Periodic solutions of a class of second–order differential equation

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We study the periodic solutions of the second–order differential equations of the form

$$\ddot{x}+3x\dot{x}+x^3+F(t)(\dot{x}+x^2)+G(t)x+H(t)=0,$$

where the functions $F(t)$, $G(t)$ and $H(t)$ are periodic of period 2π in the variable t.

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1. Introduction and statement of the main results

In this paper we shall study the existence of periodic solutions of the second–order differential equation of the form

$$\ddot{x}+3x\dot{x}+x^3+F(t)(\dot{x}+x^2)+G(t)x+H(t)=0, \tag{1.1}$$

where the dot denotes derivative with respect to the time t, and the functions $F(t)$, $G(t)$ and $H(t)$ are periodic of period 2π in the variable t.

We note that the second-order differential equation (1.1) when $F=G=H=0$ appears in the Ince's catalog of equations possessing the Painlevé property, see [5]. Moreover, the differential equation $\ddot{x}+3x\dot{x}+x^3=0$ is well known in many areas of mathematics and physics, and it possesses the algebra $\mathrm{sl}(3,\mathbb{R})$ of Lie point symmetries, see for more details the paper [6] and the references quoted there.

In a recent paper [2] (see also [3,4]) the second-order differential equation (1.1) has been studied when $F=H=0$. See also [7] for a study of coupled quadratic unharmonic oscillators in terms of the Painlevé analysis and integrability.

Here we study the periodic solutions of the second-order differential equation (1.1) when $F(t) = \varepsilon f(t)$, $G(t) = 1+\varepsilon g(t)$, and $H(t) = \varepsilon^k h(t)$ with $k = 1,2$. Our main results are the following ones.

Theorem 1.1. *We define the functions*

$$\begin{aligned} \mathscr{F}_1(X_0,Y_0) &= -\int_0^{2\pi} F(t,X_0,Y_0)\sin t\,dt, \\ \mathscr{F}_2(X_0,Y_0) &= \int_0^{2\pi} F(t,X_0,Y_0)\cos t\,dt, \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} F(t,X_0,Y_0) &= -h(t)-g(t)A(t)-f(t)B(t)-3A(t)B(t), \\ A(t) &= X_0\cos t+Y_0\sin t, \\ B(t) &= -X_0\sin t+Y_0\cos t. \end{aligned}$$

Assume that the functions $F(t) = \varepsilon f(t)$, $G(t) = 1+\varepsilon g(t)$ and $H(t) = \varepsilon^2 h(t)$ are 2π–periodic. Then for $\varepsilon \neq 0$ sufficiently small and for every (X_0^*,Y_0^*) solution of the system $\mathscr{F}_j(X_0,Y_0) = 0$ for $j = 1,2$, satisfying

$$\det\left(\frac{\partial\left(\mathscr{F}_1,\mathscr{F}_2\right)}{\partial\left(X_0,Y_0\right)}\right)\Big|_{(X_0,Y_0)=\left(X_0^*,Y_0^*\right)} \neq 0, \tag{1.3}$$

the differential equation (1.1) *has a 2π–periodic solution $x(t,\varepsilon) = \varepsilon(X_0^*\cos t+Y_0^*\sin t)+O(\varepsilon^2)$.*

Theorem 1.1 is proved in section 3 using the averaging theory described in section 2. Two applications of Theorem 1.1 are the following.

Corollary 1.1. *We consider the differential equation* (1.1) *with $F(t) = \varepsilon\ (1-\cos^2 t)$, $G(t) = 1+\varepsilon\sin^2 t$ and $H(t) = \varepsilon^2\sin t$. Then for $\varepsilon \neq 0$ sufficiently small this differential equation has a 2π–periodic solution $x(t,\varepsilon) = \varepsilon 2(\sin t-\cos t)/3+O(\varepsilon^2)$.*

Corollary 1.2. *We consider the differential equation* (1.1) *with $F(t) = \varepsilon\ (1-\cos^2 t+2\cos^4 t)$, $G(t) = 1+\varepsilon(\sin^2 t+2\sin^4 t)$ and $H(t) = \varepsilon^2(\sin t+\sin^3 t)$. Then for $\varepsilon \neq 0$ sufficiently small this differential equation has a 2π–periodic solution $x(t,\varepsilon) = \varepsilon(21\cos t-7\sin t)/20+O(\varepsilon^2)$.*

Corollaries 1.1 and 1.2 are proved also in section 3.

Theorem 1.2. *Assume that*

$$\int_0^{2\pi} h(t)\sin t\,dt = 0, \qquad \int_0^{2\pi} h(t)\cos t\,dt = 0,$$

and set

$$\begin{aligned} \mathscr{F}_1(X_0,Y_0) &= -\int_0^{2\pi} f(t,X_0,Y_0)\sin t\,dt, \\ \mathscr{F}_2(X_0,Y_0) &= \int_0^{2\pi} f(t,X_0,Y_0)\cos t\,dt, \end{aligned} \tag{1.4}$$

with

$$f(t,X_0,Y_0) = -g(t)A(t) - f(t)B(t) - 3A(t)B(t),$$
$$A(t) = X_0\cos t + Y_0\sin t - \int_0^t h(\tau)\sin(t-\tau)d\tau,$$
$$B(t) = -X_0\sin t + Y_0\cos t - \int_0^t h(\tau)\cos(t-\tau)d\tau.$$

Assume that $F(t) = \varepsilon f(t)$, $G(t) = 1+\varepsilon g(t)$ and $H(t) = \varepsilon h(t)$ are 2π–periodic functions. Then for $\varepsilon \neq 0$ sufficiently small and for every (X_0^*, Y_0^*) solution of the system $\mathscr{F}_j(X_0,Y_0) = 0$ for $j = 1,2$ satisfying (1.3)*, the differential equation* (1.1) *has a periodic solution*

$$x(t,\varepsilon) = \varepsilon\left(X_0^*\cos t + Y_0^*\sin t - \int_0^t h(\tau)\sin(t-\tau)d\tau\right) + O(\varepsilon^2).$$

Theorem 1.2 is proved in section 4. Two applications of Theorem 1.2 are the following.

Corollary 1.3. *We consider the differential equation* (1.1) *with $F(t) = \varepsilon\ (\sin(2t)+\cos(2t))$, $G(t) = 1+\varepsilon\sin t$ and $H(t) = \varepsilon\ 2\cos^2 t$. Then for $\varepsilon \neq 0$ sufficiently small this differential equation has a 2π–periodic solution*

$$x(t,\varepsilon) = \varepsilon\left((-2\cos t + 15\sin t)/31 + 2\cos^2 t(\cos t - 1)\right) + O(\varepsilon^2).$$

Corollary 1.4. *We consider the differential equation* (1.1) *with $F(t) = \varepsilon\sin t$, $G(t) = 1+\varepsilon\sin^2 t$ and $H(t) = \varepsilon 2\cos(2t)$. Then for $\varepsilon \neq 0$ sufficiently small this differential equation has a periodic solution*

$$x(t,\varepsilon) = \varepsilon\left(2(\cos t - 1)\cos(2t) - \frac{8}{5}\sin t\right) + O(\varepsilon^2).$$

Corollaries 1.3 and 1.4 are proved also in section 4.

2. Basic results on averaging theory

In this section we present the results from the averaging method that we shall need for proving our results.

We work with differential systems of the form

$$\mathbf{x}' = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \tag{2.1}$$

where ε is a small parameter, and the functions $F_0, F_1 : \mathbb{R}\times\Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^n$ are $\mathscr{C}^2$ functions, T–periodic in the variable t, and Ω is an open subset of $\mathbb{R}^n$. We assume that the unperturbed system

$$\mathbf{x}' = F_0(t,\mathbf{x}), \tag{2.2}$$

has a submanifold of dimension n filled of T–periodic orbits, i.e. of periodic orbits of period T.

Let $\mathbf{x}(t,\mathbf{z},0)$ be the solution of system (2.2) such that $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z}$. The first variational equation of system (2.2) on the periodic solution $\mathbf{x}(t,\mathbf{z},0)$ is given by

$$\mathbf{y}' = D_{\mathbf{x}}F_0(t,\mathbf{x}(t,\mathbf{z},0))\mathbf{y}, \tag{2.3}$$

where $\mathbf{y}$ is an $n\times n$ matrix. Let $M_{\mathbf{z}}(t)$ be the fundamental matrix of system (2.3) satisfying that $M_{\mathbf{z}}(0)$ is the identity matrix of $\mathbb{R}^n$.

By hypotheses there is an open set V such that $\mathrm{Cl}(V) \subset \Omega$ and for each $\mathbf{z} \in \mathrm{Cl}(V)$, $\mathbf{x}(t,\mathbf{z},0)$ is a T–periodic solution. There is the following result.

Theorem 2.1. *There exists an open and bounded set V with $\mathrm{Cl}(V) \subset \Omega$ satisfying that for each $\mathbf{z} \in \mathrm{Cl}(V)$, the solution $\mathbf{x}(t,\mathbf{z},0)$ is T–periodic, and let $\mathscr{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$ be the function*

$$\mathscr{F}(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(t) F_1(t,\mathbf{x}(t,\mathbf{z},0))dt. \tag{2.4}$$

If there exists $\alpha \in V$ with $\mathscr{F}(\alpha) = 0$ and $\det((d\mathscr{F}/d\mathbf{z})(\alpha)) \neq 0$, then there is a T–periodic solution $\mathbf{x}(t,\varepsilon)$ of system (2.1) such that $\mathbf{x}(t,\varepsilon) = \mathbf{x}(t,\mathbf{z},0) + O(\varepsilon)$.

Theorem 2.1 was proved by Malkin [8] and Roseau [9], for a new and shorter proof see [1].

3. Proof of Theorem 1.1 and its two corollaries

Proof of Theorem 1.1. Introducing the variable $y = \dot{x}$, we can write the second–order differential equation (1.1) as the following first-order differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -3xy - x^3 - F(t)(y + x^2) - G(t)x - H(t). \end{aligned} \tag{3.1}$$

Doing the rescaling $(x,y) = (\varepsilon X, \varepsilon Y)$, we obtain the system

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= -X + \varepsilon(-h(t) - g(t)X - f(t)Y - 3XY) + \varepsilon^2(-f(t)X^2 - X^3). \end{aligned} \tag{3.2}$$

System (3.2) with $\varepsilon = 0$ is the unperturbed system, otherwise system (3.2) is the perturbed system. The unperturbed system has a unique singular point, the origin of coordinates. The solution $(X(t),Y(t))$ of the unperturbed system such that $(X(0),Y(0)) = (X_0,Y_0)$ is

$$X(t) = X_0 \cos t + Y_0 \sin t, \qquad Y(t) = -X_0 \sin t + Y_0 \cos t.$$

Note that all these periodic orbits have period 2π. Using the notation introduced in section 2. We have that $\mathbf{x} = (X,Y)$, $\mathbf{z} = (X_0,Y_0)$, $F_0(\mathbf{x},t) = (Y,-X)$, $F_1(\mathbf{x},t) = (0, -h(t) - g(t)X - f(t)Y - 3XY)$ and $F_2(\mathbf{x},t) = (0, -f(t)X^2 - X^3)$.

The fundamental matrix solution $M_{\mathbf{z}}(t)$ is independent of the initial condition $\mathbf{z}$, and denoting it by $M(t)$ we obtain

$$M(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Now we compute the function $\mathscr{F}(\mathbf{z}) = (\mathscr{F}_1(X_0,Y_0), \mathscr{F}_2(X_0,Y_0))$ given in (2.4), and we get the functions (1.2) of the statement of Theorem 1.1.

By Theorem 2.1 each zero (X_0^*,Y_0^*) of system $\mathscr{F}_1(X_0,Y_0) = \mathscr{F}_2(X_0,Y_0) = 0$ satisfying (1.3), provides a 2π–periodic solution $(X(t,\varepsilon),Y(t,\varepsilon))$ of system (3.2) with $\varepsilon \neq 0$ sufficiently small such that

$$(X(t,\varepsilon),Y(t,\varepsilon)) = (X_0^* \cos t + Y_0^* \sin t, -X_0^* \sin t + Y_0^* \cos t) + O(\varepsilon).$$

Going back through the change of variables for every periodic solution $(X(t,\varepsilon),\, Y(t,\varepsilon))$ of system (3.2) with $\varepsilon \neq 0$ sufficiently small, we obtain a 2π–periodic solution $x(t,\varepsilon) = \varepsilon(X_0^* \cos t + Y_0^* \sin t) +$

$O(\varepsilon^2)$ of the differential equation (1.1) with $\varepsilon \neq 0$ sufficiently small. This completes the proof of Theorem 1.1. □

Proof of Corollary 1.1. We must apply Theorem 1.1 with

$$f(t) = 1 - \cos^2 t, \quad g(t) = \sin^2 t, \quad h(t) = \sin t.$$

We compute the functions $\mathscr{F}_1$ and $\mathscr{F}_2$ of the statement of Theorem 1.1, and we obtain

$$\mathscr{F}_1(X_0, Y_0) = \frac{\pi}{4}(4 - 3X_0 + 3Y_0), \quad \mathscr{F}_2(X_0, Y_0) = \frac{\pi}{4}(-X_0 - Y_0).$$

System $\mathscr{F}_1 = \mathscr{F}_2 = 0$ has the zero $(X_0^*, Y_0^*) = (2/3, -2/3)$. Since the Jacobian (1.3) at this zero is $3\pi^2/8$, we obtain using Theorem 1.1 the periodic solution given in the statement of the corollary. □

Proof of Corollary 1.2. We apply Theorem 1.1 with

$$f(t) = 1 - \cos^2 t + 2\cos^4 t, \quad g(t) = \sin^2 t + 2\sin^4 t, \quad h(t) = \sin t + \sin^3 t.$$

Computing the functions $\mathscr{F}_1$ and $\mathscr{F}_2$ of Theorem 1.1 we get

$$\mathscr{F}_1(X_0, Y_0) = \frac{\pi}{4}(7 - 4X_0 + 8Y_0), \quad \mathscr{F}_2(X_0, Y_0) = -\frac{\pi}{2}(X_0 + 3Y_0).$$

System $\mathscr{F}_1 = \mathscr{F}_2 = 0$ has the zero $(X_0^*, Y_0^*) = (21/20, -7/20)$. Since the Jacobian (1.3) at this zero is $5\pi^2/2$ the corollary follows. □

4. Proof of Theorem 1.2 and its corollaries

Proof of Theorem 1.2. As in the proof of Theorem 1.1 the second–order differential equation (1.1) can be written as the first order differential system (3.1). Doing the rescaling $(x, y) = (\varepsilon X, \varepsilon Y)$, we obtain the system

$$\begin{aligned} \dot{X} &= Y \\ \dot{y} &= -X - h(t) + \varepsilon(-g(t)X - f(t)Y - 3XY) + \varepsilon^2(-f(t)X^2 - X^3). \end{aligned} \tag{4.1}$$

System (4.1) with $\varepsilon = 0$ is the unperturbed system, otherwise it is the perturbed system.

The solution $(X(t), Y(t))$ of the unperturbed system such that $(X(0), Y(0)) = (X_0, Y_0)$ is

$$\begin{aligned} X(t) &= X_0 \cos t + Y_0 \sin t - \int_0^t h(\tau) \sin(t - \tau) d\tau, \\ Y(t) &= -X_0 \sin t + Y_0 \cos t - \int_0^t h(\tau) \cos(t - \tau) d\tau. \end{aligned}$$

Note that these periodic orbits have period 2π. Using the notation introduced in section 2. We have that $\mathbf{x} = (X, Y)$, $\mathbf{z} = (X_0, Y_0)$, $F_0(\mathbf{x}, t) = (Y, -X - h)$, $F_1(\mathbf{x}, t) = (0, -g(t)X - f(t)Y - 3XY)$ and $F_2(\mathbf{x}, t) = (0, -f(t)X^2 - X^3)$.

The fundamental matrix solution $M_{\mathbf{z}}(t)$ is independent of the initial condition $\mathbf{z}$ and it is

$$M(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

We compute the function $\mathscr{F}(\mathbf{z}) = (\mathscr{F}_1(X_0, Y_0), \mathscr{F}_2(X_0, Y_0))$ given in (2.4), and we get the functions (1.4) of the statement of Theorem 1.2.

By Theorem 2.1 each zero (X_0^*, Y_0^*) of system $\mathscr{F}_1(X_0,Y_0) = \mathscr{F}_2(X_0,Y_0) = 0$ satisfying (1.3), provides a 2π–periodic solution $(X(t,\varepsilon), Y(t,\varepsilon))$ of system (4.1) with $\varepsilon \neq 0$ sufficiently small such that

$$\begin{pmatrix} X(t,\varepsilon) \\ Y(t,\varepsilon) \end{pmatrix} = \begin{pmatrix} X_0^* \cos t + Y_0^* \sin t - \displaystyle\int_0^t h(\tau)\sin(t-\tau)d\tau \\ -X_0^* \sin t + Y_0^* \cos t - \displaystyle\int_0^t h(\tau)\cos(t-\tau)d\tau \end{pmatrix} + O(\varepsilon).$$

Going back through the change of variables for every periodic solution $(X(t,\varepsilon), Y(t,\varepsilon))$ of system (4.1) with $\varepsilon \neq 0$ sufficiently small, we obtain a 2π–periodic solution

$$x(t,\varepsilon) = \varepsilon \left(X_0^* \cos t + Y_0^* \sin t - \int_0^t h(\tau)\sin(t-\tau)d\tau \right) + O(\varepsilon^2)$$

of the differential equation (1.1) for $\varepsilon \neq 0$ sufficiently small. This completes the proof of Theorem 1.2. □

Proof of Corollary 1.3. We apply Theorem 1.2 with

$$f(t) = \sin(2t) + \cos(2t), \quad g(t) = \sin t, \quad h(t) = 2\cos^2 t.$$

We compute the functions $\mathscr{F}_1$ and $\mathscr{F}_2$ of the statement of Theorem 1.2, and we obtain

$$\mathscr{F}_1(X_0,Y_0) = \frac{\pi}{2}(2 + X_0 - 4Y_0), \quad \mathscr{F}_2(X_0,Y_0) = \frac{\pi}{2}(1 + 8X_0 - Y_0).$$

System $\mathscr{F}_1 = \mathscr{F}_2 = 0$ has the solution $(X_0^*, Y_0^*) = (-2/31, 15/31)$. Since the Jacobian (1.3) is $31\pi^2/4$, by Theorem 1.2 we obtain the periodic solution of the statement of the corollary. □

Proof of Corollary 1.4. We apply Theorem 1.2 with

$$f(t) = \sin t, \quad g(t) = \sin^2 t, \quad h(t) = 2\cos(2t).$$

We compute the functions $\mathscr{F}_1$ and $\mathscr{F}_2$ of the statement of Theorem 1.2, and we obtain

$$\mathscr{F}_1(X_0,Y_0) = \frac{3\pi}{4}(8 + 5Y_0), \quad \mathscr{F}_2(X_0,Y_0) = \frac{11\pi}{4}X_0.$$

System $\mathscr{F}_1 = \mathscr{F}_2 = 0$ has the solution $(X_0^*, Y_0^*) = (0, -8/5)$. Since the Jacobian (1.3) is $-165\pi^2/16$, by Theorem 1.2 we obtain the periodic solution of the statement of the corollary. □

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