its critical points is equal to 1. A vector field X is said to be bounded when any solution of $\dot{x} = X(x)$ has nonempty and compact ω-limit set.

The Poincar-Hopf Theorem solves the problem of computing $\mathrm{Ind}(X)$ in case that M is a compact and boundaryless manifold. In fact it shows that this number is a topological invariant of the manifold. However it is not very helpful when M has nonempty boundary because then the assumptions are too restrictive. To approach this problem, several authors have proved formulas to compute the index of vector fields that do not behave so simply at ∂M. The first work in this direction is due to Morse [36]. In 1929 he proved that, under more general conditions on $X|_{\partial M}$, the index of X is equal to the sum of the Euler characteristics of certain submanifolds defined by means of the behaviour of X at ∂M. Until now the generalizations of the Poincar-Hopf Theorem have followed this direction. Thus, it is to be referred the contribution of Pugh [38], who proves a similar formula to compute $\mathrm{Ind}(X)$ that can be applied to almost any vector field (that is, the condition to apply the formula is generic).

In fact there is a classical result, proved by Poincar in 1881, that allows to compute the index of the vector field in case that M is a two-dimensional disc. The idea is to relate the behaviour of X at ∂M with the existence of a special type of trajectories of $\dot{x} = X(x)$. These are the so called inner and outer contact points. Thus, if X has finitely many tangency points with the boundary of the disc, the Poincar Index Formula asserts that the sum of the local indices of X at all the critical points inside the disc is equal to

$$1 + \frac{\sigma - \nu}{2},$$

where σ and ν are respectively the number of inner and outer contact points. This shows that the notion of inner and outer contact point picks up all the information that is needed to compute the index. Llibre and Ye Yanquian [30] have generalized this result to any compact and connected two-dimensional manifold.

In Chapter 1, given a tangent vector field X on a n-dimensional manifold M (not necessarily compact), we study the flow generated by $\dot{x} = X(x)$ and we consider an attractor compact set K with region of attraction $\mathcal{A}$. The results that we obtain show that the topology of $\mathcal{A}$ is closely connected with the local indices of X at the critical points inside K. Indeed, we prove that $\mathcal{X}(\mathcal{A})$ is defined and that the sum of the local indices of X at all its critical points in K is equal to $(-1)^n\mathcal{X}(\mathcal{A})$. Moreover, when K is regular enough so that $\mathcal{X}(K)$ is well defined, we show that if K is asymptotically stable and invariant then $\mathcal{X}(K) = \mathcal{X}(\mathcal{A})$.

As a consequence of the above results, we can assert that if there is a global compact attractor for the flow generated by X then $\mathcal{X}(M)$ and $\mathrm{Ind}(X)$ are well defined and satisfying relation (1). From the dynamical point of view, this is the natural generalization of the Poincar-Hopf Theorem to noncompact manifolds. Note that when M is compact, the condition that X is never pointing outward at ∂M is equivalent to require the existence of a global compact attractor. Since this equivalence is obviously not true for noncompact manifolds, one may wonder if there is some weaker condition that yields the same relation. However there is not any. This is so because if it is only required, for example, that each solution has nonempty ω-limit set (and this implies that X never points outward at ∂M if any) then the relation does not hold any more. Even more, it is not satisfied either if it is required that X is a bounded vector field, and in Chapter 2 we will show examples of this.

In Chapter 2 we study bounded vector fields on $\mathbb{R}^n$. Our initial goal was to prove that any $\mathcal{C}^1$ bounded vector field on $\mathbb{R}^n$ satisfies that the sum of the local indices at all its critical points is equal to $(-1)^n$. We thought that this could be true because Cima and Llibre [13] proved it for any polynomial vector field if $n = 2$, and for any polynomial vector field satisfying a generic condition if $n \geq 3$. In this chapter we prove that for $\mathcal{C}^1$ vector fields the result is true in case that $n = 2$. On the other hand, when $n \geq 3$ we show that there are $\mathcal{C}^\infty$ bounded vector fields on $\mathbb{R}^n$ which do not satisfy this relation. In fact, since $\mathcal{X}(\mathbb{R}^n) = 1$, these are the examples that we have mentioned above.

In Chapter 3 we turn to the problem of computing the index of a tangent vector field on a compact manifold M, and we focus on the case in which M is two-dimensional and connected. With the same philosophy as the Poincar Index Formula, we obtain a result that allows to calculate it. Our result improves previous ones because it does not require that the vector field has finitely many tangency points with ∂M.

In the second part of this memoir we study the period function of the centers of planar analytic Hamiltonian differential systems,

$$\begin{cases} \dot{x} = -H_y(x,y), \\ \dot{y} = H_x(x,y). \end{cases} \tag{2}$$

Given a center of a planar differential system, the period function gives the period of each periodic orbit inside its period annulus. We are mainly interested in the case in which the period function is constant, the center is then said to be isochronous.

The study of isochronous systems started even before the development of the differential calculus. It goes back at least to Galileo, who found out in 1632 the isochronicity of small oscillations of a simple pendulum. Afterwards, in 1673, Huygens gave the formula of its period. Huygens also described the first nonlinear isochronous pendulum, a particle constrained to move on a cycloid under the action of gravity. He was trying to design a clock with isochronous oscillations in order to have a more accurate clock for the navigation of ships. Isochronous systems were later studied by Euler, Bernoulli and Lagrange.

The characterization of the vector fields with isochrones is still a difficult and unsolved problem. In fact, in the polynomial case, it has strong parallelisms with the so called center-focus problem. Both problems are completely solved only in the quadratic case.

Questions about the behaviour of the period function have been extensively studied by a number of authors. In relation to their applications, some works have been motivated by the desire to find conditions under which the period function is monotone (see [6, 43] for instance). This is so because monotonicity is a nondegeneracy condition for the bifurcation of subharmonic solutions of periodically forced Hamiltonian systems, and because monotonicity implies existence and uniqueness for certain boundary value problems. In fact, for these boundary value problems, the range of values of the period function determines the set of initial conditions for which the problem has at least one solution. The interested reader is referred to [7, 9, 45]. We note that in these problems isochronicity represents the worst possible setting.

On the other hand, to fully understand the properties of the period function in a family of differential systems, it is necessary to consider the bifurcation of critical points of the period function. This is done by calculating the Taylor series of the period function in a neighbourhood of the center. When this is performed on a polynomial (or even analytic) family, the coefficients of the period function are polynomials in the coefficients of the system. Once again the problem becomes more difficult in the cases in which the center is isochronous. Therefore a complete study of these bifurcations requires the identification of the subclass of systems that have an isochronous center. The work of Chicone and Jacobs in [8] is a good example of this approach.

Concerning polynomial Hamiltonian systems, Sabatini [41] shows that there is a strong connection between isochronicity and the Jacobian conjecture in $\mathbb{R}^2$. Indeed, this conjecture is equivalent to prove that if we take a polynomial canonical mapping $(x,y) \longmapsto \big(P(x,y), Q(x,y)\big)$ then the isochronous center of (2) given by

$$H(x,y) = \frac{P(x,y)^2 + Q(x,y)^2}{2}$$

is global (that is, its period annulus is the whole $\mathbb{R}^2$). By a canonical mapping we mean a mapping satisfying that the determinant of its jacobian is equal to one at any point. As a matter of fact, in the literature there are only examples of polynomial Hamiltonian systems with global isochronous centers.

In spite of the fact that the isochronicity problem has been studied by many authors, a characterization of this phenomenon is far to be found. The most significant results for general polynomial systems are due to Loud [31], who classifies the quadratic ones, and Pleshkan [37], that classifies the cubic systems with homogeneous nonlinearities. In relation to Hamiltonian systems, several authors (see [10, 20, 44]) have proved that there is not any isochronous center in those systems that have homogeneous nonlinearities. Moreover Cima, Gasull and Maosas [15] have shown recently that the only isochronous center among the polynomial systems associated to $H(x,y) = F(x) + G(y)$ is the linear one.

As we have already said, the second part of this work is devoted to study the isochronicity problem in analytic Hamiltonian differential systems, and we shall often pay attention to the polynomial case. In order to study the isochronicity in a polynomial family, one can compute the Taylor series of the period function to obtain what are called the period constants. These are strongly related to the Lyapunov constants, which give the center conditions. The vanishing of all the period constants provides necessary and sufficient conditions for isochronicity. When a family with finitely many parameters is considered then there is only a finite number of period constants that are significant (they already give the isochronicity conditions). This fact follows from the Hilbert's basis Theorem, but in general this number is unknown a priori. Consequently the method is to calculate some period constants and then to investigate if its vanishing is sufficient for isochronicity. The works of Christopher and Devlin [10] and Rousseau and Toni [39] on the Kukles system are a good example of this manner of approaching the problem. This algorithmic method is similar to search the center conditions by means of the Lyapunov constants and its applicability raises the same difficulties. Thus in most cases even the calculation of the period constants is intractable with a computer. The algebraic method described above hides completely the mechanism by which a center is isochronous. We shall not use it because the Hamiltonian systems furnish more geometric tools.

It is well known that an isochronous center of an analytic differential system is nondegenerate. There is a classical result, due to Poincar, that shows that a nondegenerate center of an analytic system can be brought by means of an analytic change of coordinates to the normal form

$$\begin{cases} \dot{x} = -y\, f(x^2+y^2), \\ \dot{y} = x\, f(x^2+y^2), \end{cases}$$

where $f(0) \neq 0$. We call normalization this coordinate transformation.

One of the purposes of Chapter 4 is to prove that for a nondegenerate center of an analytic Hamiltonian system there exists an analytic normalization being also canonical. In addition we show that any of these canonical normalizations, contrary to the usual ones, brings system (2) to the same normal form, what we call the canonical normal form (CNF in shortened form). The CNF of a nondegenerate center depends only on the function that gives the area of each periodic orbit inside the period annulus.

As a corollary of the above result we get a characterization of isochronicity for analytic Hamiltonian systems. Indeed, we prove that a center of (2) is isochronous if and only if there exists an analytic canonical mapping $(x,y) \longmapsto \big(g_1(x,y), g_2(x,y)\big)$ so that

$$H(x,y) = \frac{g_1(x,y)^2 + g_2(x,y)^2}{2}. \tag{3}$$

Concerning this last result, it is worth mentioning that Sabatini [40] proves the existence of a $\mathcal{C}^\infty$ canonical mapping such that.

In Chapter 5 we study the isochronicity problem in the Hamiltonian systems of the form "kinetic+potential". More concretely, the ones associated to

$$H(x,y) = \frac{y^2}{2} + V(x),$$

where V is an analytic function on $\mathbb{R}$. The aim of this chapter is not to give new results but to approach the problem in a more geometric manner. This approach allows us to give a simple proof of two results due to Chicone and Jacobs [8] and Urabe [50]. We relate the isochronicity of the center to the shape of the periodic orbits inside the period annulus. We also show the strong connection between isochronicity and involutions on $\mathbb{R}$.

As it has been said before, in Chapter 4 we show that for any analytic Hamiltonian system with an isochronous center there exists an analytic canonical mapping so that the Hamiltonian can be written in the form (3). At this point it is natural to investigate if it occurs the same in the polynomial case. That is, given a polynomial Hamiltonian system having an isochronous center, does there exist a polynomial canonical mapping so that (3) holds? We note that all the examples appearing in the literature can be obtained in this way. In fact this is precisely the reason why there is not any example of polynomial Hamiltonian system with a nonglobal isochronous center (such a system would give a counterexample to the Jacobian conjecture).

In Chapter 6 we show that the question posed above has a negative answer, and we give examples of polynomial Hamiltonian systems with a nonglobal isochronous center. This is achieved by studying the Hamiltonian systems associated to

$$H(x,y) = A(x) + B(x)\,y + C(x)\,y^2,$$

where A, B and C are analytic functions on $\mathbb{R}$. These systems include the ones considered in Chapter 5, and we call them quadratic-like Hamiltonian systems. We assume that the origin is a center and we give necessary and sufficient conditions for its isochronicity. In the polynomial case, a complete classification of the isochronous centers is given when C has at most degree three.

Finally, in Chapter 7 we characterize the cubic Hamiltonian isochronous centers. More precisely, we determine the fourth degree polynomials $H(x,y)$ for which the differential system (2) has an isochronous center. To this end we begin by showing some necessary conditions for the isochronicity of a polynomial Hamiltonian system of arbitrary degree. These conditions enable us to suppose without loss of generality that any cubic Hamiltonian system with an isochronous center is quadratic-like. We then take advantage of the results proved in Chapter 6. It is to be noted that in [32] it is given a characterization of the cubic Hamiltonian isochronous centers which are Darboux linearizable.

The two parts that constitute this memoir can be read independently. Each chapter has an introduction which explains in detail the questions that are discussed, and the main theorems are enumerated alphabetically.

The results of Chapter 1 have appeared in *Topology* under the title “A Poincar-Hopf Theorem for noncompact manifolds,” and the ones of Chapter 2 constitute a work entitled “On bounded vector fields” that will appear in *Rocky Mountain Journal of Mathematics.* The results of Chapters 5, 6 and 7 form a work entitled “Isochronicity for several classes of Hamiltonian systems” that have been accepted for publication in *Journal of Differential Equations.* These three papers are a join work with Anna Cima and Francesc Maosas. The results that appear in Chapter 3 are a join work with Jaume Llibre and have been published in *Differential and Integral Equations* under the title “A Poincar Index Formula for surfaces with boundary.”