Topological Attractors Of Quasi-periodically Forced One-dimensional Maps

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Chapter 1

Introduction

This memoir is concerned with the topological attractors of some quasi-periodically forced one-dimensional maps. Our investigations display some very essential features of the dynamics of this kind of systems. The quasi-periodically forced systems consist of two generic types, the pinched ones and the non-pinched. The dynamical behaviours of these two types have an intrinsic distinction, presented on their attractors. Roughly speaking, the dynamical structures of the unforced one-dimensional maps, can be seen preserved by suitable forcing term in non-pinched systems, but this is not the case in pinched ones. Our Theorem A says that the attractors of a pinched system must be in one piece, with the unique ω-limit set of pinched points inside all of them. Hence, if there are different attracting orbits in the unforced one-dimensional map, all of them can be represented by corresponding invariant subsets in the non-pinched systems; however in the case of a pinched system, the situation depends much on the pinched condition. These features of two types of systems are exhibited clearly by the typical families that we choose as examples.

More concretely, we study two types of typical families, whose unforced one-dimensional maps are already very well studied. The first type consists of two different quasi-periodically forced increasing real maps, both of them are classic examples of saddle-node bifurcation. We elaborate different states of their attractors by Theorem B and Theorem C respectively in the third chapter. They show evidently that the qualitative behaviours of non-pinched systems are exactly the same with the corresponding unforced real systems, which the pinched ones are affected in a great degree by the pinched conditions. The second type of families is the quasi-periodically forced S-unimodal maps. S-unimodal maps are prototypes of periodic behaviours. We propose the mechanism for the states of periodicity in forced systems according to the forced terms, which is based on rigorous analysis of the S-unimodal maps and is substantiated by numerical evidences.

Moreover, our analysis of S-unimodal maps also demonstrates the mechanism of bifurcations that happen on the attractors of cycles of chaotic intervals. This is a new result for us about these well studied systems, given by Theorem D. Bifurcations of this type have been reported for decades, but they are only described in physical context. We explain their mechanism mathematically and show each of them is the reverse of a corresponding bifurcation of periodic orbit. A special significance of these reverse bifurcations is that, each correspondence pairs of them forms a unit of similarity in the transition of an S-unimodal family. Below we give more detailed introduction of this memoir.

The attractors are one of the main subjects of dynamical systems and chaos theory. They are invariant subsets of state spaces that the asymptotic motion of the points in their neighbourhood follows them. Hence the long-term evolution behaviours of dynamical systems are mainly represented by their attractors, and the knowledges of attractors are the key for the understanding of the whole system. There are many important and impressive results on attractors, such as the famous Lorenz's attractor [51], Hénon's attractor [36], Feigenbaum's attractor [22, 23]. They are all important strange attractors, and each of them stands for an important achievement in the field of dynamical systems. Here by strangeness it means that they own some fractal features in the geometry aspect. A chaotic attractor is the one which is "sensitive dependence on initial conditions", usually measured by a positive Lyapunov exponent. In early literature, a strange attractor often refers to the chaotic attractor, since all known chaotic attractors are with the strange feature together.

Our motivation is the problem of strange nonchaotic attractors(SNAs for short), they are strange attractors which are nonchaotic instead. In 1984 Grebogi, Ott, Pelikan and Yorke [29] found examples of strange nonchaotic attractors in quasi-periodically forced skew product systems. A noticeable work [44] by Keller in 1996 gives an elegant mathematical proof on the existence of SNA in modified systems of [29]. In recent decades, a lot of works were devoted to find and to study SNAs. However the concrete mechanism for the birth of SNAs remains unclear yet. In this memoir, rather than focusing only on how to find SNAs, we extend our perspective to the general dynamical behaviours of quasi-periodically forced systems, in which typical SNAs occur. It provides us some basic cognition on the possible mechanism of SNAs by studying some representative forced systems.

The main content of this memoir consists of three chapters. The basic notions and background knowledges are introduced in the second chapter. We start from the elementary concepts of dynamical systems, orbits, ω-limit sets and attractors. Next we make a short summary on ergodic theory and Lyapunov exponent, which is the major measurement of the chaoticity of attractors. These materials are all standard, we include them so that this memoir is more self-contained.

The third section is devoted to an introduction of some fundamental and important issues about SNAs. We first introduce the quasi-periodically forced systems, they belong to a special form of the skew products. In this memoir, we focus only on quasi-periodically forced one-dimensional maps, which are on cylinder $\mathbb{S}^1 \times I \subset \mathbb{S}^1 \times \mathbb{R}$ of the form

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1, \\ x_{n+1} = f(\theta_n, x_n). \end{cases} \tag{1.1}$$

Here $\theta \in \mathbb{S}^1$ the unit circle, ω is a fixed irrational real number and $I \subset \mathbb{R}$ represents the forced space which is called fibre of the skew product.

The SNA problem on such systems consists of the chaoticity and the strangeness of their attractors. In the second subsection, we summarize the paper [1] of Alsedà and Costa for their discussion on the definition of SNA. The two important examples of Grebogi, Ott, Pelikan and Yorke [29] and Keller [44] are presented in the third subsection, these examples help us to obtain an intuitive understanding on such attractors. The last subsection is devoted to some arguments involving the strangeness of attractors, which is the most confusing and difficult part about confirming an SNA. We summarize the results of Stark [74] on the regularity of invariant graphs in quasi-periodically forced one-dimensional systems, and the elaborate discussions of fractalization mechanism by Jorba and Tatjer [41].

Our investigations are presented in the next two chapters. In the third chapter we first explore shortly some general topological structures of pinched closed invariant subsets of systems (1.1), which are sets possibly strange. By pinched we mean that the intersection of the set with some fibre contains only one point, which is called a pinched point. We prove that, the ω-limit set of pinched points is the unique minimal set in a pinched closed invariant set, and any continuous graph contained in a pinched set must be invariant. Such an invariant graph is then the most possible closed invariant subset in a quasi-periodically forced system. A system (1.1) itself is called pinched, if there is one fibre who is wholly mapped to one point. Any closed invariant set in a pinched system is certainly pinched. Theorem A, our first main result shows that, in a pinched system the ω-limit set of pinched points is the only minimal set which must be contained in every closed invariant set. Hence the pinched points plays a crucial role in a pinched system, since all the other orbits can only go around their ω-limit set.

This special role of the pinched orbits in pinched systems is clearly presented in the two models we study next in this chapter. They are families of forced monotone increasing maps with two parameters, both in form of

$$F(\theta, x) = (\, \theta + \omega \bmod(1),\, \lambda f(x + a) \cdot g(\theta)\,), \tag{1.2}$$

with f a real function defined on $\mathbb{R}$ which is continuous, strictly increasing and satisfies $f(0) = 0$. The two real number a and $\lambda > 0$ are parameters, and $g(\theta) \geq 0$ is continuous from $\mathbb{S}^1$ to $\mathbb{R}$.

Moreover, for the first family, we ask f to be bounded, α-concave for $x \geq 0$ with some $\alpha > 0$, and is β-convex for $x \leq 0$ with some $\beta > 0$. While for the second, f is assumed to α-concave or β-convex on whole $\mathbb{R}$ for some $\alpha > 0$ or $\beta > 0$. Thus, in both cases, the real systems given by $\lambda f(x + a)$ are typical examples of the saddle-node bifurcations, due to the concave and convex structures of f. Precisely, if we let λ increases from 0 with $a \neq 0$ fixed in the first case, the number of fixed points changes from 1 to 3 after the bifurcation. In the second case, the number of fixed points are 0 and 2 at the two sides of the bifurcation respectively.

In the second section, before proving rigorously the states of their dynamics, we develop some general and common properties derived from the monotonicity, and the concavity and convexity respectively. These properties provide us the basic tools of the investigations of systems with such structures on their fibre maps. Next, the complete dynamics of the two families are given by the main theorem in each of the following two sections. Briefly saying, in both of their non-pinched cases, only the fixed points of the one-dimensional maps $\lambda f(x + a)$ are replaced by continuous invariant graphs, so they have totally analogous dynamics and exhibit the same type of bifurcation with invariant graphs.

For the pinched systems of the first model, the corresponding bifurcation is totally destroyed for any $a \neq 0$. With any value of λ, there exists a unique continuous graph which is invariant and attracts all the points in the system. In the second model, different situations occur according to the cases of $g(\theta)$. When $g(\theta) = 0$ on a positive measure set of $\theta \in \mathbb{S}^1$, the system has a unique invariant graph which attracts all the points for any λ and a, and hence no bifurcation takes place. Otherwise, when $g(\theta) = 0$ only on a zero measure set of $\mathbb{S}^1$, there is similar bifurcation as the unforced real system $\lambda f(x + a)$. However, if the value of λ is big such that the graph $x = 0$ is invariant and repelling at $a = 0$, then the critical value of bifurcation is fixed at $a = 0$, with the attractor being an SNA.

From these examples, we see that all the regular types of bifurcations of one-dimensional maps may happen correspondingly in pinched systems, that is, the pitchfork, saddle-node, and transcritical ones. We also give an example of the period-doubling one. Notice that, it is possible for all of these types to give the birth of an SNA. However, one cannot just expect such a result by simply resembling any a bifurcation of the real map to the pinched one, because it happens only when the pinched orbits locate appropriately in a relative special position.

The last chapter is devoted to the quasi-periodically forced S-unimodal maps. We propose the mechanism of the change of periodicity of their attractors, which is based on the structures of the restrictive intervals of the unforced S-unimodal maps. Such structures also exhibit the reason of the reverse bifurcations in S-unimodal families, which we present in Theorem D. For a more intuitive introduction of these problems and our results, the reader can refer directly to the first section of this chapter.

Just simply, we consider system

$$\begin{cases} \theta_{n+1} &= \theta_n + \omega \pmod 1, \\ x_{n+1} &= \psi(\theta_n, x_n), \end{cases} \tag{1.3}$$

where $(\theta, x) \in \mathbb{S}^1 \times I$, the function $\psi(\theta_n, x_n)$ is continuous on both x and θ, which is in form of $\psi_\theta(x) = f(x) \cdot g_\epsilon(\theta)$ or $\psi_\theta(x) = f(x) + g_\epsilon(\theta)$ with f an S-unimodal map defined on $I \subseteq \mathbb{R}$. Here $\epsilon \geq 0$ is used as a parameter to control the perturbation given by the forcing function $g(\theta)$. Moreover, we suppose that $\psi_\theta(x) = f(x)$ for all $\theta \in \mathbb{S}^1$ if $\epsilon = 0$, and in case of $\psi_\theta(x) = f(x) \cdot g_\epsilon(\theta)$, $g_\epsilon(\theta) \geq 0$ so that the S-unimodal structure can be maintained in fibre maps.

The crucial concept of this chapter is the block structures of restrictive intervals for S-unimodal maps, that we develop in the second section. S-unimodal maps are popular and well studied already, they are unimodal maps with negative Schwarzian derivative in every point except for the critical point. For any periodic orbit of period n of a unimodal map f, we show that there is a set of K intervals with $K = n$ or $K = 2n$, each of them has an endpoint of this periodic orbit. This set of intervals are called to be restrictive if their union is forward invariant under f. The restrictive intervals own very nice properties as follows. By the definition, for any two sets of restrictive intervals, one must be contained in the other, hence has the block structure over it, which is represented by extension pattern. Notice that, all the restrictive intervals of an S-unimodal map are nested, and their intersection is also forward invariant. We prove further that, the only attractor of a generic S-unimodal is contained in their intersection, and each one of three cases of restrictive intervals corresponds actually to one type of topological attractors: an attracting periodic orbit, a solenoidal (Feigenbaum-like) attractor, or a cycle of chaotic intervals. This characterization of attractors by restrictive intervals provides us a convenient criterion to confirm the state of an attractor.

Next, the bifurcation issues in S-unimodal families are discussed in the third section. We have known already that, each new periodic orbit comes from a bifurcation, and the theories on bifurcations of these periodic orbits are classic already. In Theorem D we demonstrate another kind of bifurcation, which happens for the attractor in form of cycle of chaotic intervals, at the value of a set of restrictive intervals becoming non-restrictive. Since every set of restrictive intervals occurs together with the unique periodic orbit whose points are their endpoints, this bifurcation at the end of being restrictive behaves actually to be reverse of the bifurcation which generates the corresponding periodic orbit. Furthermore, a particular fact is that, the transition of an S-unimodal family f_μ between each pairs of such corresponding bifurcations is shown to be a unit of similarity, because f_μ^K turns out to be a full family on each one of these restrictive intervals.

Finally we analyse the quasi-periodically forced S-unimodal maps in the last section. In system (1.3), we let ϵ increase from 0 with the S-unimodal map f being fixed, then the attractor shows somehow similar behaviour as the reverse bifurcation of cycle of chaotic intervals in S-unimodal families. Precisely, for a fixed S-unimodal map f, its block structure of restrictive intervals is specific, and every block corresponds to an invariant region in $\mathbb{S}^1 \times I$ which limits the attractor of (1.3) when $\epsilon = 0$. With the increasing of ϵ, both these regions and the attractor change continuously, and it can be seen that the attractor increases its size and finally goes beyond these limit regions one by one. So at each time the attractor increases over the boundary of a limit region, a similar behaviour like the reverse bifurcation can be observed.

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Chapter 2

Preliminary and background

In this chapter we introduce the basic definitions and notions that we need throughout this thesis. The reader with a good background on dynamical systems, particularly on strange nonchaotic attractors, can skip this chapter and come back only when some of these are needed. This introduction is tried to keep the "minimum amount", starting with some fundamental and essential concepts in the study of dynamical systems. The purpose of these preliminaries is to make this work as self-contained as possible, so that it allows the reader to follow the thesis without consulting other books or articles frequently. The issue of Strange Nonchaotic Attractors(SNAs for short) is the motivation of this memoir, hence the contents are all close related on it.

More precisely, the first section covers basic definitions and notions about general dynamical system, including orbits, limit sets and attractors. In the second section we introduce some of basic measure theory and the concept of Lyapunov exponent, which are usually used as measurement of the chaoticity. The third section is devoted to the problem of SNAs, where we present the notions on quasi-periodically forced skew product systems, a short discussion on the definition of SNA, and the typical examples including the famous Keller's model. Its last subsection is focused on the strangeness of attractors. The noticeable works by Stark [74] and by Jorba and Tatjer [41] on fractalization mechanism are summarized.

2.1 Basic definition on dynamical systems

The contents of this section are some of the most fundamental concepts of dynamical systems. We start from the definition of a dynamical system, followed by the notion of the orbit of a point, which is the natural object for the study of dynamical systems. The efforts to understand the orbits lead to the concepts of the periodic orbit, the ω-limit set and the attractor of a system step by step.

2.1.1 dynamical Systems

A dynamical system is one whose state changes with time t under some deterministic law. All the possible states of the system form a set X, which is called the *state space*, or *phase space*. Mathematically, a state space can be a metric, topological space or manifold and so on. Any point of X is called a *state*.

According to the time variable t, there are two main types of dynamical systems: those for which the time variables are continuous ($t \in \mathbb{R}$), and those for which they are discrete ($t \in \mathbb{Z}$). A continuous dynamical system is usually described by a differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = F(x) \quad (t \in \mathbb{R})\,, \tag{2.1}$$

while discrete one is in the form of the iteration of a map, i. e.,

$$x_{t+1} = F(x_t) \quad (t \in \mathbb{Z}). \tag{2.2}$$

In each of the above situations, x represents the state of the systems and takes value in the state space X. The differential equation in (2.1), or the map F in (2.2) represents the law which determines uniquely the evolution behaviour of the system.

Moreover, in order to form a dynamical system, either the solutions of differential equation or the iteration of a map must have some group structure, that is, it must satisfy the following definition (see, for instance [43, 48]).

Definition 2.1.1. A *dynamical system* is a triple $(\mathrm{X}, \mathrm{T}, \mathrm{F})$, where

(1) X is a set of states (normally with some special structure);

(2) T is a set of times, such that $0 \in \mathrm{T}$ and for any $s, t \in \mathrm{T}$, $s + t \in \mathrm{T}$;

(3) $\mathrm{F}\colon \mathrm{X} \times \mathrm{T} \longrightarrow \mathrm{X}$ is a function satisfying the group property $\mathrm{F}^0 = \mathrm{Id}$, $\mathrm{F}^s \circ \mathrm{F}^t = \mathrm{F}^{s+t}$.

In the above definition, if the time variable t belongs only to the set of nonnegative real number or the nonnegative integer number, and the solution or the iteration of a map satisfies the corresponding property of item (3), the system $(\mathrm{X}, \mathrm{T}, \mathrm{F})$ is called a *semi-dynamical system*. In this case, F needs not be invertible. Notice that semi-dynamical systems are often called dynamical systems in the literature by abuse of notation. In this memoir, we only work with discrete semi-dynamical systems, that is, with $\mathrm{T} = \mathbb{Z}^+ \cup \{0\}$. So we use (X, F) to denote such a system, and call it dynamical system whenever there is no possible ambiguity.

2.1.2 Orbits and ω-limit

The basic goal of the study of dynamical systems is to understand how the states of all points change with time under the action of given mathematical laws, that is, their *orbits* or *trajectories*.

Definition 2.1.2. The *(forward) orbit*, or *trajectory*, of a point $x \in \mathrm{X}$ is the set

$$\mathrm{Orb}(x) = \mathcal{O}^{+}(x) = \{F^n(x) : n \in \mathbb{Z}^{+} \cup \{0\}\}.$$

In general, the orbit of a point can be a very complicated set. However, there may exist some simple ones among all the orbits, which play an important role in the study of the whole system.

Definition 2.1.3. A point x is a *periodic point of period* n if there is an $n \in \mathbb{N}$ such that $f^n(x) = x$, and $f^m(x) \neq x$ for each $m < n$. If $n = 1$, then x is called a *fixed point*. Moreover, if a periodic point has a neighbourhood such that all the points in this neighbourhood eventually approach to its orbit, then it is called an *attracting periodic point*. On the contrary, if all the points in some of its neighbourhood leave this neighbourhood under the action of f^n, that is, x is the only point which stays always inside this neighborhood, then it is *repelling*.

If a point x is a periodic point of period n, let the number $\lambda(x) = \frac{d}{dx}f^n(x)$. If $|\lambda(x)| < 1$, then x is an attracting one. On the other hand, if $|\lambda(x)| > 1$ then x is repelling. Whenever $|\lambda(x)| \neq 1$, the orbit of x is called *hyperbolic*.

It is clear that if x is a periodic point of period n, then $\mathrm{Card}(\mathrm{Orb}(x)) = n$. Since $\{f^n(x)\}$ is a repeating sequence of these n points, the behaviour of x is well-understood. For those points which are not periodic, it is useful to understand their limiting behaviour. This gives rise to the notion of ω-limit set.

Definition 2.1.4. Let (X, F) be a dynamical system, and $x \in \mathrm{X}$. The *ω-limit set* of x, denoted by $\omega(x)$, is the set of the limit points of $\mathrm{Orb}(x)$, that is, $\omega(x) = \{y :$ there is a subsequence $\{n_j\}$ of $\{n\}$ such that $f^{n_j}(x) \to y$ as $n_j \to \infty\}$.

ω-limit sets have particular significance in the study of dynamical systems, they are very important invariant sets in the systems. A set $A \subseteq X$ is *(forward) invariant* in a system F if $F(A) \subseteq A$. particularly, if $F(A) = A$, then A is called *strongly invariant*. Normally, we only clarify a strongly invariant set in time of necessary.

Obviously, any orbits will stay in an invariant set eventually since they enter it. Thus it forms a special system of it own. This display the special role of such set.

Remark 2.1.1. It is well-known (see for instance [67]) that the ω-limit set of a point has the following properties.

(i) For any $x \in$ X, $\omega(x) = \cap_{n \geq 0} \mathrm{Cl}(\mathrm{Orb}(f^n(x)))$;

(ii) For all $y \in \mathrm{Orb}(x)$, $\omega(y) = \omega(x)$;

(iii) For any point x, $\omega(x)$ is closed and forward invariant, that is, $f(\omega(x)) \subset \omega(x)$;

(iv) Moreover, if $\mathrm{Orb}(x)$ is contained in some compact subset of X, then $\omega(x)$ is nonempty, compact and (totally) invariant, that is, $f(\omega(x)) = \omega(x)$;

(v) If $D \subset X$ is closed and forward invariant and $x \in D$, then $\omega(x) \subset D$. In particular, if $y \in \omega(x)$, then $\omega(y) \subset \omega(x)$.

$\diamond$

Obviously, if x is periodic, then $\omega(x) = \mathrm{Orb}(x)$.

2.1.3 Definition of attractors

In the study of dynamical systems, the main attention is devoted to the eventual behaviour of most of points in the state space. The ω-limit set can give us this information about a point, but it is not enough for the whole system. For this reason we need some more global notion.

Notice that the orbit of a fixed or a periodic point has only finite points, so its dynamics is simple and well-understood. Moreover, if such a point is attracting, then its orbit plays an important role in the study of dynamical systems. Geometrically, we can say that the asymptotic motion of these points in the neighbourhood follows the orbit of this periodic point in state space. Thus, the dynamics of all these points are well-understood through the dynamics of this attracting periodic point. Generalizing this idea, we can see that, an invariant subset of the state space with such properties also plays the same role in the study of dynamical systems. This leads to the concept of attractors. For a concrete mathematical meaning of attractor, we adopt the definition given by Milnor in [58]. We should point out that, there are many other definitions of attractors in the literature. Milnor also makes a survey of some of them in his article.

Definition 2.1.5 (Milnor). Let (X, F) be a dynamical system where X is a smooth, compact manifold endowed with a measure μ equivalent to Lebesgue measure when it is restricted to any coordinate neighbourhood. A closed subset $A \subset X$ is called an *attractor* if it satisfies the following two conditions:

(1) The set $\varrho(A) := \{x : \omega(x) \subset A\}$ has strictly positive measure;

(2) there is no strictly smaller closed set $A' \subset A$ so that $\varrho(A')$ coincides with $\varrho(A)$ up to a set of measure zero.

The set $\varrho(A)$ is called *realm of attraction of* A, it can be defined for every subset of X. When it is open, it is called *basin of attraction of* A.

Remark 2.1.2. In the above definition, the first condition assures that there is some positive possibility, such that a randomly chosen point will be attracted to A, so the realm is visible in this sense. The second is a minimality condition, that is, every part of A should play an essential role.$\diamond$

In [58], Milnor also pointed out some properties of the attractor based on this definition, and proved the following result about the existence of attractor. With this result, it is often convenient to ensure that there is an attractor in a system.

Theorem 2.1.1. *Let S be a compact subset of X such that $\mu(S) > 0$ and $f(S) \subset S$, then S contains at least one attractor.*

In this spirit, in the topological space X, the (topological) attractor refers to a set with dynamical structure similar to the metric one.

Definition 2.1.6. A closed invariant set $A \subseteq X$ is called a *topological attractor* of f if
(i) $rl(A)$ is a set of second Baire category;
(ii) for any proper closed invariant subset $A' \subset A$, the set $rl(A) \smallsetminus rl(A')$ is of second Baire category as well.

Here $rl(A) = \{x : \omega(x) \subseteq A\}$ is its"realm of attraction".

One can see from the definition of the attractors, the long-term evolution behaviours of dynamical systems are mainly represented by their attractors. Hence the knowledge of the dynamical behaviours of the attractors is the key for the understanding of the whole system, and the study of the attractors is one of the main subjects of dynamical systems and chaos theory. The improvement on this study is always closely related with the development of the whole field. In the passed half century, many impressive discoveries were presented. These results not only interested a great number of scientists, they even aroused the public's enormous enthusiasm of this field because of the beauty of their wonderful forms.

In 1963, Lorentz published his famous and historical paper "Deterministic Nonperiodic Flow" [51], and reported the Lorentz's attractor obtained from a very rough model simulating the atmospheric motion. In this paper, he rediscovered the phenomenon of "sensitive dependence on initial conditions" described by Poincaré in his book [63] more than half century ago. Just like Poincaré pointed out, the systems with this character always exhibit complicated behaviours, and predication is impossible. It is notable that such phenomena were found to exist in many systems with very simple mathematical forms, see for instance, May [54], Li and Yorke [49], where the term "chaos" was coined. Here in this context, when we refer to chaotic attractors, we mean that there is sensitive dependence on initial conditions. The chaoticity is given in terms of positive Lyapunov exponents.

Later than Lorentz, in 1971 Ruelle and Takens [70] suggested that the turbulent motion of a fluid could be explained in terms of strange attractors. For them, a strange attractor is a chaotic attractor. While in this memoir, by strangeness we mean that the attractors have nonelementary geometrical properties, such as noninteger fractal dimension, or nowhere differentiability. But at the early time in this field, almost all the important chaotic attractors own the features of strangeness in geometry structure. So in the early literature, when people spoke of "strange attractor", sometimes they spoke of the strangeness in geometry, sometimes they referred to the chaotic behaviour of the attractor, and sometimes both. As the study of this phenomenon was proceeded further, people realized that a chaotic attractor may not be strange or fractal in its geometry form. There is a simple example of a chaotic attractor which is a nicely smooth manifold (see [58, Appendix 3]). On the other hand, a strange attractor needs not be chaotic either. The earliest literature we found about this is the works of V.M. Millionščikov [56, 57] and R. E. Vinograd [76], in which there are some constructions of continuous flows containing such attractors. The term of Strange Nonchaotic Attractor(SNA for short) was introduced and coined by Grebogi, Ott, Pelikan and Yorke in 1984. In their paper [29], they gave out models of two and three dimensional systems, and proved both numerically and theoretically that the attractors are nonchaotic with geometry structure of nowhere differentiability. So far, the Strange Nonchaotic Attractors are reported to typically appear in the quasi-periodically forced skew product systems. In the remaining of this memoir, we will focus our study on this particular class of dynamical systems, so we give an special introduction of the SNAs and quasi-periodically forced systems with the third section. Before we this, we need some knowledges on ergodic theory and Lyapunov exponent.

2.2 Ergodic theory and Lyapunov exponents

When one investigates a complicated system, a more global point of view is usually very helpful. Ergodic theory provides the measure-theoretic approach to reveal the statistical properties of the systems, which is very necessary for the understanding of general behaviour of those complicated systems. We summarize the basic materials on it first in the second subsection. One of the important application of ergodic theory is that, it can be used to simplify the calculation of Lyapunov exponents. Lyapunov exponent is another important notion in the study of dynamical system, which is often used to indicate the degree of chaoticity. We discuss it at the last of this section. All these materials are quite standard, so we just present them briefly.

At the beginning of the study of dynamical systems, people tried to use strictly analytical tools to find the precise solutions, so that they can get the information of the orbits of the points in the system. But such methods can only be used to study particular solutions, and often work only locally. Whereas a more global picture of the system is often needed, particularly, for the study of long-term asymptotic behaviour and of its qualitative aspects. It is in the late nineteenth century, Henri Jules Poincaré introduced geometrical and topological methods into the study of dynamical systems (see [63]), which began the history of the modern theory of dynamical systems. The geometrical and topological approaches do not rely on explicit calculation of solutions, they are the tools to make the system visualized. The attractor in the preceding section is just a notion from this point of view.

In addition to the qualitative study of a dynamical system, the measure-theoretic approaches are also very utilized to overcome the difficulties that arise in using strict analysis, especially for those very complicated systems. People have some complicated systems could even behave with some probabilistic characters. Around 1900, Gibbs suggested looking at what happens to subsets of state space, for instance, the probability that a subset is in another subset of the space at a given time t, and the average time that the subset spends inside of the other one, so one can discover statistical properties of dynamical systems. Such questions motivate the type of study undertaken in ergodic theory. In general, ergodic theory is the study of transformations and flows from the viewpoint of recurrence properties, mixing properties, and other global dynamical properties connected with asymptotic behaviour. In this section, we introduce some essential notions of ergodic theory. All the materials here in this section are very standard in ergodic theory, and come from the textbooks of Walters [77] or Mañe [53].

We start with some basic definitions of measure theory.

Definition 2.2.1. Let X be a set, a σ-algebra $\mathcal{B}$ over X is a nonempty collection of subsets of X, which is closed under complementation and countable unions of its members. That is, the following properties hold:

(1) If $B \in \mathcal{B}$, $X \setminus B \in \mathcal{B}$.

(2) If B_n is a sequence of elements of $\mathcal{B}$, then $\bigcup B_{i\in\mathbb{N}} \in \mathcal{B}$.

From the definition, it follows that X and the empty set are in $\mathcal{B}$, and that the σ-algebra is also closed under countable intersections. Moreover, the intersection of σ-algebras is also a σ-algebra. One can also see that, for any collection of subsets of X, there is always a σ-algebra containing it, namely, the power set of X. Thus, if $\mathcal{S}$ is a collection of subsets of X, by taking the intersection of all σ-algebras containing $\mathcal{S}$, we obtain the smallest such σ-algebra, which is called the *σ-algebra generated by $\mathcal{S}$*.

The main use of σ-algebras is in the definition of measures on X. We give this definition as below.

Definition 2.2.2. If $\mathcal{B}$ is a collection of subsets of X which forms a σ-algebra, then the pair $(X, \mathcal{B})$ is called a *measurable space*. A measure on the measurable space $(X, \mathcal{B})$ is a set function $\mu : \mathcal{B} \to \mathbb{R}^+$ satisfying the following conditions:

(1) $\mu(\varnothing) = 0$,

(2) if $B_1, B_2, \ldots, B_m, \ldots$ is a countable collection of pairwise disjoint elements of $\mathcal{B}$, $\mu(\bigcup_{i=1}^{\infty} B_i) = \Sigma_{i=1}^{\infty}\mu(B_i)$.

The triple $(X, \mathcal{B}, \mu)$ is then called a *measure space*.

In addition, if $\mu(X) = 1$, then we say that μ is a *probability measure*. Most of the times, we work with probability measures on finite dimensional topological spaces equipped with a *Borel* σ-algebra, that is, the σ-algebra generated by the topology.

Definition 2.2.3. Let $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ be two measurable spaces. A transformation $T : (X_1, \mathcal{B}_1) \longrightarrow (X_2, \mathcal{B}_2)$ is a *measurable transformation* if $T^{-1}(B) \in \mathcal{B}_1$ whenever $B \in \mathcal{B}_2$. We denote it briefly by $T : X_1 \longrightarrow X_2$ as long as the σ-algebras are clear for us.

Moreover, a measurable transformation $T : (X_1, \mathcal{B}_1, \mu_1) \longrightarrow (X_2, \mathcal{B}_2, \mu_2)$ is *measure-preserving* or *invariant* if $\mu_1(T^{-1}(B)) = \mu_2(B)$ for every $B \in \mathcal{B}_2$.

In the study of dynamical systems and ergodic theory, we are mainly interested in self-transformations of probability measure spaces. That is, $(X_2, \mathcal{B}_2, \mu_2) = (X_1, \mathcal{B}_1, \mu_1)$. According to this situation, we have the following definition.

Definition 2.2.4. A measure μ is called T-invariant for a transformation $T : (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ if $\mu(T^{-1}(B)) = \mu(B)$ whenever $B \in \mathcal{B}$.

The set of all invariant probability measures on X is denoted by $\mathcal{M}_{inv}(T)$, or $\mathcal{M}_{inv}$ briefly. The Krylov-Bogolubov Theorem guarantees that there always exists an invariant measure for a compact topological space with a continuous map on it, so $\mathcal{M}_{inv}$ is non-empty. Moreover, the set $\mathcal{M}_{inv}$ is convex and compact with the weak* topology.

It may be difficult to check directly from the definition whether a measure is invariant or not, since we usually do not have explicit knowledge of all members of $\mathcal{B}$. The following proposition is a useful tool to help us to simplify this work.

Proposition 2.2.1 (**Characterization of invariant measures**). *A measure $\mu \in \mathcal{M}_{inv}(T)$ if and only if $\int_M g \circ T \mathrm{d}\mu = \int_M g \mathrm{d}\mu$ for all $g \in L^1(X, \mathcal{B}, \mu)$.*

In a dynamical system, we usually have a space X with some structure on it, and a transformation T of X which preserves this structure, for instance, a topological space and a continuous map on it. To apply the measure theoretic methods to the study of such system, we need an invariant measure for the transformation T which acts "nicely" with respect to the structure on X. Ergodic measure is just such a concept.

Definition 2.2.5. An invariant probability measure $\mu \in \mathcal{M}_{inv}$ is called *ergodic* if whenever $T^{-1}B = B$ for some $B \in \mathcal{B}$, then either $\mu(B) = 0$ or $\mu(B) = 1$. We denote the set of ergodic measures by $\mathcal{M}_{erg} = \{\mu \in \mathcal{M}_{inv} : \mu \text{ is ergodic}\}$.

Remark 2.2.1. Ergodicity is a concept of irreducibility for the given system from the measure theory point of view. Since, if $T^{-1}(B) = B$ for some $B \in \mathcal{B}$, then $T^{-1}(X \backslash B) = X \backslash B$, and we can study the action of T on B and $X \backslash B$ separately. If $0 < \mu(B) < 1$, this would simplify the study of the given system into two proper separated parts. We can see that this cannot happen to a system with ergodic measure. $\diamond$

We have known that the Krylov-Bogolubov Theorem guarantees that $\mathcal{M}_{inv}$ is nonempty for a continuous map on a compact topological space, while the ergodic measures are precisely those extremal points of $\mathcal{M}_{inv}$, so $\mathcal{M}_{erg}$ is also non-empty (refer to [77, 64] for details). Moreover, this fact makes the following definition allowable.

Definition 2.2.6. If a transformation T has a unique invariant measure, then this measure must be ergodic. Such a transformation T is called *uniquely ergodic.*

A classic example of this is the irrational rotation on $\mathbb{S}^1$ with the Haar-Lebesgue measure.

Analogous to Proposition 2.2.1, ergodicity can be characterized in terms of properties of functions too. We are going to see that this characterization is very useful in application of ergodic theorem.

Proposition 2.2.2 (**Characterization of ergodic measures**). *The following conditions are equivalent:*

(1) μ is T-ergodic;

(2) If a measurable function f is T-invariant, that is, $f \circ T = f$, then f is constant almost everywhere;

(3) If a measurable function f is T-invariant almost everywhere, then f is constant almost everywhere.

The best known and major result in ergodic theory is the Birkhoff Ergodic Theorem proved by Birkhoff in 1931.

Theorem 2.2.3 (Birkhoff Ergodic Theorem). *Let $T : X \to X$ be a transformation on a measurable space $(X, \mathcal{B})$, and $m \in \mathcal{M}_{inv}(T)$. For any $f \in L^1(X, \mathcal{B}, m)$, there is an integrable function f^* such that*

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) \longrightarrow f^*(x), \quad \text{as } n \to +\infty,$$

for almost all points $x \in X$ (with respect to m). Moreover, f^* is T-invariant, and if $m(X) < \infty$,

$$\int_X f^* \mathrm{d}m = \int_X f \mathrm{d}m.$$

Remark 2.2.2. By Proposition 2.2.2, if the invariant measure m is also ergodic, then f^* must be constant m-almost everywhere. So if $m(X) < \infty$, then $f^* = \frac{1}{m(X)}\int_X f \mathrm{d}m$ m-almost everywhere. Consequently, if m is ergodic, the "temporal averages" of f,

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$$

and the "spatial averages"

$$\frac{1}{m(X)}\int_X f \mathrm{d}m$$

coincide. ◇

Another important ergodic theorem is the so-called Subadditive Ergodic Theorem. It is not only utilized in the proofs of a number significant ergodic theorems, but also very pregnant in many applications.

Theorem 2.2.4 (Subadditive Ergodic Theorem). *Let $T : X \to X$ be a transformation on a measurable space $(X, \mathcal{B})$, and $m \in \mathcal{M}_{inv}(T)$. If $\{\varphi_n\}$ is a sequence of integrable functions satisfying*

$$\varphi_{n+m}(x) \le \varphi_n(x) + \varphi_m(T^n x),$$

then for m-almost every x,

$$\lim_{n\to\infty} \frac{1}{n}\varphi_n(x) = \overline{\varphi}(x),$$

where the function $\overline{\varphi}$ is T-invariant and integrable. Moreover, if m is ergodic, $\overline{\varphi}$ is constant m-almost everywhere.

A notion which is closely related with ergodic theorems is the Lyapunov (characteristic) exponent. The definition of Lyapunov exponents goes back to the dissertation of Lyapunov in 1892 (see [67]). It gives the averaged rate of exponential divergence (or convergence, according as it is positive or negative) from perturbed initial conditions. Nowadays, it is commonly used as the measure of the degree of chaoticity of a system. The precise definition is as follows:

Definition 2.2.7. Let $f : M \to M$ be a differentiable map on a manifold of dimension k. For each $(x, v) \in M \times T_xM$ the *Lyapunov exponent* of (x, v) is defined as

$$\lambda(x, v) := \lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n v\|,$$

where $\|\cdot\|$ is a norm on the tangent space induced by a Riemannian metric on M.

Note that the above definition of Lyapunov exponent measures the exponential divergence rate of nearby trajectories in the v-direction. In general, the rates of separation can be different for different orientations of initial separation vectors. Hence, there is a whole spectrum of Lyapunov exponents, the number of them is equal to the number of dimensions of the phase space. This fact is just what the famous Oseledets' Multiplicative Ergodic Theorem shows. This theorem is a very important theorem on the existence and properties of Lyapunov exponents, which is given by Oseledets in 1968 [61]. One can also refer to [69] for this theorem.

Theorem 2.2.5 (Oseledets' Multiplicative Ergodic Theorem). *Let $(M, \mathcal{B})$ be a measurable space, where M is k-dimensional, $f : M \longrightarrow M$ a measurable transformation and μ an f-invariant probability measure. Then for μ-almost every $x \in M$ there exists $s(x) \le n$ and numbers*

$$-\infty \le \lambda_1(x) \le \ldots \le \lambda_{s(x)}(x)$$

with a sequence of subspaces

$$\{0\} = V_0(x) \subsetneq V_1(x) \subsetneq \ldots \subsetneq V_{s(x)}(x) = T_xM$$

such that

(1) $\lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n v\| = \lambda_i(x)$ *for every* $v \in V_i(x) \backslash V_{i-1}(x)$;

(2) $D_x f V_i(x) = V_i(f(x))$;

(3) $\lambda_i(x)$, $s(x)$ *and* $V_i(x)$ *are* μ*-measurable;*

(4) if we denote $E_i(x) = V_i(x) \backslash V_{i-1}(x)$ *and* $k_i(x) = \dim E_i(x)$ *for every* $i = 1, \ldots, s(x)$*, then*

$$\lim_{n\to\infty} \frac{1}{n} \log |\det D_x f^k| = \sum_{i=0}^{s(x)} k_i(x)\lambda_i(x)$$

and

$$T_x M = E_1(x) \oplus \ldots \oplus E_{s(x)}(x).$$

Moreover, if the measure μ *is* f*-ergodic, then* $\lambda_i(x)$*,* $s(x)$ *and* $V_i(x)$ *do not depend on* x*.*

Remark 2.2.3. We should point out that it is useful to use an axiomatic definition of Lyapunov exponents for the study their properties and the proof of this theorem. For this concept, see [6]. $\diamond$

It is common to just refer to the largest one of these exponents, i.e. to the *Maximal Lyapunov exponent*, because it determines the predictability of a dynamical system. The maximal Lyapunov exponent at a point $x \in M$ is normally given by

$$\lambda_{max}(x) := \limsup_{n\to\infty} \frac{1}{n} \log \|D_x f^n\|,$$

where $\|\cdot\|$ is a matrix norm compatible with the vector norm defined in the tangent space induced by the Riemannian metric on M. A positive maximal Lyapunov exponent is usually taken as an indication that the system is chaotic.

2.3 Quasi-periodically forced skew products and SNAs

This section is devoted an exclusive introduction of the quasi-periodically forced skew product systems. Found SNAs are usually pinched closed invariant sets in such systems, so we introduce some of their essential ingredients in the first subsection. Next we turn to the SNAs issues. A precise definition of SNAs and arguments involved are given in the second subsection, this will help us to clarify unnecessary confusion on this notion. The third subsection contains two concrete examples of SNAs by Grebogi, Ott, Pelikan and Yorke [29] and Keller [44]. In the last subsection, it covers two noticeable papers on the strangeness problem. They are on the regularity of invariant graphs by Stark [74], and the works on the fractalization mechanism by Jorba and Tatjer [41].

2.3.1 Quasi-periodically forced skew products

The essence of a forced systems is to have two or more equations which are coupled together in some way. This type of coupling is known as a *skew product*. It is written in the form

$$\begin{cases} x_{n+1} = f(x_n), \\ y_{n+1} = g(x_n, y_n), \end{cases} \tag{2.3}$$

where $x_n \in \mathrm{X}$ represents the state of the forcing (driving) system, and $y_n \in \mathrm{Y}$ represents the state of the forced (driven) system, influenced by the dynamics of the forcing system. Another way of denoting system (2.3) is by a map $F : X \times Y \longrightarrow X \times Y$, with

$$F(x, y) = (f(x), g(x, y)). \tag{2.4}$$

The space Y is called the *fibre* of the skew product, and the space X is called the *base*. f is a map of X which makes (X, f) a dynamical system in its own way, its dynamical behaviour influence the combined dynamics of x and y given by the skew product (2.3). The case that (X, f) induces a periodic influence on the skew product has been extensively studied for a long time. On the contrary, when this influence is quasi-periodic — a quasi-periodically forced system — it is in general much more poorly understood yet. Such kind of systems received more and more attention in the last three decades, because of its close relation with the occurrence of SNAs.

In this memoir, we focus only on quasi-periodically forced one-dimensional skew products. Precisely, all the bases of such systems are given by a simple one-dimensional quasi-periodically system, an irrational rotation in $\mathbb{S}^1$. For those fibres of the skew product, they are also taken as one-dimensional space X (usually $\mathbb{R}$). Hence the state space of the whole system is a cylinder $\mathbb{S}^1 \times \mathbb{R}$ if $X = \mathbb{R}$, and a system is of the form

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1, \\ x_{n+1} = f(\theta_n, x_n), \end{cases} \tag{2.5}$$

where $\theta \in \mathbb{S}^1$, ω is a fixed irrational real number and X represents the forced space. Physically, this class of dynamical systems models physical systems subject to external quasi-periodic perturbations of frequency ω.

In such a quasi-periodically forced dynamical system, due to the irrational rotation of the forcing system, there cannot be any fixed or periodic points. The simplest invariant closed subset, and hence the attractor, can only be the graph of a map from the base $\mathbb{S}^1$ to the fibre X, that is, the graph of a map $\varphi : \mathbb{S}^1 \longrightarrow X$. So it can be denoted by $\mathcal{A} = \{(\theta, \varphi(\theta) : \theta \in \mathbb{S}^1\}$. Moreover, this graph $\mathcal{A}$ must be *invariant* under the action of system (2.5), that is, $\varphi(\theta + \omega \pmod 1)) = f(\theta, \varphi(\theta))$ for any $\theta \in \mathbb{S}^1$. For the existence of such a graph as an attractor, a simple condition can be obtained from the Hirsch-Pugh-Shub stability theory [38, 39] (see also [67]). We may also call it an *invariant curve* when we focus on its geometric structure.

With respect to the ergodicity of the skew product dynamical system (2.3), it can be shown that the invariant and ergodic measures of the whole system are related with the ones for the system defined on the base space. This is given by the next proposition, the reader can see [10] for its proof.

Proposition 2.3.1. *In a skew product dynamical system $F : X \times Y \longrightarrow X \times Y$ given by $F(x, y) = (f(x), g(x, y))$, if m is F-invariant, then $m \circ \pi^{-1} \in \mathcal{M}_{inv}(f)$, where $\pi : X \times Y \longrightarrow X$ is the projection over the first component. Furthermore, if m is F-ergodic then $m \circ \pi^{-1}$ is f-ergodic.*

Remark 2.3.1. In most of the literature of the study of quasi-periodically forced dynamical systems (2.5), people use the notion of the horizontal and vertical Lyapunov exponents, which correspond the two component directions, that is, the θ and the x directions respectively. They claim that the horizontal one is zero a.e. for any invariant measure due to the constant rotation on $\mathbb{S}^1$. While for every point $z = (\theta, x) \in M$, the vertical one, $\lambda_V(z) = \lambda(z, (0, 1)^t)$, is given by

$$\lambda_V(z) = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial x_n}{\partial x} \right| .$$

Many authors believe that they are indeed the Lyapunov exponents of such systems, but it may not be the case. In fact, it is not an easy problem like it looks, and we should be careful on this assertion. For a discussion of the problem on Lyapunov exponents, see [1] and references therein. ◇

2.3.2 A definition of Strange Nonchaotic Attractor

The existence of Strange Nonchaotic Attractors interested mathematicians and physicists working on dynamical systems greatly. Following the work of Grebogi et al., there have been lots of papers related with SNAs. In those papers, some authors try to report the existence of SNAs in some models by numerical experiments or by theoretical proof (see [19, 7, 26]), some try to explain the mechanisms of the creation of SNAs (see, for instance, [25, 35, 42, 45, 46, 60]). But this problem is still far from being solved now. Most of the reports of the existence of SNAs are only based on rough numerical experiments and are proved to be smooth curves by further research recently. Moreover, since there is no a common precisely formulated definition of an SNA, even the question of what an SNA is is still a problem. The authors use their own intuitive idea on what an SNA is. To clarify this notion, we first give a brief introduction of the definition of SNAs in this subsection.

Here we adopt the definition given by Sara Costa in her master thesis [10](see also [1]). This definition is the most common one in the literature, but is not the only one. There are several other definitions known in the literature, for instance, using Hausdorff dimension to define the strangeness in [11, 66]. For the details and the relation of these definitions, we refer the reader to Sara's thesis, in which she investigated this problem comprehensively and made an integrated survey about it.

Definition of nonchaoticity: Let $\mathcal{A}$ be an attractor and $\varrho(\mathcal{A})$ be its realm of attraction. The attractor $\mathcal{A}$ is said to be *nonchaotic* if the set of points in the realm of attraction whose maximal Lyapunov exponent is positive, that is,

$$\left\{ x \in \varrho(\mathcal{A}) : \limsup_{n\to\infty} \frac{1}{n} \log \| D_x f^n \| > 0 \right\}$$

has zero Lebesgue measure.

Remark 2.3.2. The reason that we use such a condition is to guarantee the nonchaoticity is Lebesgue observable. Notice that we use lim sup to compute the maximal Lyapunov exponent, it must exist for every point $x \in X$. Then the observability of the nonchaoticity is guaranteed by the fact that $\varrho(\mathcal{A})$ has positive Lebesgue measure from the definition of the attractor. ◇

Definition of Strangeness: For a quasi-periodically forced system of the form (2.5), an attractor, which is given by the invariant graph of a map φ from the forcing space to the forced space, is called to be strange if it is neither finite nor piecewise differentiable.

2.3.3 Examples of Strange Nonchaotic Attractors

To date, the model given by Grebogi et al. is still the most classic one, and the works on the rigorous proofs of the existence of SNA are still very few. The best results that we known on the proof of such problem is given by Keller [44] and Bezhaeva and Oseledets [7] separately, based on a generalized model of Grebogi et al.. In this subsection, we have a brief look at the model of Grebogi et al. and Keller to make the abstract notion more clear.

Example 2.3.3 (Grebogi et al.). Consider the dynamical system

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1, \\ x_{n+1} = f(\theta_n, x_n) = 2\sigma \cos(2\pi\theta) \tanh(x) \end{cases} \tag{2.6}$$

where $\omega \in \mathbb{R} \setminus \mathbb{Q}$. The authors take $\omega = \frac{\sqrt{5}-1}{2}$, the golden mean, for numerical computations. ◇

There are two Lyapunov exponents in this system. The authors claim that the horizontal one (according to the θ direction) is always zero, and the vertical one (according to the x direction) is

$$h = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \left| \frac{\partial}{\partial x} f(\theta_k, x_k) \right| .$$

Obviously, the θ-axis, i.e., $x = 0$ is invariant under the map of system (2.6). Whether it is an attractor or not is determined by its stability. Note that two orbits on $x = 0$ maintain a constant separation, thus if $h > 0$ for the $x = 0$ orbits, the nearby points can only diverge from the θ-axis, so it is unstable and cannot be an attractor.

To calculate h for the $x = 0$ orbit, applying the Birkhoff Ergodic Theorem for m-a.e. θ and $x = 0$, the authors obtain that the vertical Lyapunov exponent is

$$\lambda_V(\theta, 0) = \int_{\mathbb{S}^1} \log \left| \frac{\partial}{\partial x} f(\theta, 0) \right| d\theta = \log |\sigma|.$$

Consider for the parameter $|\sigma| > 1$, $x = 0$ is not an attractor. On the other hand, since $|x_n| \leq 2|\sigma| < \infty$ for every $n \in \mathbb{N}$, there must exist an attractor because the orbit of any point is in this compact subset of the state space. Moreover, the measure on the attractor generated by an orbit is uniform in θ, because of the ergodicity in θ. The authors note that in this system, for any point $p = (1/4, x)$ or $p = (3/4, x)$, there must be $f(p) = 0$. So the attractor must contain the points $(1/4+\omega, 0)$ and $(3/4+\omega \pmod 1, 0)$ and must not contain any other point in $\theta = 1/4+\omega$ and $\theta = 3/4+\omega \pmod 1$. Consequently, every $(1/4 + k\omega \pmod 1, 0)$, $(3/4 + k\omega \pmod 1, 0)$ for $k \in \mathbb{N}$ belong to the attractor. Hence there is a subset of points which is dense both in the attractor and in $x = 0$, which is not an attractor itself.

The authors draw the picture of the attractor for $\sigma = 1.5$ (see Figure 2.1) and calculate the vertical Lyapunov exponent by numerical method. It is seen from the picture of the attractor that there are points off $x = 0$ as expected, and the vertical Lyapunov exponentis $h \approx -1.059$. So the attractor is an example of Strange Nonchaotic Attractor.

To verify that h must be negative so that the attractor is indeed nonchaotic, the authors consider the points that $x \neq 0$. Note that the function tanh is increasing in $\mathbb{R}$, concave in $(0, \infty)$ and convex in $(-\infty, 0)$, so

$$0 \leq \frac{\mathrm{d}}{\mathrm{d}x} \tanh x \leq \frac{\tanh x}{x},$$

for every $x \neq 0$. That is,

$$\left| \frac{\partial}{\partial x} f(\theta, x) \right| \leq \left| \frac{f(\theta, x)}{x} \right|$$

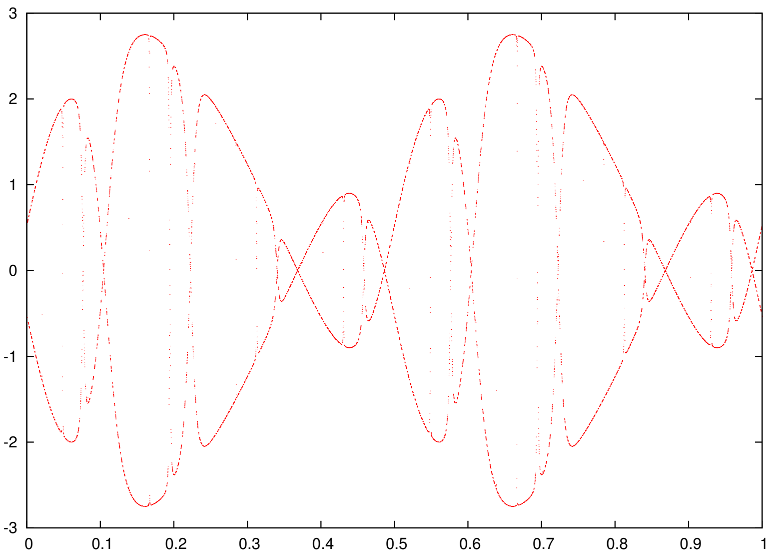


Figure 2.1: SNA of the model of Grebogi et al..

for all $\theta \in \mathbb{S}^1$ and every $x \neq 0$. In particular,

$$\left|\frac{\partial}{\partial x} f(\theta_n, x_n)\right| \leq \left|\frac{f(\theta_n, x_n)}{x_n}\right| = \left|\frac{x_{n+1}}{x_n}\right|.$$

From this inequality, the authors obtain

$$\begin{aligned} \lambda_V(\theta, x) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \left|\frac{\partial}{\partial x} f(\theta_k, x_k)\right| \\ &\leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \left|\frac{x_{k+1}}{x_k}\right| \\ &= \lim_{n\to\infty} \frac{1}{n} (\log |x_n| - \log |x_0|) \\ &\leq \lim_{n\to\infty} \frac{1}{n} \log 2|\sigma| = 0 \end{aligned}$$

for every $\theta \in \mathbb{S}^1$ and $x \neq 0$. So the vertical Lyapunov exponent $\lambda_V(\theta, x)$ is nonpositive for every $x \neq 0$ and m-a.e. $\theta \in \mathbb{S}^1$. Since in the attractor, those points who are on $x = 0$ form only a zero measure subset, this assertion of SNA is valid.

Example 2.3.4 (**Keller**)**.** Later in 1996, Keller gave out an elegant analytical proof for the existence of SNA in a generalized model. The model that he considered is

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1, \\ x_{n+1} = f(x_n) g(\theta_n) \end{cases} \tag{2.7}$$

where $\omega \in \mathbb{R}\backslash\mathbb{Q}$ as before; $f : [0,\infty) \longrightarrow [0,\infty)$ is $\mathcal{C}^1$, bounded, increasing and strictly concave, moreover $f(0) = 0$; and $g : \mathbb{S}^1 \longrightarrow [0,\infty)$ is continuous. Thus it corresponds to the model of Grebogi et al. by changing the quasi-periodically forced map defined on $\mathbb{S}^1 \times [0,\infty)$ and replacing $\cos(2\pi\theta)$ by $|\cos(2\pi\theta)|$. $\diamond$

Keller shows the existence of an SNA by studying the properties of the invariant graph of the given model (2.7). He successfully constructs a decreasing sequence of continuous functions and proves that the attractor of this model is the graph of a map $\psi : \mathbb{S}^1 \longrightarrow [0,\infty)$ to which that sequence of functions converges. His complete theorem is as follows:

Theorem 2.3.2 (**Keller**). *Let us consider the two-dimensional discrete dynamical system $T : \mathbb{S}^1 \times [0,\infty) \longrightarrow \mathbb{S}^1 \times [0,\infty)$ given by*

$$T(\theta, x) = (\theta + \omega, f(x) \cdot g(\theta))$$

where $\omega \in \mathbb{R}\backslash\mathbb{Q}$; $f : [0,\infty) \longrightarrow [0,\infty)$, bounded, increasing, strictly concave and $f(0) = 0$; and $g : \mathbb{S}^1 \longrightarrow [0,\infty)$ is continuous. Then there is an upper semi-continuous function $\varphi : \mathbb{S}^1 \longrightarrow [0,\infty)$ with an invariant graph such that:

(1) $\lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1}|x_k - \varphi(\theta_k)| = 0$ for m-a.e. $\theta \in \mathbb{S}^1$ and all $x > 0$, where m is the Lebesgue measure on $\mathbb{S}^1$. In particular, the Lebesgue measure on $\mathbb{S}^1$ "lifted" to the graph of φ is a SRB (Sinai-Ruelle-Bowen) measure for T, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \nu(T^k(\theta, x)) = \int_{\mathbb{S}^1} \nu(\theta, \varphi(\theta))\mathrm{d}\theta$$

for all $\nu \in \mathcal{C}(\mathbb{S}^1 \times [0,\infty))$ and for a.e. $(\theta, x) \in \mathbb{S}^1 \times [0,\infty)$.

Define

$$\lambda_\varphi = \int_{\mathbb{S}^1} \log g(\theta)\mathrm{d}\theta + \int_{\mathbb{S}^1} \log f'(\varphi(\theta))\mathrm{d}\theta,$$

and consider the parameter

$$\sigma := f'(0)\exp\left(\int_{\mathbb{S}^1} \log g(\theta)\mathrm{d}\theta\right).$$

(2) If $\sigma \leq 1$, then $\varphi \equiv 0$ and $\lambda(\theta, x) = \lambda_\varphi = \log\sigma$ for m-a.e. $\theta \in \mathbb{S}^1$ and each $x \geq 0$.

(3) If $\sigma > 1$, then $\lambda(\theta, x) = \lambda_\varphi < 0$ for m-a.e. $\theta \in \mathbb{S}^1$ and all $x > 0$. The set $\{\theta : \varphi(\theta) > 0\}$ has full Lebesgue measure. Furthermore,

(3.1) if $g(\hat{\theta}) = 0$ for at least one $\hat{\theta} \in \mathbb{S}^1$, then the set $\{\theta : \varphi(\theta) > 0\}$ is at the same time meagre and φ is m-a.e. discontinuous.

(3.2) if $g(\theta) > 0$ for all $\theta \in \mathbb{S}^1$, then $\varphi(\theta) > 0$ for all $\theta \in \mathbb{S}^1$. In this case, φ is continuous, and if g is $\mathcal{C}^1$, then so is φ.

(4) If $\sigma \neq 1$, then $|x_n - \varphi(\theta_n)| \to 0$ exponentially fast for m-a.e. $\theta \in \mathbb{S}^1$ and each $x > 0$.

The idea of the proof is the following: taking a horizontal line which is higher than the upper bound of $f \cdot g$ and iterating it, the monotonicity of the model ensures that there exists a limit of this decreasing sequence of continuous graphs, which is an upper semicontinuous graph. The uniqueness of this semicontinuous graph and its attraction for all upper points are the consequence of the concavity of f. This semicontinuous graph, when $x = 0$ is attracting with the nonpositive vertical Lyapunov exponent, is just the invariant graph $x = 0$. Otherwise, it must be $x > 0$ almost everywhere.

The interesting case is in (3.1). The upper semi-continuous function φ in the theorem is m-a.e. discontinuous, while the closure of the graph of φ contains the circle $\{(\theta, x) : x = 0\}$, and it is the ω-set of m-a.e. points in the state space, so it is a strange attractor. Moreover, the vertical Lyapunov exponent of the points in this graph is $\lambda_\varphi < 0$, this shows that the closure of this graph is an SNA.

2.3.4 Regularity, fractalization and strangeness

In this subsection we summarize related results on the strangeness of invariant curves in quasi-periodically forced systems, which are from two papers by Stark [74] and Jorba and Tatjer [41]. The works of Stark give some sufficient conditions on the smoothness of invariant curves in skew product systems according the Lyapunov exponents. Jorba and Tatjer point out that, even a smooth invariant curve may get extremely wrinkled during the process that it goes close to the critical value 0 of its Lyapunov exponent. This process is called as fractalization, it reveals the difficulties on judging the strangeness of the curve in such situation. At last, we mention the most reliable and easy ways on deciding the continuity of an invariant curve.

There have been a lot of studies of these objects for several kinds of quasi-periodically forced dynamical systems (see, for instance, [16, 17, 18, 35, 42, 46, 68] and references therein). In these papers, the authors tried to report the existence of SNAs and to characterize them through several kinds of properties of topological, spectral and dimensional nature as well as other ones. However, until now, rigorous mathematical results are still scarce. Statements on the existence of an SNA in some papers are just based on very rough numerical evidences, and turned to be wrong in later research.

For this reason, it is necessary and helpful to obtain a better understanding on the properties of the attractors in the quasi-periodically forced skew product systems first. Here we only mention the results related close to our subsequent investigations, and omit all the details on the proofs. Both the papers are elaborate and technological, which cover more results we present here. The interested reader can refer to them directly.

Regularity of invariant curves In [74] Stark study the regularity of invariant curves in general skew product systems. Particularly for quasi-periodically forced one-dimensional systems, he proves (more than) that a continuous invariant graph with a negative vertical Lyapunov exponent is as smooth as the fibre function of the system. This is the theorem below.

In a quasi-periodically forced dynamical system of the form

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1, \\ x_{n+1} = f(\theta_n, x_n), \end{cases} \tag{2.8}$$

with $\theta \in \mathbb{S}^1$, ω is irrational, and $x \in X$. We have,

Theorem 2.3.3 (Stark). *Suppose that Φ is a continuous invariant graph of (2.8) such that its largest Lyapunov exponent in the x direction is negative, then Φ is as smooth as f. In particular, if X is one-dimensional and Φ is a continuous invariant graph with a negative Lyapunov exponent in the x direction, then Φ is as smooth as f.*

Fractalization of continuous curves Now we know that the negative vertical Lyapunov exponent can guarantee the continuity of a curve. However, a smooth curve may also be a highly oscillating one. If the oscillation degree is extremely high, then it may be very difficult to detect its smoothness by only the numerical methods. Jorba and Tatjer prove that this fractalization mechanism does exist on some so-called nonreducible attracting invariant graphs for some quasi-periodically forced one-dimensional systems. Notice that fractalization is usually regard as one of the process which results in SNAs in physical contexts.

Precisely, in some quasi-periodically forced one-dimensional system, there is a continuous change of the attracting invariant curve, which makes them more and more wrinkled as the the system parameter varies. When this happens, the attracting invariant graph may look strange with only usual numerical methods. But in reality, it can continue being smooth as long as its vertical Lyapunov exponent is negative. This procedure of fractalization is described mathematically as below.

Consider a family of one-dimensional quasi-periodically forced dynamical systems F_μ of the form

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1, \\ x_{n+1} = f_\mu(\theta_n, x_n), \end{cases} \tag{2.9}$$

with $x \in \mathbb{R}$, $\theta \in \mathbb{S}^1$, ω an irrational number, and f_μ a smooth function of both x and θ. Here $\mu \in \mathbb{R}$ is a parameter, on which f_μ depends continuously. Moreover, the function f_0 does not depend on θ, that is, $f_0(\theta, x) = g(x)$ for some smooth function $g : \mathbb{R} \to \mathbb{R}$.

For a given value of $\mu = \mu_0$, assume that system (2.9) has an invariant curve $x = u_{\mu_0}(\theta)$ and the curve is of class $\mathcal{C}^r$ for some $r \geq 0$. Without loss of generality, it can be taken $\mu_0 = 0$. Then the invariant curve $u_0(\theta)$ must satisfy the functional equation $F(u_0, 0) = 0$, where $F : \mathcal{C}^r(\mathbb{S}^1, \mathbb{R}) \times \mathbb{R} \to \mathcal{C}^r(\mathbb{S}^1, \mathbb{R})$ is given by

$$F(u, \mu)(\theta) = f_\mu(u(\theta), \theta) - u(\theta + \omega), \tag{2.10}$$

for any $(u, \mu) \in \mathcal{C}^r(\mathbb{S}^1, \mathbb{R}) \times \mathbb{R}$. Next, use the Implicit Function Theorem to study the continuation of this curve with respect to the parameter μ. That is, look for a regular function $\mu \mapsto u_\mu$, which is defined for $|\mu|$ small enough, such that $F(u_\mu, \mu) = 0$.

On the Banach space $\mathcal{C}^r(\mathbb{S}^1, \mathbb{R})$ endowed with the standard $\mathcal{C}^r$ norm, it is not difficult to see that such an F is differentiable, and the function $D_u F(u, \mu) v \in \mathcal{C}^r(\mathbb{S}^1, \mathbb{R})$ is given by

$$[D_u F(u, \mu) v](\theta) = D_x f_\mu(u(\theta), \theta) v(\theta) - v(\theta + \omega) \tag{2.11}$$

for any $(u, \mu) \in \mathcal{C}^r(\mathbb{S}^1, \mathbb{R}) \times \mathbb{R}$, and any $v \in \mathcal{C}^r(\mathbb{S}^1, \mathbb{R})$. It is immediate to verify that $D_u F(u, \mu) v$ is a bounded operator.

Assume that an invariant curve x or $u_0(\theta)$ is of class $\mathcal{C}^r$, with $r \geq 0$, its linearized normal behaviour is described by the following linear skew product system

$$\begin{cases} \theta_{n+1} = \theta_n + \omega, \\ x_{n+1} = a(\theta) x_n. \end{cases} \tag{2.12}$$

where $a(\theta) = D_x f_0(u_0(\theta), \theta)$ is of class $\mathcal{C}^r$ too, $x \in \mathbb{R}$ and $\theta \in \mathbb{S}^1$. Moreover, assume that the invariant curve is not degenerate, in the sense that the function $a(\theta)$ is not identically zero. For the invariant curves, it will turn out that there is important effect on their behaviours according to whether the linear system (2.12) can be reduced to a form with a constant coefficient or not. That is, whether the system verifies the property of reducibility. The definition of reducibility is given by the following:

Definition 2.3.1. The system (2.12) is called *reducible* if and only if there exists a change of variable $x = c(\theta)y$ (which may be complex), continuous with respect to θ, such that (2.12) becomes

$$\begin{cases} \theta_{n+1} = \theta_n + \omega, \\ x_{n+1} = bx_n, \end{cases} \tag{2.13}$$

where b does not depend on θ.

Jorba and Tatjer proved that, under suitable conditions, the reducibility of (2.12) is equivalent to the fact that $a(\theta)$ has no zeros. Then the fractalization mechanism for nonreducible invariant curves is defined by:

Definition 2.3.2. A curve is undergoing a fractalization mechanism if its $\mathcal{C}^1$ norm – taken on any closed nontrivial interval for θ – goes to infinity much faster that its $\mathcal{C}^0$ norm, that is,

$$\limsup_{\alpha \to \alpha_0} \frac{\|x'_\alpha\|_{I,\infty}}{\|x_\alpha\|_\infty} = +\infty,$$

where $\|\cdot\|_{I,\infty}$ denotes the sup norm on a nontrivial closed interval I.

In a family of the form

$$\begin{cases} \theta_{n+1} = \theta_n + \omega, \\ x_{n+1} = \alpha a(\theta) x_n + b(\theta), \end{cases} \tag{2.14}$$

where $x \in \mathbb{R}$, $\theta \in \mathbb{S}^1$, ω is an irrational number as usual, $a(\theta)$ and $b(\theta)$ are $\mathcal{C}^r$ functions, and α is a real positive parameter. Clearly, the linearized normal behaviour around it is described by

$$\begin{cases} \theta_{n+1} = \theta_n + \omega, \\ x_{n+1} = \alpha a(\theta) x_n. \end{cases} \tag{2.15}$$

If an invariant curve of system (2.14) exists, its vertical Lyapunov exponent over this curve is given by

$$\Lambda = \ln \alpha + \int_{\mathbb{S}^1} \ln |a(\theta)| \mathrm{d}\theta. \tag{2.16}$$

If the above integral exists, then set

$$\alpha_0 = \exp\left(- \int_{\mathbb{S}^1} \ln |a(\theta)| \mathrm{d}\theta \right). \tag{2.17}$$

For the values of $\alpha < \alpha_0$, the vertical Lyapunov exponent is negative. Particularly, this implies that this invariant curve is globally attracting, hence it must be unique. We also know that there is a continuous change of this curve with respect to the parameter α when $\alpha < \alpha_0$. Let $x_\alpha(\theta)$ denote the solution of (2.14) for $\alpha < \alpha_0$ and a given continuous function $b(\theta)$. The next theorem describes the fractalization process of a nonreducible system.

Theorem 2.3.4. *Assume that $a(\theta), b(\theta) \in \mathcal{C}^1(\mathbb{S}^1, \mathbb{R})$ and that (2.15) is not reducible. Then,*

(1) if

$$\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_\infty < +\infty,$$

and $b \in D_1$, where D_1 is some residual set, we have

$$\limsup_{\alpha \to \alpha_0^-} \|x'_\alpha\|_{I,\infty} = +\infty,$$

for any nontrivial closed interval $I \subset \mathbb{S}^1$;

(2) if

$$\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_\infty = +\infty,$$

then for any nontrivial closed interval $I \subset \mathbb{S}^1$, we have

$$\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_{I,\infty} = +\infty, \quad \textit{and} \quad \limsup_{\alpha \to \alpha_0^-} \frac{\|x'_\alpha\|_{I,\infty}}{\|x_\alpha\|_\infty} = +\infty.$$

Remark 2.3.5. If in Theorem 2.3.4 the system (2.15) is reducible, the situation is different. In this case, if ω is Diophantine and a, b are $\mathcal{C}^r$ for r large enough, then

(i) If $\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_\infty < +\infty$, then $\limsup_{\alpha \to \alpha_0^-} \|x'_\alpha\|_\infty < +\infty$.

(ii) If $\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_\infty = +\infty$, then $\limsup_{\alpha \to \alpha_0^-} \dfrac{\|x'_\alpha\|_\infty}{\|x_\alpha\|_\infty} < +\infty$.

◇

Supplementary Finally we supplement two methods which can be used to prove the continuity and strangeness respectively. They are simple, traditional, and easy to use, which we will use subsequently.

It is well-known that a continuous graph is both upper and lower semicontinuous. Moreover, the limit of a decreasing sequence of continuous graphs is upper semicontinuous, and the limit of an increasing sequence of continuous graphs is lower semicontinuous. Hence, if a curve is proved to be the limits of both a decreasing and an increasing sequences of continuous graphs, it is proved that it is continuous.

For the strangeness, we known that, if two invariant graphs intersect in a quasi-periodically forced system, they must intersect in a dense set of $\theta \in \mathbb{S}^1$ by the irrational rotation on the base. This means that, when this happens, at least one of them cannot be continuous. This is the most reliable way for the proof of strangeness so far.

Chapter 3

Pinched invariant sets and quasi-periodically forced increasing systems

In this chapter we investigate the basic topological structure of pinched invariant subset in quasi-periodically forced systems, particularly the crucial role of pinched orbits in those pinched systems. We also elaborate the dynamics of two concrete families as examples, which exhibit intuitively and clearly the effects of the pinched orbits and the differences with the non-pinched cases that they bring.

More precise, we first discuss some essential topological properties of pinched invariant sets in quasi-periodically forced systems. A pinched invariant set is a forward invariant set that there is only one point in some fibre. Such sets have special significance for the existence of strange attractors, since there can exist at most one continuous graph in a pinched set. We notice that, if a continuous graph is contained in a pinched invariant set, then this graph must be invariant and be the ω-limit set of all pinched points.

Particularly, any compact invariant subset in a pinched system must be a pinched set. By a pinched system we mean that, there is at least one fibre in the system which is mapped into one single point, which is called a pinched point of the system. We prove by Theorem A that, in such a system, the unique ω-limit set of all pinched points is the only possible minimal subset which is contained in any invariant set of the system. This fact implies that its dynamics must take place around this ω-limit set of pinched points, which is a distinctive feature of the pinched systems.

Next, We demonstrate exhaustively the overall dynamics of two families of quasi-periodically forced systems. The differences between the pinched and non-pinched cases of the same family show explicitly how the pinched orbits affect dynamical behaviours. Concretely, both the families are given by maps $F\colon \mathbb{S}^1 \times X \to \mathbb{S}^1 \times X$ in form of

$$F(\theta, x) = (\,\theta + \omega \bmod(1),\ \lambda f(x+a) \cdot g(\theta)\,).$$

Here the base map is an irrational rotation on the unit circle $\mathbb{S}^1$ by a fixed angle ω. The real function $f(x)$ is continuous on $\mathbb{R}$, and is forced by a continuous map $g\colon \mathbb{S}^1 \to \mathbb{R}$ with $g(\theta) \geq 0$. Two real numbers $\lambda > 0$ and a are used as parameters. For both of our two examples, we assume that f is strictly increasing and satisfies $f(0) = 0$. The basic dynamics of the forced system depends certainly first on the structures of corresponding one-dimensional system given by f, whose detailed structures for the two specific models are as follows.

Besides being monotone increasing, in the first family we require that f is bounded, and that $f(x)$ is α-concave for $x \geq 0$ and β-convex for $x \leq 0$. The function f is α-concave if $f_\alpha(x) = f(x) + \alpha x^2$ is concave for some $\alpha > 0$, and is β-convex if $f_\beta(x) = f(x) - \beta x^2$ is convex for some $\beta > 0$. As a family of monotone interval maps $\lambda f(x+a)$ with such conditions, it has simple and evident dynamics. If $a = 0$, then $x = 0$ is a fixed point for any $\lambda > 0$ which is unique when λ is small enough. With the increasing of λ, there can appear two other fixed points respectively, each at one side of $x = 0$. They are the results of the pitchfork bifurcations. For a fixed $a \neq 0$, it is a typical example of saddle-node bifurcation with the increasing of λ. In such system, there exists one attracting fixed point when the value of λ is small, and two more fixed points occur simultaneously after that λ increases over the critical value of bifurcation.

In the second family, f is assumed to be β-convex or α-concave on $\mathbb{R}$. The family $\lambda f(x+a)$ in this case is also a representative of saddle-node bifurcation. Precisely, for any $\lambda > 0$, if f is convex(concave), there is some $a_\lambda \geq 0 (a_\lambda \leq 0)$ which is the critical value of bifurcation, such that there are two fixed points when $a < a_\lambda (a > a_\lambda)$ and that there is no any fixed point for a at the other side of a_λ.

In the corresponding two quasi-periodically forced families, there exists a common feature of their dynamics for both the non-pinched cases. Briefly saying, the forced terms do not affect the qualitative behaviours. Generally, the dynamics of any non-pinched forced system is essentially same with the one-dimensional family of maps $\lambda f(x+a)$, only the fixed points of unforced interval maps are replaced by the corresponding invariant graphs. With the parameters change in the same way above for the unforced interval systems, there are the same types of bifurcations in non-pinched forced systems, which occur in form of smooth invariant graphs.

However for their pinched cases, the dynamical behaviours depend not only on the structures of f itself, but also on how the pinched conditions act on it. We can see in the first model that, the pinched condition totally destroys the saddle-node bifurcation in the non-pinched system, and there always exists just one invariant graph for any $\lambda > 0$ if $a \neq 0$. For the second one, the situations are more diverse, which correspond to different cases of the forcing function g and the value of parameter λ. There may exist no bifurcation as the first family, or there may happen that the bifurcation values are changed to $a_\lambda = 0$ for a large set of λ. The detailed statements of the general dynamics of these two families are given in Theorem B and C, the main results of the last two sections respectively.

This chapter consists of four sections. In the first one we discuss properties of the pinched invariant sets and the pinched orbits in pinched systems. The arguments display their most general and essential topological structures. The other three sections are devoted to the two quasi-periodically forced monotone increasing systems. In the second section, we develop some concepts and properties which are basic tools for the investigation of general forced monotone increasing systems. After that, each of the two specific models are treated in one of the last two sections respectively, where we present a main theorem with elaborated proofs, which describes the complete dynamics of the whole family with respect to its two parameters.

3.1 Pinched invariant sets and pinched systems

In this section, we give out two theorems on the topology of quasi-periodically forced systems. The first is concerned with continuous graphs in pinched invariant subsets of forced systems, which is the issue of the first subsection. Theorem A in the second subsection is our first main result of this memoir and is also the core theorem of this chapter, it exhibits the most essential features particularly on the dynamics of pinched systems.

The reason for considering the continuous graphs in pinched invariant sets is that, as we see in Keller's model, there is a notable difference in appearance between the pinched and non-pinched cases. Namely, there may exist an invariant graph in a pinched system which is not continuous and displays complicated geometric shape. However, we know from literature that, not only in pinched systems, it is also possible that there exists a pinched invariant subset in a non-pinched system, which admits strange invariant graph too. In examples by Heagy and Hammel [35] and by Bjerklöv [8], they show that two different invariant graphs intersect at a dense orbit of some $\theta \in \mathbb{S}^1$ in a non-pinched system, this results in a pinched invariant subset with at least one strange graph, since there can exist at most one continuous graph in such a set. Therefore we discuss first the most basic topological structures of those pinched invariant sets in general quasi-periodically forced systems. Our theorem shows that, if there exists a continuous graph inside a pinched compact invariant set, this graph must be invariant and in fact be the ω-limit set of all pinched points.

The second theorem focuses on dynamics of pinched systems, which demonstrates that: there may exist only one minimal invariant subset in a pinched system, which is the unique ω-limit set of all those pinched orbits, and is contained in any invariant compact subset of the systems. This theorem implies that all other points in such a system have to go around the pinched orbits, which therefore are the key of dynamics of the whole pinched systems. This role of pinched orbits is helpful to understand a remarkable difference between the dynamics of pinched and non-pinched systems. That is, the dynamical behaviours of non-pinched ones in version of invariant graphs or subsets are basically close to the corresponding unforced maps; but in pinched systems, some behaviours of unforced maps cannot be found due to the action of pinched orbits. The dynamical phenomena behind this abstract theorem are exhibited concretely by the two families that we investigate in the sequel of this chapter.

3.1.1 Continuous graphs in pinched invariant sets

In this subsection we first introduce briefly some necessary definitions and then review some known results about pinched invariant sets. Next, after a simple Lemma 3.1.2, we prove our theorem about the continuous graph in a pinched invariant set.

The notions of the ω-limit set and the invariant set are standard. The *ω-limit set* of a point p, denoted by $\omega(p)$, is the set of the limit points of $\mathrm{Orb}(p)$, the orbit of point p. If a subset $\mathcal{A}$ of the system (3.1) satisfies $F(\mathcal{A}) \subseteq \mathcal{A}$, then it is called an *invariant subset*.

Generally a map $F\colon \mathbb{S}^1 \times X \to \mathbb{S}^1 \times X$:

$$F(\theta, x) = (\, R(\theta), G(\theta, x)\,) \tag{3.1}$$

gives a quasi-periodically forced dynamical system. Here $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is the unit circle, and X denotes some interval of $\mathbb{R}$. The function $R\colon \mathbb{S}^1 \to \mathbb{S}^1$ denotes an irrational rotation of the circle $\mathbb{S}^1$ by a fixed angle ω. Due to it, there cannot be any fixed or periodic points in such systems. If an invariant set is compact, then the projection of this set to $\mathbb{S}^1$ must be the whole circle. The simplest invariant closed subset can only be the graph of a map from $\mathbb{S}^1$ to X. If we denote a map by Φ, its graph is the set $\mathcal{A} = \{(\theta, \Phi(\theta) : \theta \in \mathbb{S}^1\}$. To be invariant under the action of system (3.1), the graph need to satisfy the invariant equation $\Phi(R(\theta)) = G(\theta, \Phi(\theta))$ for any $\theta \in \mathbb{S}^1$. We abuse the notation and call Φ an *invariant graph*, or an *invariant curve*.

Let $\mathcal{A} \subset \mathbb{S}^1 \times X$ be a compact invariant set for F, the *lower* and *upper boundaries* of $\mathcal{A}$ are respectively the functions $\mathcal{A}^-$, $\mathcal{A}^+ : \mathbb{S}^1 \to X$ given by

$$\mathcal{A}^-(\theta) = \inf\{x \in X\colon (\theta, x) \in \mathcal{A}\},$$

$$\mathcal{A}^+(\theta) = \sup\{x \in X\colon (\theta, x) \in \mathcal{A}\}.$$

Since $\mathcal{A}$ is compact in $\mathbb{S}^1 \times X$, it is bounded and closed, so these functions are well-defined, and $(\theta, \mathcal{A}^-(\theta)) \in \mathcal{A}$, $(\theta, \mathcal{A}^+(\theta)) \in \mathcal{A}$. The *pinched compact invariant subset* is a compact F-invariant set $\mathcal{A}$ such that $\mathcal{A} \cap \{\theta_0 \times X\}$ contains only one point for at least one $\theta_0 \in \mathbb{S}^1$, that is, $\mathcal{A}^+(\theta_0) = \mathcal{A}^-(\theta_0)$ for such $\theta_0 \in \mathbb{S}^1$. We call such points $(\theta_0, \mathcal{A}(\theta_0))$ the *pinched points*, and denote the set of all the pinched points by $P(\mathcal{A})$. Due to the continuity of F and the invariance of $\mathcal{A}$, it is easy to see that $P(\mathcal{A})$ is also an F-invariant set.

There are already some known properties on the topological structures of compact invariant sets in the literature (see [21] and reference therein), which we summarize as the proposition below.

Proposition 3.1.1. *Suppose $\mathcal{A}$ is a compact F-invariant set, its upper and lower boundaries $\mathcal{A}^+$ and $\mathcal{A}^-$ are upper and lower semicontinuous graphs respectively. If it is pinched, then $\mathcal{A}^+(\theta) = \mathcal{A}^-(\theta)$ for a residual set of $\theta \in \mathbb{S}^1$. Moreover, both $\mathcal{A}^+$ and $\mathcal{A}^-$ are continuous at all these $\theta \in \mathbb{S}^1$.*

In fact, we can say a little more about the continuity of the pinched compact invariant set, which is derived from the property below.

Lemma 3.1.2. *Let $p = (\theta, x)$ be a point in the quasi-periodically forced dynamical system* (3.1). *If its orbit* $\mathrm{Orb}(p) \subset \Psi$ *where Ψ is a continuous graph, then Ψ is an invariant graph and $\Psi = \omega(p)$.*

Proof. We only need to prove that $\Psi = \omega(p)$, because the F-invariance of Ψ just follows the F-invariance of $\omega(p)$.

First we prove that $\Psi \subset \omega(p)$, that is, taking any point $q = (\theta_q, \Psi(\theta_q)) \in \Psi$, we have $q \in \omega(p)$. Denote $F^n(p)$ by (θ_n, x_n). Since the set $\{F^n(p)\} \subset \Psi$ by assumption, we have $x_n = \Psi(\theta_n)$ for all $n \geq 0$. Notice that $F(\theta, x) = (\, R(\theta), G(\theta, x)\,)$ with R an irrational rotation on $\mathbb{S}^1$, this implies that $\{\theta_n\}$ is a dense set in $\mathbb{S}^1$. So we can choose a subsequence (θ_{n_j}) of (θ_n) such that its limit is θ_q. Thus, $(x_{n_j} = \Psi(\theta_{n_j}))$ must converge to $\Psi(\theta_q)$ by the continuity of Ψ. This shows q is a limit point of $\mathrm{Orb}(p)$, that is $q \in \omega(p)$.

Next we show that $\omega(p) \subset \Psi$. As a continuous graph, Ψ is a closed subset in $\mathbb{S}^1 \times X$. Thus $\mathrm{Orb}(p) \subset \Psi$ implies any limit point of $\mathrm{Orb}(p)$ must also belong to this closed set Ψ. □

Next we prove our theorem which show the property of continuous invariant graph in a pinched compact invariant set.

Theorem 3.1.3. *Suppose $\mathcal{A}$ is a pinched compact F-invariant subset of system* (3.1), *and let $p = (\theta_p, x_p) \in P(\mathcal{A})$ be any pinched point. If there exists a continuous graph $\Psi \subset \mathcal{A}$, then Ψ must be invariant and $\Psi = \omega(p)$.*

Proof. $\mathcal{A}$ is pinched means that $\mathcal{A}^+(\theta_p) = \mathcal{A}^-(\theta_p) = x_p$ for any $p \in P(\mathcal{A})$. Since $\Psi \subset \mathcal{A}$, we have $\mathcal{A}^+(\theta_p) \geq \Psi(\theta_p) \geq \mathcal{A}^-(\theta_p)$ by the definition of $\mathcal{A}^+$

and $\mathcal{A}^-$. Notice that $\mathcal{A}^+(\theta_p) = \mathcal{A}^-(\theta_p)$ since p is a pinched point, so $p \in \Psi$ because $x_p = \Psi(\theta_p)$. Thus we have proved that $P(\mathcal{A}) \subset \Psi$. This implies $\mathrm{Orb}(p) \subset \Psi$ since $\mathrm{Orb}(p) \subset P(\mathcal{A})$ for any pinched point p. Now the results follow directly from Lemma 3.1.2 above. □

Remark 3.1.4. This theorem demonstrates a mechanism for the creation of geometric strangeness in pinched invariant sets. In fact, there are two possibilities: one is that, if a pinched compact invariant set itself is not the ω-limit set of the pinched points, then there must be some strange geometry of this invariant set (at least one of its boundaries cannot be continuous, like Keller's model); the other one is more difficult to analyze and it is still an open problem, which is the case that a pinched compact invariant set is just the ω-limits set of the pinched points. In this case, either the pinched set itself is just a continuous graph, or there exists no any continuous graph inside it. One cannot ignore the possibility of the latter, although we have not seen an explicit example of such strange set yet.

We summarize some useful results on this problem in the literature. In [3], this ω-limits set of the pinched points is treated as a pseudo curve, which must be a curve if it contains any piece of curve. Stark([73]) has proved that, if the projection of the set of pinched points is with full measure on $\mathbb{S}^1$, and the normal Lyapunov exponent of the unique invariant measure supported on this pinched set is negative, then this pinched set must be a smooth curve provided that F is a C^1 map.

In general, it is still not known if all the ω-limit sets of the pinched points are continuous curves or not. Particularly, even in the case that a curve is smooth indeed, its shape may be extremely wrinkled by undergoing the fractalization mechanism when its Lyapunov exponent approaches to zero. There are some works which claim such curves are not continuous based on their numerical looking, but are reported as mistakes (see, for example [12, 27, 62]). A detailed theoretical discussion on fractalization mechanism is given in [41], which we have summarized in the last section of the previous chapter. Finally, some examples on the situation of the Lyapunov exponent at critical value zero can be found in [37]. ◇

3.1.2 Orbits of pinched points in pinched systems

This subsection is focused on the dynamics of pinched systems. We prove by Theorem A that, the unique ω-limit set of all the system pinched points is the only minimal invariant subset in whole system, which must be a subset for any compact invariant sets. This means that all the points have to come back arbitrarily close to the pinched points, so all the interesting dynamics can only happen around them. This distinctive feature exhibits the crucial role of the pinched orbits in those pinched systems. We give a particular example to display such feature at the end of this subsection, more detailed effects by these pinched points are exhibited by examples of the following sections.

For the bifurcation mechanism which can produce a pinched invariant set, those *pinched systems* are natural and important candidates. A dynamical system given by the map $F\colon \mathbb{S}^1 \times X \to \mathbb{S}^1 \times X$ in (3.1) is *pinched* if there is at least one fibre who is mapped to a single point. That is, if there exists some $\theta_0 \in \mathbb{S}^1$ such that $G(\theta_0, x) = c$ is a constant for all $x \in X$. The point $(R(\theta_0), c)$, who is the image of a pinched fibre, is called a *(system) pinched point*. We reserve the notation p for (system) pinched points in the following of this chapter. Obviously, in a pinched system, the compact invariant set must also be pinched, hence our arguments of the previous subsection are also valid here. Furthermore, such systems have more properties of their own, which are given by the following lemmas and theorem.

First we show that, unless going to infinity, all the points have to come back arbitrarily close to the pinched points.

Lemma 3.1.5. *In a pinched system, $p \in \omega(x)$ for any point x who doesn't go to infinity.*

Proof. Denote p by (θ_p, x_p) and $F^n(x)$ by (θ_n, x_n). Because R is an irrational rotation on $\mathbb{S}^1$, this implies that $\{\theta_n\}$ is a dense set in $\mathbb{S}^1$. Choose a subsequence (θ_{n_j}) of (θ_n) such that the limit of (θ_{n_j}) is just $R^{-1}(\theta_p)$ and (x_{n_j}) is bounded. If we cannot do it, this implies that $(F^n(x))$ goes to infinity. Otherwise, because F is continuous and maps the whole fibre over $R^{-1}(\theta_p)$ to p, it must be that p is the limit point of $(F^{n_j+1}(x))$. □

Corollary 3.1.6. *It is easy to see that:*

(1) if there is some system pinched point p who goes to infinity, then all the points in the system also go to infinity;

(2) the ω-limit set for all the pinched points is unique.

Proof. The first claim is trivial from the above lemma, in such case the ω-limit of any point is empty. Now we consider the case that the pinched points do not go to infinity. If p_1 and p_2 are two different pinched points, Lemma 3.1.5 gives both $p_1 \in \omega(p_2)$ and $p_2 \in \omega(p_1)$. By the property of the ω-limit sets, if $y \in \omega(x)$, then $\omega(y) \subset \omega(x)$ for any points x and y. Thus $\omega(p_1) \subset \omega(p_2)$ and $\omega(p_2) \subset \omega(p_1)$, we have $\omega(p_1) = \omega(p_2)$. □

Therefore we can use $\omega(p)$ to denote the ω-limit set of any pinched point in a pinched system, it doesn't matter which one the point p is. Now, it is almost trivial for us to get our theorem below.

Theorem A. *In the pinched quasi-periodically forced systems, if $\omega(p)$ is not empty, then it is the only minimal compact invariant set in the whole system, and is a subset for any compact invariant set.*

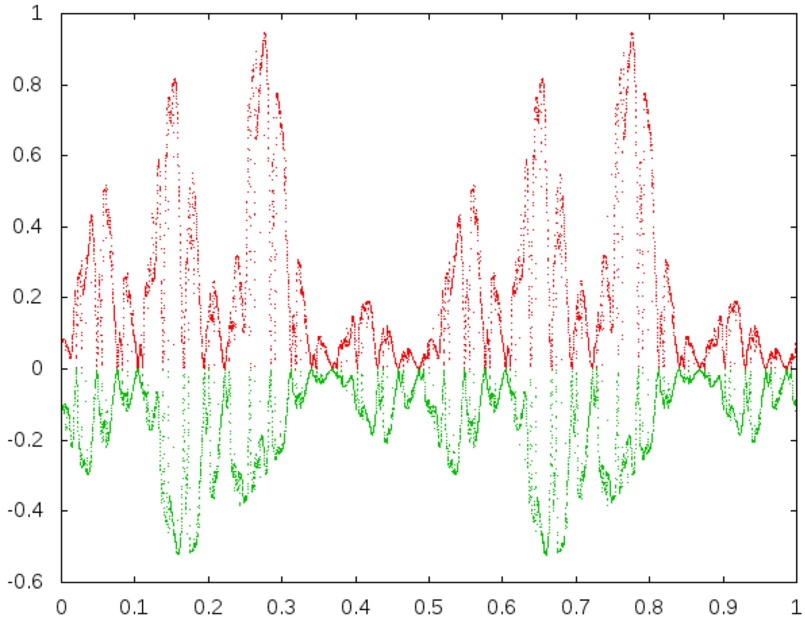


Figure 3.1: Period doubling bifurcation at $x = 0$, obtained with $G(\theta, x) = 2.2\,|\cos(2\pi\theta)|\,x(x-1)$ and ω equals the golden mean.

Proof. To see that $\omega(p)$ is minimal, we need to prove it is the ω-limit set of all its points. Taken any $x \in \omega(p)$, the property of the ω-limit set gives $\omega(x) \subset \omega(p)$. We have also $\omega(p) \subset \omega(x)$ because $p \in \omega(x)$ by Lemma 3.1.5. Hence $\omega(x) = \omega(p)$ as required.

Now let $\mathcal{A}$ be a compact invariant set, for any point $x \in \mathcal{A}$, we have also $p \in \omega(x)$ by Lemma 3.1.5. This means $\omega(p) \subset \omega(x)$. On the other hand, the ω-limit set of any point in a compact invariant set must be its subset, which is $\omega(x) \subset \mathcal{A}$. So we have $\omega(p) \subset \omega(x) \subset \mathcal{A}$. Notice that this works for any $x \in \mathcal{A}$, which implies $\omega(p)$ is unique. $\square$

Remark 3.1.7. Notice that, inside each pinched compact invariant set in non-pinched systems, the ω-limit set of its pinched points is also its unique minimal subset. Using arbitrary point x and pinched point of the pinched set instead, the arguments above work the same for this case. That is, the "set" pinched point must be in the ω-limit set of any points of this set, and then the result follows. $\diamond$

Theorem A is simple but instructive, it displays the special significance of orbits of pinched points on the dynamics of pinched systems. Naturally we can hope that, a forced system owns the similar dynamics with the unforced interval map when the perturbation given by forced term is relative small. This is really the case for the non-pinched systems, in which there can exist several disjoint compact invariant sets, each one corresponds to a different fixed or periodic point of the unforced interval maps. But it is generally not true for the pinched systems. Theorem A implies that any two compact invariant sets cannot be disjoint, hence all interesting dynamics can only be in a piece which is around the ω-limit set of the pinched points.

We will discuss some monotone interval families exhaustively in the following sections. They illustrate more clearly this special significance of the pinched systems by pinched orbits and their difference with the non-pinched ones. Before that, we present a simple example to show that, how the position of the pinched orbits in a pinched system affects the general dynamical behaviour of the system.

Example 3.1.8. Consider the system

$$F(\theta, x) = (\,\theta + \omega \bmod(1),\, 2.2\,|\cos(2\pi\theta)|\, x(x-1)\,),$$

which is a pinched one whose attractor is plotted in Figure 3.1.

In [4], the authors study the pinched quasi-periodically forced unimodal map

$$F(\theta, x) = (\,\theta + \omega \bmod(1),\, \mu x(1-x)\cdot g(\theta)\,),$$

with $g = 0$ at some θ. They propose a question that, if there are some periodic invariant graphs in such system, just like the unimodal maps which can go into the period doubling cascade. Notice that $x = 0$ is invariant and contains all the pinched points, hence the dynamics of this system happens only around it. But $x = 0$ is not the place where the period doubling takes place for the unimodal map $\mu x(1-x)$, that should be at the other fixed point $1 - 1/\mu$.

If we replace $\mu x(1-x)$ by $\mu x(x-1)$, then Figure 3.1 shows clearly that the period doubling occurs. This is because that, for this new map $\mu x(x-1)$, $x = 0$ is just the right place for the first period doubling. In fact, it is equivalent to that there is a change of variable, so that the fixed point $1 - 1/\mu$ of the common logistic map $\mu x(1-x)$ is move to $x = 0$ of this new model $\mu x(x-1)$. $\diamond$

3.2 Transfer operator and contraction due to concavity or convexity

In this section we develop some general properties of the quasi-periodically forced monotonically increasing systems. They are derived from the monotonicity, and from the concavity and convexity respectively. These properties provide the basic tools for the investigations of systems with such structures on their fibre maps, not only for both the two models whose fibres are set as $\mathbb{R}$ in the next two sections, but also for systems whose fibre maps are finite interval maps, or even for cases of being just locally with these structures around some attractors as well.

Precisely, in the first subsection we discuss the notations of the forward and backward transfer operators. They are very useful tools in the study of quasi-periodically forced systems, obtained by considering the systems from the function space point of view. Some properties of the monotone increasing systems and their operators are given in the second subsection. Finally, the third one is devoted to concepts of concavity and convexity, where we introduce the concepts of α-concavity and β-convexity, and prove some contraction results of the fibre maps with such structures.

3.2.1 Transfer operators

If we consider that, let the system in form of (3.1) act on functions from $\mathbb{S}^1$ to $\mathbb{R}$, we get a functional version of the system. Then an invariant function is a fixed point in this functional version of system. For the skew products (3.1), an *invariant function* is a function $\varphi\colon \mathbb{S}^1 \to \mathbb{R}$ that satisfies the following *invariance equation*

$$\varphi(R(\theta)) = G(\theta, \varphi(\theta)).$$

Recall that we abuse terminology and refer an invariant graph φ to a graph of an invariant function φ. Thus its graph is kept to be fixed under the action of (3.1). An easy example is the function $\varphi = 0$ in Keller's model. This idea leads to the important tools for the study of invariant graphs, which are the *transfer operators* defined as follows.

Let $\mathcal{P}$ be the space of all functions (not necessarily continuous) from $\mathbb{S}^1$ to $\mathbb{R}$, $\psi \in \mathcal{P}$. The *(forward) transfer operator* $\mathcal{T}\colon \mathcal{P} \to \mathcal{P}$ of the skew product (3.1) is defined as:

$$(\mathcal{T}\psi)(\theta) = G(R^{-1}(\theta), \psi(R^{-1}(\theta))).$$

In this memoir, we only consider transfer operators for systems given by

$$F(\theta, x) = (\, \theta + \omega \text{ mod}(1),\ \lambda f(x + a) \cdot g(\theta)\,), \tag{3.2}$$

here we have $R(\theta) = \theta + \omega \pmod 1$, and $G(\theta, x) = \lambda f(x + a) \cdot g(\theta)$. In the map $G(\theta, x)$, λ and a are used as real parameters. We assume that $g\colon \mathbb{S}^1 \to \mathbb{R}$ is continuous and $g(\theta) \geq 0$. The real function $f(x)$ is strictly increasing, which satisfies $f(0) = 0$.

Then the transfer operator of system (3.2) is given by

$$(\mathcal{T}\psi)(\theta) = \lambda f(\psi(\theta - \omega) + a) \cdot g(\theta - \omega). \tag{3.3}$$

From now on, we save the notation $\mathcal{T}$ for operators given by (3.3) above, and call them just as transfer operators for short. Notice that, the graph of $\mathcal{T}\psi$ is the image of the graph of ψ under F, and φ is invariant if and only if $\mathcal{T}\varphi = \varphi$. Moreover, when we want to indicate clearly a transfer operator that corresponds to a system given by some specific parameter, for instance to a, we then denote it as

$$(\mathcal{T}_a\psi)(\theta) = \lambda f_a(\psi(\theta-\omega)) \cdot g(\theta-\omega). \tag{3.4}$$

When considering the backward iterate or the preimage of a function ψ in the system (3.2), we need the *backward transfer operator* $\mathcal{R}$ (only defined when $g(\theta) > 0$):

$$(\mathcal{R}_a\psi)(\theta) = f_a^{-1}\left(\frac{\psi(\theta+\omega)}{\lambda g(\theta)}\right) = f^{-1}\left(\frac{\psi(\theta+\omega)}{\lambda g(\theta)}\right) - a. \tag{3.5}$$

Notice that this operator is well-defined if $\lambda g(\theta) \neq 0$ for all $\theta \in \mathbb{S}^1$, since we require f to be strictly increasing in (3.2).

3.2.2 Some facts due to monotonicity

Next, we show some simple facts of the system (3.2), coming from the monotonicity of f and $g(\theta) \geq 0$. They will be frequently used later when we deal with the models in the next two sections.

Observation 3.2.1. *The fibre map $\lambda f(x+a)g(\theta)$ is also monotone increasing for each $\theta \in \mathbb{S}^1$. That is, taking (θ_0, x_0) and (θ_0, y_0) for any $\theta_0 \in \mathbb{S}^1$ with $x_0 \geq y_0$, then*

$$\lambda f_a(x_0)g(\theta_0) \geq \lambda f_a(y_0)g(\theta_0).$$

Furthermore, if $g(\theta_0) > 0$, this fibre map $\lambda f(x+a)g(\theta_0)$ is also strictly increasing.

For a point $(\theta, x) \in \mathbb{S}^1 \times \mathbb{R}$, denote $(\theta_n, x_n) = F^n(\theta, x)$ for $n \geq 0$ as well.

Observation 3.2.2. *If $a \geq 0$ and $x_k \geq -a$ for some $k \geq 0$, the monotonicity of these fibre maps implies that $x_{k+n} \geq 0$ for all $n \geq 1$.*

Proof. First note that, if $x_k \geq 0$ for some $k \geq 0$, we have $f(x_k+a) \geq f(a) \geq f(0) = 0$, which implies $x_{k+1} = \lambda f(x_k+a)g(\theta_n) \geq 0$. Hence $x_{k+n} \geq 0$ for all $n \geq 1$.

Moreover, if $x_k \geq -a$, then $x_{k+1} \geq 0$ since $f(x_k+a) \geq f(0) = 0$. □

Remark 3.2.3. The above observations imply that, if $a \geq 0$ in the system (3.2), then $\mathbb{S}^1 \times [0, +\infty)$ is invariant, and any point with initial value $x \geq -a$ enters this invariant region after one iterate. Notice that it is the same case for $\mathbb{S}^1 \times (-\infty, 0]$ if $a \leq 0$. ◇

In terms of transfer operators, the monotonicity of fibre maps in system (3.2) gives the following lemma.

Lemma 3.2.4. *Let $\psi, \varphi \in \mathcal{P}$, and $\psi \leq \varphi$. Then*

(1) $\mathcal{T}^n\psi \leq \mathcal{T}^n\varphi$ for all $n \geq 1$;

(2) $\mathcal{R}^n\psi \leq \mathcal{R}^n\varphi$, for all $n \geq 1$, whenever $\mathcal{R}$ is well-defined;

(3) particularly, if $\mathcal{T}\psi \leq \psi$ ($\mathcal{T}\psi \geq \psi$), then $\mathcal{T}^{n+1}\psi \leq \mathcal{T}^n\psi$ ($\mathcal{T}^{n+1}\psi \geq \mathcal{T}^n\psi$) for all $n \geq 1$. It is also the same case for the backward transfer operator $\mathcal{R}$.

Proof. Here $\mathcal{T}\psi \leq \mathcal{T}\varphi$ follows clearly from monotonicity of all the fibre maps in Observation 3.2.1, then we have $\mathcal{T}^n\psi \leq \mathcal{T}^n\varphi$ by induction. All the arguments go the same way for backward transfer operator $\mathcal{R}$. □

Remark 3.2.5. Particularly in case of $\mathcal{T}\psi \leq \psi$, the above lemma shows that $\mathcal{T}^{n+1}\psi \leq \mathcal{T}^n\psi$ for all $n \geq 1$, which results a decreasing sequence of graphs $\{\mathcal{T}^n\psi\}$. It is commonly known that, if the pointwise limit of a decreasing or an increasing sequence of continuous functions exits, this limit is an upper or a lower semicontinuous function respectively. Hence, if a function is proved to be the limit of both a decreasing and an increasing sequence at the same time, this function must be a continuous one. ◇

3.2.3 Contraction due to concavity or convexity

Now we discuss a property derived from the concave or convex structure of the monotonic function f in system (3.2). This property implies contraction of such interval map f in an invariant region, which is also the reason of the existence and the attraction of invariant graphs in some models of form (3.2) that we will investigate later.

Our treatment here follows the arguments in [5], which extends their result to a more general setting. We start from the definition of concavity and convexity first, then introduce the concepts of α-concavity and β-convexity, and finally show that, for increasing maps with α-concavity or β-convexity structures in some invariant intervals, all the points go to a unique limit point with only one possible exception of an endpoint.

A real continuous function f defined on an interval $I \subset \mathbb{R}$ is *concave*, if and only if for any x and y in the interval I,

$$f\Big(\frac{x+y}{2}\Big) \geq \frac{f(x)+f(y)}{2}. \tag{3.6}$$

Or equivalently, for any three points $t_1 < t_2 < t_3$ in I, it is

$$\frac{f(t_2)-f(t_1)}{t_2-t_1} \geq \frac{f(t_3)-f(t_1)}{t_3-t_1}. \tag{3.7}$$

If the symbol "$\geq$" in (3.6) and (3.7) above is changed to be "$\leq$', then f is said to be *convex*. Moreover, if the inequalities of definitions are strict, then f is also strictly concave or strictly convex.

Next, let us introduce the notions of α-concavity and β-convexity.

Definition 3.2.6. Let f be a continuous real-valued function on a closed interval $I \subset \mathbb{R}$ and let $\alpha \geq 0$. The function f will be called *α-concave* if the function f_α, given by

$$f_\alpha(x) = f(x) + \alpha x^2,$$

is concave.

The following properties of an α-concave function f follow immediately from the definition:

(1) f is concave;

(2) if $\alpha > 0$ then f is strictly concave;

(3) if $0 \leq \gamma \leq \alpha$ then f is γ-concave.

Similarly, we define a β-convex function as follows.

Definition 3.2.7. Let f be a continuous real-valued function on a closed interval $I \subset \mathbb{R}$ and let $\beta \geq 0$. The function f will be called *β-convex* if the function f_β, given by

$$f_\beta(x) = f(x) - \beta x^2,$$

is convex.

A β-convex function f also satisfies:

(1) f is convex;

(2) if $\beta > 0$ then f is strictly convex;

(3) if $0 \leq \gamma \leq \beta$ then f is γ-convex.

Notice that the properties of concavity and convexity are kept under the change of variable, this is also true for α-concavity and β-convexity. Taking the α-concavity as example, we have the following.

Lemma 3.2.8. *If $f(x)$ is an α-concave function defined on a closed interval $I = [t_1, t_2]$ (the endpoints may be infinity), then $f(x+a)$ is also α-concave on $[t_1 - a, t_2 - a]$ for any $a \in \mathbb{R}$.*

Proof. By the definition, to prove $f(x+a)$ is α-concave, we need to show that $f(x+a) + \alpha x^2$ is concave on $[t_1 - a, t_2 - a]$.

Since $f(x)$ is α-concave on $I = [t_1, t_2]$, $f_\alpha(x+a) = f(x+a) + \alpha(x+a)^2$ is concave on $[t_1 - a, t_2 - a]$. Hence by (3.6), the definition of concavity, for any x_1 and x_2 on the interval $[t_1 - a, t_2 - a]$, it must be

$$f_\alpha\Big(\frac{(x_1+a)+(x_2+a)}{2}\Big) \geq \frac{f_\alpha(x_1+a)+f_\alpha(x_2+a)}{2}. \tag{3.8}$$

Notice that, for the left hand side of (3.8),

$$f_\alpha\Big(\frac{(x_1+a)+(x_2+a)}{2}\Big) = f\Big(\frac{(x_1+a)+(x_2+a)}{2}\Big) + \alpha\Big(\frac{(x_1+a)+(x_2+a)}{2}\Big)^2$$
$$= f\Big(\frac{(x_1+a)+(x_2+a)}{2}\Big) + \alpha\Big(\big(\frac{x_1+x_2}{2}\big)^2 + a(x_1+x_2) + a^2\Big),$$

and for the right hand side, it is

$$f_\alpha\Big(\frac{(x_1+a)+(x_2+a)}{2}\Big) = \frac{f(x_1+a)+\alpha(x_1+a)^2+f(x_2+a)+\alpha(x_2+a)^2}{2}$$
$$= \frac{f(x_1+a)+f(x_2+a)}{2} + \alpha\Big(\frac{x_1^2+x_2^2}{2} + a(x_1+x_2) + a^2\Big).$$

The last two terms in the above two expressions of both sides are the same, deleting them gives

$$f\Big(\frac{(x_1+a)+(x_2+a)}{2}\Big) + \alpha\big(\frac{x_1+x_2}{2}\big)^2 \geq \frac{f(x_1+a)+f(x_2+a)}{2} + \alpha\frac{x_1^2+x_2^2}{2},$$

which means that $f(x+a) + \alpha x^2$ is concave on $[t_1 - a, t_2 - a]$. □

For β-convexity the situation is exactly the same. That is, if $f(x)$ is a β-convex function defined on a closed interval $I = [t_1, t_2]$, then $f(x+a)$ is also β-convex on $[t_1 - a, t_2 - a]$ for any $a \in \mathbb{R}$. For simplicity, we will only consider concave functions in the sequel and will state the corresponding results of convex functional only when necessary, since their arguments are all analogous.

A contraction property of some increasing real maps with a concave structure will be shown in Remark 3.2.10, it is derived from the inequality (3.9) below.

Let t be a real number, given two points u and v with $(u-t)(v-t) > 0$, we define

$$\kappa(u, v) := \frac{|v-u|}{\min\{|u-t|, |v-t|\}}.$$

Lemma 3.2.9. *Assume that a real increasing map f satisfies $f(t) \geq t$, and f is also α-concave on $[t, y]$, then for any $y > x > t$,*

$$\frac{\kappa(f(x), f(y))}{\kappa(x, y)} \leq \frac{f(y)-f(t)}{f(y)-f(t)+\alpha(y-t)^2}. \tag{3.9}$$

Proof. Particularly for $y > x > t$, we have

$$\kappa(y, x) = \frac{y - x}{x - t}.$$

We know that f is α-concave on $[t, y]$, which means $f_\alpha(x) = f(x) + \alpha x^2$ is concave on $[t, y]$. So by definition (3.7), if $y > x > t$ we have

$$\frac{f_\alpha(x) - f_\alpha(t)}{x - t} \geq \frac{f_\alpha(y) - f_\alpha(t)}{y - t}. \tag{3.10}$$

Notice that,

$$\frac{f_\alpha(x) - f_\alpha(t)}{x - t} = \frac{f(x) + \alpha x^2 - f(t) - \alpha t^2}{x - t} = \frac{f(x) - f(t)}{x - t} + \alpha(x + t),$$

so (3.10) is

$$\frac{f(x) - f(t)}{x - t} + \alpha(x + t) \geq \frac{f(y) - f(t)}{y - t} + \alpha(y + t),$$

which implies that

$$\frac{f(x) - f(t)}{x - t} \geq \frac{f(y) - f(t)}{y - t} + \alpha(y - x), \tag{3.11}$$

and hence,

$$f(x) - f(t) \geq \frac{x - t}{y - t}(f(y) - f(t)) + \alpha(y - x)(x - t).$$

The above formula gives,

$$\begin{aligned} f(y) - f(x) &= f(y) - f(t) - (f(x) - f(t)) \\ &\leq (y - x)\left(\frac{f(y) - f(t)}{y - t} - \alpha(x - t)\right). \end{aligned}$$

Now we have that,

$$\frac{f(y) - f(x)}{y - x} \cdot \frac{x - t}{f(x) - t} \leq \left(\frac{f(y) - f(t)}{y - t} - \alpha(x - t)\right) \cdot \frac{x - t}{f(x) - t}.$$

Moreover, it is

$$\frac{f(x) - t}{x - t} \geq \frac{f(x) - f(t)}{x - t}$$

since $f(t) \geq t$, then by (3.11)

$$\begin{aligned} \frac{\kappa(f(x), f(y))}{\kappa(x, y)} &= \frac{f(y) - f(x)}{f(x) - t} \cdot \frac{x - t}{y - x} \leq \frac{\frac{f(y)-f(t)}{y-t} - \alpha(x - t)}{\frac{f(y)-f(t)}{y-t} + \alpha(y - x)} \\ &= 1 - \frac{\alpha(y - t)}{\frac{f(y)-f(t)}{y-t} + \alpha(y - x)} \leq 1 - \frac{\alpha(y - t)}{\frac{f(y)-f(t)}{y-t} + \alpha(y - t)} \\ &= \frac{\frac{f(y)-f(t)}{y-t}}{\frac{f(y)-f(t)}{y-t} + \alpha(y - t)} = \frac{f(y) - f(t)}{f(y) - f(t) + \alpha(y - t)^2}. \end{aligned}$$

□

Remark 3.2.10. Assume that a real map f is increasing on a closed invariant interval I with its left endpoint being t and $f(t) \geq t$. If f is also α-concave for some $\alpha > 0$, then there can exist at most one fixed point in $I \setminus \{t\}$. Furthermore, this fixed point attracts all other points in $I \setminus \{t\}$ if it exists.

This is because that, the same as in [5], for any two points $x < y$ inside $I \setminus \{t\}$, denote $x_n = f^n(x)$ and $y_n = f^n(y)$, then we have

$$|x_n - y_n| = \min\{x_n, y_n\}\kappa(x_n, y_n) = \min\{x_n, y_n\}\kappa(x_0, y_0) \prod_{k=0}^{n-1} \frac{\kappa(x_{k+1}, y_{k+1})}{\kappa(x_k, y_k)}.$$

For all $0 \leq k < n$, by (3.9) of Lemma 3.2.9,

$$\frac{\kappa(x_{k+1}, y_{k+1})}{\kappa(x_k, y_k)} \leq \frac{f(y_{k+1}) - f(t)}{f(y_{k+1}) - f(t) + \alpha(y_{k+1} - t)^2},$$

and the right hand sides for all $0 \leq k < n$ are smaller than a constant who is less than 1. Hence, the products in the above equality go to 0 as $n \to \infty$, it means that

$$\lim_{n\to\infty} |x_n - y_n| = 0.$$

That is, any two points x and y inside $I \setminus \{t\}$ have the same limit.

For the dynamics of such interval maps f, this implies that, if $f(t) = t$ and this left endpoint t is also attracting, then it is the only fixed point of I which attracts all points of I. Otherwise, when $f(t) > t$ or $f(t) = t$ with t repelling, all points of $I \setminus \{t\}$ have the same limit other than t, which is attracting if it is a fixed point. Notice that, if I is a finite interval, then this attracting fixed point must exist.

Analogously, provided $\beta > 0$, in the case that f is increasing and β-convex on a closed invariant interval I with its right endpoint being t and $f(t) \leq t$, it has the same dynamics as the concave case above. This is derived from the inequality below, that is, for any $y < x < t$

$$\frac{\kappa(f(x), f(y))}{\kappa(x, y)} \leq \frac{f(t) - f(y)}{f(t) - f(y) + \beta(y - t)^2}, \tag{3.12}$$

which is totally analogous to (3.9). ◇

Instead of the interval maps, here the systems that we are interested in are those quasi-periodically forced one. However, if their forced interval maps have the same structures, the contraction properties are basically same. Below we take only one case as instance, all the other cases can be treated in exactly the same way, so we do not repeat.

For the function $f(x + a)$ in quasi-periodically forced system (3.2) with $a \geq 0$, when $f(x)$ is α-concave on $[0, +\infty)$, we have an inequality as below.

Lemma 3.2.11. *Assume that $f(x)$ in (3.2) is α-concave on $[0,+\infty)$, and $a \geq 0$. Let $0 < x < y$, then,*

$$\frac{\kappa(f(x+a), f(y+a))}{\kappa(x,y)} \leq \frac{f(y+a)}{f(y+a)+\alpha(y+a)^2}.$$

Proof. Since f is α-concave on $[0,+\infty)$, we have $f(x+a)$ is also α-concave on $[-a,+\infty)$ by Lemma 3.2.8. Take the left endpoint t in Lemma 3.2.9 as $-a$, since $f(-a+a) = f(0) = 0 \geq -a$, we have by (3.9) that

$$\frac{\kappa(f(x+a), f(y+a))}{\kappa(x,y)} \leq \frac{f(y+a)}{f(y+a)+\alpha(y+a)^2}.$$

as required. □

Unlike a map on a fixed interval, in a quasi-periodically forced system each iterate goes to a different fibre. However, with assumptions of above lemma, the contraction exists also for every pair of points in the same fibre.

Corollary 3.2.12. *For two initial points (θ_0, x_0) and (θ_0, y_0) in the same fibre, denote $(\theta_n, x_n) = F^n(\theta_0, x_0)$ and $(\theta_n, y_n) = F^n(\theta_0, y_0)$ where F is a map of (3.2) with $f(x)$ α-concave on $[0,+\infty)$. If $a \geq 0$ and $g(\theta_k) \neq 0$ for all $0 \leq k \leq n$, then for any $0 < x_0 < y_0$, we have $0 < x_{k+1} < y_{k+1}$ and*

$$\frac{\kappa(x_{k+1}, y_{k+1})}{\kappa(x_k, y_k)} = \frac{\kappa(f(x_k+a), f(y_k+a))}{\kappa(x_k, y_k)} \leq \frac{f(y_k+a)}{f(y_k+a)+\alpha(y_k+a)^2}. \quad (3.13)$$

Proof. $0 < x_{k+1} < y_{k+1}$ follows directly from Observation 3.2.1, while the strict inequality is due to $g(\theta_k) \neq 0$ for all $0 \leq k \leq n$.

Next, since

$$\frac{|x_{k+1} - y_{k+1}|}{\min\{|x_{k+1}|, |y_{k+1}|\}} = \frac{\lambda g(\theta_k)|f(x_k+a) - f(y_k+a)|}{\lambda g(\theta_k)|f(x_k+a)|} = \frac{|f(x_k+a) - f(y_k+a)|}{|f(x_k+a)|},$$

we have $\kappa(x_{k+1}, y_{k+1}) = \kappa(f(x_k+a), f(y_k+a))$, this gives the equality in (3.13) first. The inequality is just the result of Lemma 3.2.11. □

Remark 3.2.13. Analogously, if f in (3.2) is β-convex on $(-\infty, 0]$ with $\beta > 0$, then for any $y < x < 0$ and a function $f(x+a)$ with $a \leq 0$, we also have

$$\frac{\kappa(x_{k+1}, y_{k+1})}{\kappa(x_k, y_k)} = \frac{\kappa(f(x_k-a), f(y_k-a))}{\kappa(x_k, y_k)} \leq \frac{f(y_k-a)}{f(y_k-a)+\beta(y_k-a)^2}. \quad (3.14)$$

Its proof goes literally the same as the concave case above. ◇

3.3 First monotonic increasing model

Each of this and the next section is devoted to a concrete family of quasi-periodically forced increasing real maps respectively. Both of these families are given by maps $F\colon \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ in form of (3.2), with f being monotonically increasing and having some concave or convex structures. They also can be viewed as generalization of Keller's model to whole cylinder. For the family that we discuss in this section, f is assumed to have a symmetric-like structure: its upper part is kept to be concave like Keller's, the lower part is set to be convex.

The dynamical behaviours of this type of models are based first on the structure of map f, and finally are decided by the situations of parameters λ and a and by the case of function g being pinched or non-pinched. Especially when $a \neq 0$, there is particular behaviour distinctively different with Keller's model, which is presented clearly in our Theorem B.

We divide this section into two subsections. The first one is dedicated to a general introduction of the complete dynamics of this type of families. Their proofs are given in the second subsection.

3.3.1 The model and its dynamics

Our first family generalizes Keller's model to the whole cylinder $\mathbb{S}^1 \times \mathbb{R}$ in the following way:

$$F(\theta, x) = (\, \theta + \omega \bmod(1),\, \lambda f(x+a) \cdot g(\theta)\,), \tag{3.15}$$

with f a real function which satisfies:

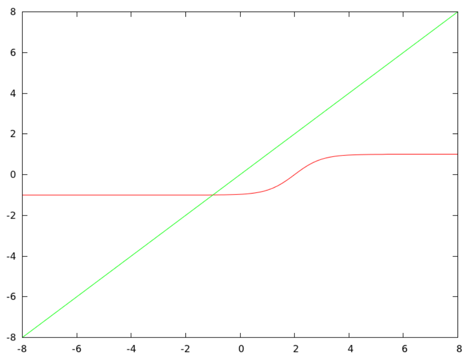
(1) f is continuous and bounded;

(2) f is strictly increasing on $\mathbb{R}$ and $f(0) = 0$;

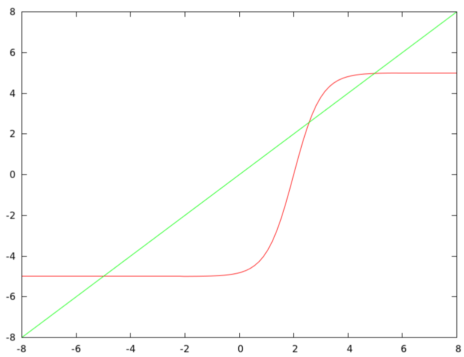
(3) f is α-concave for $x \geq 0$ with some $\alpha > 0$, and is β-convex for $x \leq 0$ with some $\beta > 0$.

Hence the unforced interval map $f(x)$ is the same with Keller's model for $x \geq 0$, but is extended to the negative part with a convex curve. Here both λ and a are real numbers which are used as parameters. We let $\lambda > 0$ so that each fibre map is kept to be increasing. Different with Keller, another parameter a is added in this model. The reason of this setting on fibre maps of $f(x+a)$ is that, we hope to study the general dynamics of any maps with this type of shape, particularly without 0 being fixed. With the change of a, the zero and fixed points of $f(x+a)$ change also, notice that this makes $x = 0$ no longer invariant in the forced system as long as $a \neq 0$.

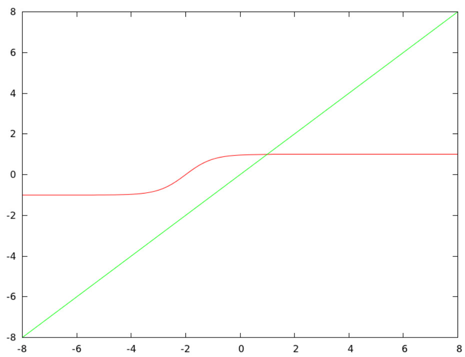
First we have a look at the dynamics of interval map $\lambda f(x+a)$, which is the basis of the behaviour of the forced system. The family of this kind



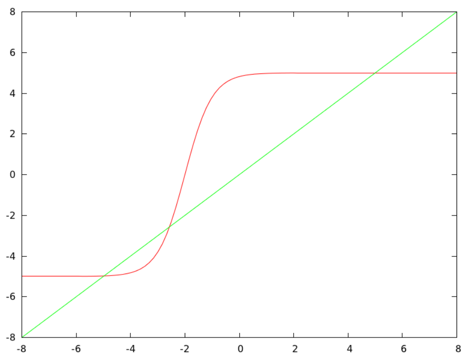
(a) $a < 0$ and λ is small.



(b) $a < 0$ and λ is big.



(c) $a > 0$ and λ is small.



(d) $a > 0$ and λ is big.

Figure 3.2: Graphs of $\lambda f(x + a)$ for different cases.

of unforced interval maps $\lambda f(x+a)$ is a typical example of the saddle-node bifurcation of one-dimensional systems when $a \neq 0$. In Figure 3.2 we plot the pictures for the cases of λ big and small with $a < 0$ and $a > 0$ respectively. They display the situations before and after the saddle-node bifurcation with the increasing of λ. The dynamics of $\lambda f(x+a)$ can be easily understood via graph analysis. Notice that, the cases of $a < 0$ and of $a > 0$ are in fact essentially the same, except for the directions that those points go are in an inverse way. We do not plot the graphs of case $a = 0$, that $x = 0$ is always fixed, and there is a pitchfork bifurcation occurs from it with the increasing of λ.

Corresponding to the cases of unforced interval maps $\lambda f(x+a)$ above, now we present the forced dynamical systems (3.15) with parameters a fixed and λ increasing. Similar as the maps $\lambda f(x+a)$, their behaviours are also different for the cases of $a = 0$ and $a \neq 0$, we describe each of them separately.

Case $a = 0$: The forced systems in case of $a = 0$ are in fact just Keller's models, that we have known already.

(1) $x = 0$ is invariant in the system for any λ. Moreover, if λ is relatively small, $x = 0$ is the only invariant graph which attracts all other points in the system.

(2) With the increasing of λ, two invariant graphs other than $x = 0$ will bifurcate out from above and below respectively, when $x = 0$ becomes repelling in each of these two directions. The two new invariant graphs attract those points with positive and negative initial x values separately.

(3) If $g(\theta) > 0$ for all $\theta \in \mathbb{S}^1$, all the invariant graphs are as smooth as g; if $g(\theta_0) = 0$ for some $\theta_0 \in \mathbb{S}^1$, both the invariant graphs bifurcate out from $x = 0$ are strange.

Remark 3.3.1. This case is a rather special case for general families (3.15), since $x = 0$ is always fixed. It is clearly a directly generalization of Keller's model, hence the dynamical behaviours can be easily understood via Keller's theorem without new proof. Precisely, the parts of $x \geq 0$ and $x \leq 0$ are two subsystems individually. The upper one is exactly Keller's model; while for the lower one, the only difference with the upper is its convexity instead. However the dynamics goes analogously with the upper part due to the symmetric structure, so does its proof.

It should be noticed that, here by symmetry we don't mean the two parts are strictly geometrically symmetric, we only refer that their general structures are symmetric in the topological sense. Hence the bifurcations of these two individual subsystems may happen at two different values of λ. $\diamond$

Generally speaking, in the case of $a = 0$, for both the pinched and non-pinched cases there are all the pitchfork type of bifurcations in form of invariant graphs, and the only difference between them is the strangeness of the graphs that bifurcate out. However, we can see that the situations of $a \neq 0$ are not so, the bifurcation behaviours are noticeably distinct and there only exist bifurcations in the non-pinched cases.

Case $a \neq 0$: In this case, $x = 0$ is no longer invariant. The complete dynamical behaviours according to parameters λ and a are as below.

Theorem B. *If $a \neq 0$ in family* (3.15)*, the dynamics of the system is the following:*

(1) For any the values of a and λ, there is an invariant graph Φ_λ in the system, which is attracting and continuous. $\Phi_\lambda \geq 0$ when $a > 0$; $\Phi_\lambda \leq 0$ when $a < 0$.

(2) If there exists $g(\theta_0) = 0$ for some $\theta_0 \in \mathbb{S}^1$, Φ_λ is the only *invariant graph who attracts all points of the whole system.*

(3) If $g(\theta) > 0$ for all $\theta \in \mathbb{S}^1$, then there is a critical value of λ, denoted by λ_0, such that

   (a) if $\lambda < \lambda_0$, Φ_λ is the only invariant graph which attracts all the points in whole system $\mathbb{S}^1 \times \mathbb{R}$;

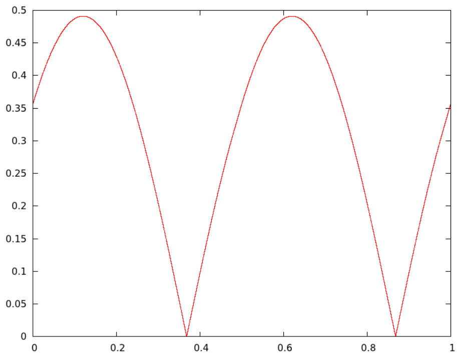
   (b) if $\lambda > \lambda_0$, there exist another two continuous invariant graphs other than Φ_λ, one of them is repelling and the other one is attracting. Denote this new attracting graph by Ψ_λ and the repelling one by Γ_λ, it is $\Psi_\lambda < -a$ when $a > 0$ ($\Phi_\lambda \geq 0$); and $\Psi_\lambda > -a$ when $a < 0$ ($\Phi_\lambda \leq 0$). In both these two cases, the repelling graphs lie between Φ_λ and Ψ_λ.

Remark 3.3.2. We notice the following facts which lead to a simpler dynamical interpretation of the general situations:

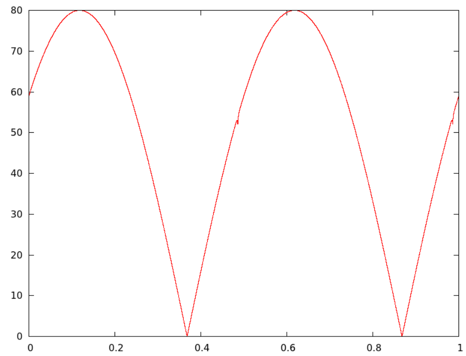
(1) The existence of one attracting invariant graph, Φ_λ namely, depends neither on if the system is pinched or not, nor on that λ is small or big. But the bifurcations of new graphs do depend on them.

(2) For the non-pinched cases, there is a bifurcation occurs at the opposite side of Φ_λ with the increasing of λ. The type of such bifurcation is exactly corresponding to the saddle-node bifurcation of the unforced interval maps $\lambda f(x + a)$.

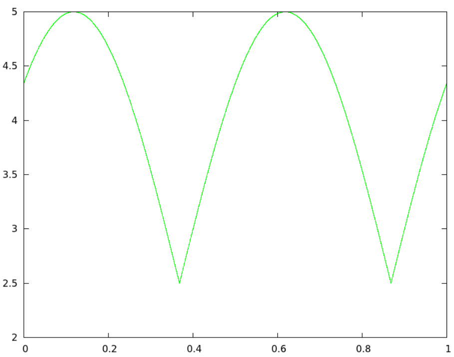
(3) Instead, there is **no any** bifurcation in any pinched systems of this case $a \neq 0$. Notice that this is also different with the pinched case of $a = 0$, which must have bifurcation for big enough λ. Particularly, it implies that there is no strange curve in any cases of $a \neq 0$.



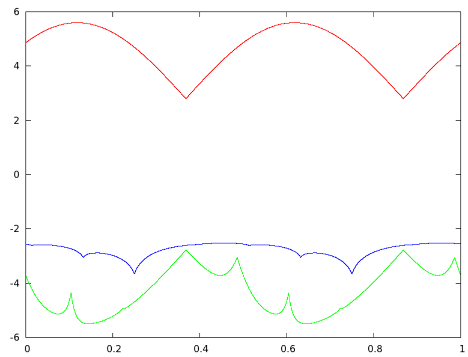
(a) The attractor obtained by model with $\lambda g(\theta) f(x + a) = 0.5\,|\cos(2\pi\theta)|\tanh(x + 2)$.



(b) The attractor obtained by model with $\lambda g(\theta) f(x + a) = 80\,|\cos(2\pi\theta)|\tanh(x + 2)$.



(c) The attractor obtained by model with $\lambda g(\theta) f(x + a) = 2.5(1 + |\cos(2\pi\theta)|)\tanh(x + 2)$. No repellor founded in system.



(d) The two attractors and repellor obtained by model with $\lambda g(\theta) f(x + a) = 2.8(1 + |\cos(2\pi\theta)|)\tanh(x + 2)$. The top and bottom curves are attractors, the middle one is repellor.

Figure 3.3: Graphs of different cases of the first family.

$\diamond$

In Figure 3.3 we plot the invariant graphs for all typical situations for the case of $a \neq 0$, which correspond to the pinched and non-pinched cases, with λ values small and big respectively. These invariant graphs are obtained by a concrete model with $f(x) = \tanh(x)$ and $a = 2$. In this memoir, we take the golden mean as ω in all the examples that we compute. We can see from the pictures at the bottom row that, the dynamics of the non-pinched cases are exactly similar to the unforced interval maps $\lambda f(x + a)$, only that the fixed points are replaced by invariant graphs. While for the pinched systems whose pictures are at the top row, even though the λ value is taken to be very big as in Figure 3.3(b), there is still only one invariant graph in the system.

Finally, we summarize briefly the character of the general dynamics of this forced model (3.15). Comparing with the family of unforced interval maps $\lambda f(x + a)$ and viewing the invariant graphs of forced systems as the fixed points of unforced interval maps, the basic features of this family are as follows.

- For the systems non-pinched, their dynamical behaviours are exactly the same with the unforced system $\lambda f(x+a)$, from the point of view of the bifurcations with respect to parameter λ. That is, pitchfork type for $a = 0$, and saddle-node for $a \neq 0$.
- In the pinched case, the dynamics is distinctive for $a \neq 0$, in which case the bifurcation is totally destroyed; if $a = 0$, the dynamics is the same as the interval maps with a pitchfork bifurcation, the particular point is that the invariant graphs other than $x = 0$ are strange.

The reason for the destruction of bifurcation is the pinched orbits, which can be seen clearly in the proof at the next subsection.

3.3.2 Proof of Theorem B

Theorem B consists of several assertions, we prove them one by one for the case $a > 0$ with a series of propositions below. For $a < 0$, the proof goes analogously with a change of sign and the exchange of concavity and convexity. So we omit it.

More precisely, first we prove the existence of the unique invariant graph $\Phi_\lambda \geq 0$ in any cases and its attraction in the invariant region $\mathbb{S}^1 \times [-a, +\infty)$. Next we show that, this graph Φ_λ is also the unique attracting invariant graph of the systems for both the pinched case and small enough λ. This is because that, in these two situations, all the points of the systems enter eventually the region $\mathbb{S}^1 \times [-a, +\infty)$. Finally, we prove that, there exists a bifurcation in the region $\mathbb{S}^1 \times (-\infty, -a)$ with the increasing of λ in the non-pinched case. This completes the proof of the whole theorem.

Existence of Φ_λ as attractor

One way for this proof is just to follow Keller. Take an upper bound M of all the fibre maps, and let the transfer operator on the function space act on $x = M$. This produces a decreasing sequence of continuous curves. Meanwhile the same action on $x = 0$ gives an increasing sequence. These two sequences have the same limit Φ_λ, which must be continuous.

Our investigations in this memoir involve neither the detailed regularity of the invariant curves, nor the estimation of the convergent speed of the points to invariant curves. These issues can refer to Keller [44] directly. Here we prove the existence of the invariant graph Φ_λ as the attractor by a straightforward method, using the contraction from the concavity.

First we prove a lemma which is derived from Corollary 3.2.12.

Lemma 3.3.3. *Take (θ_0, x_0) and (θ_0, y_0) for any $\theta_0 \in \mathbb{S}^1$, if both $x_0 \geq -a$ and $y_0 \geq -a$, then*

$$\lim_{n\to\infty} |x_n - y_n| = 0.$$

Proof. By Observation 3.2.2, if $-a \leq x_0 < 0$, then $0 \leq x_1$, and then $x_2 > 0$ if $g(\theta_1) > 0$. Thus we can start the following arguments from x_1 or x_2 if necessary.

If there is any n such that $g(\theta_n) = 0$, the result is true trivially because $x_{n+1} = y_{n+1} = 0$. Otherwise, $g(\theta_n) > 0$ for all $n \in \mathbb{N}$, then we have

$$|x_n - y_n| = \min\{x_n, y_n\}\kappa(x_n, y_n) = \min\{x_n, y_n\}\kappa(x_0, y_0)\prod_{k=0}^{n-1}\frac{\kappa(x_{k+1}, y_{k+1})}{\kappa(x_k, y_k)}.$$

Corollary 3.2.12 tells us that, each $\frac{\kappa(x_{k+1},y_{k+1})}{\kappa(x_k,y_k)}$ is smaller than a constant number which is smaller than 1. Hence, the products in the above equality goes to 0 as $n \to \infty$. □

Proposition 3.3.4. *There is a unique invariant graph Φ_λ in the region $\mathbb{S}^1 \times [0, +\infty)$, which is continuous and attracts all the points in the region $\mathbb{S}^1 \times [-a, +\infty)$.*

Proof. The above lemma says that, for all the initial values (θ_0, x_0) with $x_0 > -a$, they go into $\mathbb{S}^1 \times [0, +\infty)$ with one iterate. Moreover, there is only one limit point left in each fibre after iterations in $\mathbb{S}^1 \times [0, +\infty)$ eventually. So we obtain a function by corresponding each $\theta \in \mathbb{S}^1$ with the limit point left in this fibre, which is the Φ_λ that we claim.

By definition Φ_λ must be unique and attract all the points in $\mathbb{S}^1 \times [-a, +\infty)$. In view of the transfer operator, Φ_λ is also the only possible limit of $\mathcal{T}^n(x = -a)$ and $\mathcal{T}^n(x = c)$ with c any constant larger than the upper bound of all fibre maps. These two sequences of graphs are monotone increasing and decreasing respectively, so their common limit Φ_λ must be invariant and continuous. □

Remark 3.3.5. We point out two facts here.

(1) First, by the proof of this proposition, neither the existence of Φ_λ nor its properties of attraction and continuity depend on whether the system (3.15) is pinched or not.

(2) It can also be seen that f is not necessarily bounded. In fact, if there is big enough $c > 0$ which satisfies that $\mathcal{T}(x = c) \le c$, the arguments of our proof still work. This is indeed the case for a large number of unbounded maps which are increasing and α-concave.

$\diamond$

Bifurcation problem

The essential difference between the second and third items of our theorem is that, whether there exists a bifurcation at the negative part. Our theorem says that the bifurcation can occur at the negative part if and only if the system (3.15) is non-pinched.

In the following, we prove first that, Φ_λ is the unique invariant graph who attracts all the points in the pinched system, so there is no any bifurcation possible. The non-pinched case is considered after that.

Pinched case: bifurcation destroyed

Proposition 3.3.6. *If there is some $\theta_0 \in \mathbb{S}^1$ such that $g(\theta_0) = 0$, then all points $(\theta, x) \in \mathbb{S}^1 \times \mathbb{R}$ enter $\mathbb{S}^1 \times [-a, +\infty)$ eventually, hence Φ_λ is the only invariant graph and attracts all the points in the whole system.*

Proof. We just prove that all the points must enter the region $\mathbb{S}^1 \times [-a, +\infty)$ with this pinched condition, then Proposition 3.3.4 implies that all the points must be eventually attracted by Φ_λ.

Let M denote any upper bound of $|f|$. Since g is continuous and $g(\theta_0) = 0$, there must be an interval $J = (\theta_0 - \delta, \theta_0 + \delta) \in \mathbb{S}^1$ such that, for any $\theta \in J$, $g(\theta) < \frac{a}{2\lambda M}$. Then we have $-a < \frac{-a}{2} < \lambda f(x + a)g(\theta) < \frac{a}{2}$ for any $x \in \mathbb{R}$.

Due to the ergodicity of the irrational rotation orbit, all the orbits eventually enter $J \times \mathbb{R}$, and then enter $\mathbb{S}^1 \times [-a, +\infty)$ at the next iterate. $\square$

Non-pinched case: saddle-node bifurcation For the one-dimensional family $\lambda f(x + a)$ with $a > 0$, the dynamics of its negative part is a typical example of the saddle-node bifurcation. When the forced system is non-pinched, this saddle-node bifurcation is exact what happens at the negative part too. That is, for λ small enough, there is no invariant graph at the negative part, and all points enter $\mathbb{S}^1 \times [0, +\infty)$; with the increasing of λ

over a critical value, there occur two new invariant graphs in the system, one is attracting, the other is repelling.

The proofs of these results go in the following way: first we show that there is no invariant graph in $\mathbb{S}^1 \times (-\infty, 0]$ for small enough λ; next we prove that there are indeed two invariant graphs in this negative part if λ is big enough; and then we prove the facts that, if there are two invariant graphs in $\mathbb{S}^1 \times (-\infty, 0]$ at some value of λ, then this is the case for all bigger λ too; while for the case of no invariant graph at some value of λ, there is also no any invariant graph for all smaller λ; these facts together with the properties of invariant graphs in monotone system indicate the bifurcation can only happen at one value of λ.

First We give some notations that we need for the proofs. Recall that non-pinched case means $g(\theta) > 0$ for all $\theta \in \mathbb{S}^1$. We denote that $m_g = \min\{g(\theta)\}, M_g = \max\{g(\theta)\}$, both are well-defined since $\mathbb{S}^1$ is compact and g is continuous on it. Clearly $M_g \geq m_g > 0$. Let $l < 0$ be some lower bound of f, thus $l \leq f(x)$ for all $x \in \mathbb{R}$.

Now we begin our proofs. First we show that, if λ is small enough, for the same reason of the pinched case, all points enter $\mathbb{S}^1 \times [0, +\infty)$, then Φ_λ is the only invariant graph in the whole system.

Proposition 3.3.7. *If $\lambda < \frac{-a}{M_g l}$, all initial points go eventually into $\mathbb{S}^1 \times [-a, +\infty)$, thus Φ_λ is the only invariant graph and the attractor in the whole system.*

Proof. We show that for points with any initial value of x, their values of x are larger than $-a$ under one iterate. Let $\lambda < \frac{-a}{M_g l}$, which is equivalent to $\lambda l M_g > -a$. By definition of l and M_g, $\lambda f(x+a) g(\theta) \geq \lambda l M_g > -a$ as required. $\square$

Next, we show that there are two other invariant graphs in the region of $\mathbb{S}^1 \times (-\infty, -a)$ for λ big enough. Therefore there are two attracting invariant graphs and one repelling in the system simultaneously.

Proposition 3.3.8. *For any $c > a$, let $\lambda \geq \frac{-c}{m_g f(-c+a)}$, then there exist an invariant curve Ψ_λ with $\Psi_\lambda \leq -c$, which attracts all the points $(\theta, x) \in \mathbb{S}^1 \times (-\infty, -c)$. Moreover, there is a repelling invariant graph Γ_λ, which lies in $\mathbb{S}^1 \times (-c, -a)$.*

Proof. Notice that $\lambda \geq \frac{-c}{m_g f(-c+a)}$ is equivalent to $\lambda m_g f(-c+a) \leq -c$. Take any points on $x = -c$, they all have $\lambda f(-c+a) g(\theta) \leq \lambda m_g f(-c+a) \leq -c$. It implies that, in terms of transfer operator, we have $\mathcal{T}(-c) \leq -c$. Hence $\mathcal{T}^{n+1}(-c) \leq \mathcal{T}^n(-c)$ for all $n \geq 0$. We define $\Psi_\lambda = \lim_{n\to+\infty} \mathcal{T}^n(-c)$. Moreover, for every iterate, the fibre map satisfies the condition of (3.12), hence the convexity gives the contraction which makes Ψ_λ an attracting invariant graph. This implies that Ψ_λ must also be the limit of $T^n(l)$, which is monotone increasing. So Ψ_λ is continuous as the common limit of these two sequences.

Using the backward transfer operator on the graphs $\mathcal{T}(-c)$ and $x = -c$, since $\mathcal{T}(-c) \leq -c$ we have $\mathcal{R}(\mathcal{T}(-c)) = -c \leq \mathcal{R}(-c)$. Hence $\mathcal{R}(-c) \leq \mathcal{R}^2(-c)$, which follows then $\mathcal{R}^n(-c) \leq \mathcal{R}^{n+1}(-c)$ for all $n \geq 0$. Similarly, we also have $\mathcal{R}^{n+1}(-a) \leq \mathcal{R}^n(-a)$ for all $n \geq 0$. The limit of these two sequences must also be the same, which is just the repelling invariant graph Γ_λ.

This is because that, $\mathbb{S}^1 \times (-c, -a)$ is invariant under the backward iterate of F, and the fibre maps of backward iterate in this region are all β-concave, thus the same as the arguments of Remark 3.2.10 of the fixed interval map, we know that there is only one limit point in every fibre. Since this only limit point is attracting under backward iterate, it is repelling for the forward. □

Finally, we show that, these two new invariant graphs come from a bifurcation at some critical value of λ. This can be derived from the fact that, in such kind of systems, the invariant graphs also have some monotonicity with respect to the parameters.

Proposition 3.3.9. *If the invariant graphs Ψ_{λ_0} and Γ_{λ_0} in Proposition 3.3.8 exit for some parameter value λ_0, then the invariant graphs Ψ_λ and Γ_λ also exit for any $\lambda > \lambda_0$. Moreover, $\Psi_\lambda < \Psi_{\lambda_0}$ and $\Gamma_\lambda > \Gamma_{\lambda_0}$.*

Proof. Ψ_{λ_0} is invariant under the value λ_0 means it satisfies the invariant equation below,

$$(\mathcal{T}_{\lambda_0}\Psi_{\lambda_0})(\theta + \omega) = \lambda_0 f(\Psi_{\lambda_0}(\theta) + a)g(\theta) = \Psi_{\lambda_0}(\theta + \omega).$$

We know that $\Psi_{\lambda_0} < -a$, this implies $f(\Psi_{\lambda_0}(\theta) + a) < 0$. For any $\lambda > \lambda_0$, now in the system F_λ given by function $\lambda g(\theta) f(x + a)$, let the transfer operator act on the curve Ψ_{λ_0}, we have

$$(\mathcal{T}_\lambda\Psi_{\lambda_0})(\theta + \omega) = \lambda f(\Psi_{\lambda_0}(\theta) + a)g(\theta) < \lambda_0 f(\Psi_{\lambda_0}(\theta) + a)g(\theta) = \Psi_{\lambda_0}(\theta + \omega).$$

That is, $\mathcal{T}_\lambda\Psi_{\lambda_0} < \Psi_{\lambda_0}$, what follows is $\mathcal{T}_\lambda^{n+1}\Psi_{\lambda_0} \leq \mathcal{T}_\lambda^n\Psi_{\lambda_0}$ for all $n \geq 0$. Now define $\Psi_\lambda = \lim_{n\to+\infty} \mathcal{T}_\lambda^n\Psi_{\lambda_0}$, according to the proof of the above proposition, this limit Ψ_λ is just the invariant graph under the value λ.

Analogously, under the action of the backward operator $\mathcal{R}_\lambda$, it is easy to show $\mathcal{R}_\lambda\Gamma_{\lambda_0} > \Gamma_{\lambda_0}$. Define $\Gamma_\lambda = \lim_{n\to+\infty} \mathcal{R}_\lambda^n\Gamma_{\lambda_0}$ then the statement on invariant graphs Γ_λ follows also. □

Now the last fact we show is that, there is a critical value λ_0 of parameter λ at which a saddle-node bifurcation takes place with invariant graphs. Hence the general dynamical behaviours of the non-pinched systems are as follows.

Proposition 3.3.10. *For any fixed $a > 0$, there is a value λ_0 such that: when $\lambda < \lambda_0$, all the points of system are attracted by the only invariant graph $\Phi_\lambda > 0$; when $\lambda > \lambda_0$, there are two other invariant graphs occurring in the region $\mathbb{S}^1 \times (-\infty, -a)$. The lower one is attracting and is decreasing with respect to the increasing of λ; the upper one is repelling and is increasing with respect to the increasing of λ.*

Proof. The existence of some λ_c with two invariant graphs at the negative part is proved in Proposition 3.3.8, Proposition 3.3.9 shows that, such two graphs then exist and have monotonicity with respect to all $\lambda > \lambda_c$. Thus we can define by λ_l, the greatest lower bound of all values of λ which admit two invariant graphs in $\mathbb{S}^1 \times (-\infty, -a)$. That is, for any $\lambda \in (\lambda_l, +\infty)$ we have two invariant graphs in $\mathbb{S}^1 \times (-\infty, -a)$.

On the other hand, assume that λ_1 is some value for which all the points in $\mathbb{S}^1 \times (-\infty, -a)$ go eventually up to $x = -a$, that is, for any $(\theta, x) \in \mathbb{S}^1 \times (-\infty, -a)$, there exists a smallest $n \in N$ such that $(\theta_n, x_n)_{\lambda_1} = F^n_{\lambda_1}(\theta, x) \in \mathbb{S}^1 \times [a, +\infty)$. Notice that, for any $\lambda_2 < \lambda_1$, whenever $f(x + a) < 0$ we have $\lambda_2 f(x+a)g(\theta) > \lambda_1 f(x+a)g(\theta)$, which implies $(\theta_n, x_n)_{\lambda_2} = F^n_{\lambda_2}(\theta, x) \in \mathbb{S}^1 \times [a, +\infty)$ too. Hence these values of λ for which all the points in $\mathbb{S}^1 \times (-\infty, -a)$ go eventually up to $x = -a$ also form an interval $(0, \lambda_h)$. Next, we show that these two bounds can only be $\lambda_l = \lambda_h$, which indicates our claim.

In proof of Proposition 3.3.9, the monotonicity of the invariant graphs with respect to the parameter λ depends, in fact only on the forward or the backward transfer operator used to an invariant graph at $\mathbb{S}^1 \times (-\infty, -a)$. Precisely, if φ_{λ_0} is any invariant graph at $\mathbb{S}^1 \times (-\infty, -a)$ for some λ_0, then for any $\lambda > \lambda_0$ we all have $\mathcal{T}_\lambda \varphi_{\lambda_0} < \varphi_{\lambda_0}$ and $\mathcal{R}_\lambda \varphi_{\lambda_0} > \varphi_{\lambda_0}$. This mens that, whenever there is a value λ_0 such that there exists an invariant graph in this region, there must be two invariant graphs $\Psi_\lambda < \varphi_{\lambda_0}$ and $\Gamma_\lambda > \varphi_{\lambda_0}$ for all $\lambda > \lambda_0$. Hence $[\lambda_l, \lambda_h]$ can only consist of a single point, which is the critical value λ_0 of bifurcation. □

3.4 The second model and its dynamics

In this final section we investigate another forced monotonic family in form of (3.2) too, which is also generalized from Keller's model. The fibre maps of this family are kept to be concave or convex on whole $\mathbb{R}$, whose concrete conditions are given below. In Theorem C, we describe the complete dynamics of this family. We can see that the dynamics of non-pinched systems is also analogous graphs version of the unforced interval maps, but for the pinched cases there exist diverse situations this time.

As we have said, the family that we consider is also in the form of

$$F(\theta, x) = (\, \theta + \omega \bmod(1),\ \lambda f(x + a) \cdot g(\theta)\,), \tag{3.16}$$

with f strictly increasing on $\mathbb{R}$ and $g(\theta) \geq 0$ continuous. However, we remove the restriction of f being bounded, and let it be concave or convex on whole $\mathbb{R}$. Precisely, in (3.16) the map f is a real function which satisfies:

(1) f is continuous;

(2) f is strictly increasing on $\mathbb{R}$ and $f(0) = 0$;

(3) f is α-concave or β-convex on $\mathbb{R}$ for some $\alpha > 0$ or $\beta > 0$.

We take only the model of f being convex as example. The details of the dynamics of such families are in Theorem C below.

Theorem C. *In the case of $g(\theta) \neq 0$ almost everywhere, for any $\lambda > 0$ fixed, the family of* (3.16) *goes through a graph version saddle-node bifurcation with respect to the increasing of a. The details are the following.*

- *There is a critical value $a_0 \geq 0$ such that, if $a > a_0$, there is no any invariant graph and all points in the system go to infinity.*

- *For $a < a_0$, there exist two invariant graphs φ_a and ψ_a with $\varphi_a > \psi_a$. The upper one φ_a is repelling and all the points up to it go to infinity; the lower one ψ_a is attracting which attracts all the points below φ_a.*

  Particularly in the pinched system, φ_a is the graph of a measurable function which is not defined on a set of measure zero, unless $\varphi_a = 0$.

  Let a increase from $a < 0$ to a_0, φ_a goes down and ψ_a goes up with respect to it. Actually, it is $\varphi_{a_2} \geq \varphi_{a_1} > \psi_{a_1} \geq \psi_{a_2}$ for any $a_2 < a_1 < a_0$.

  The detailed dynamical behaviours in this process are:

  (1) when $a < 0$, $\varphi_a > 0$ and $\psi_a \leq 0$. Particularly, $\psi_a < 0$ if $g(\theta) > 0$ for all $\theta \in \mathbb{S}^1$;

  (2) at $a = 0$, either $\varphi_a = 0$ or $\psi_a = 0$;

  (3) when $0 < a < a_0$,

  (a) if $g(\theta) > 0$ for all $\theta \in \mathbb{S}^1$, either $\varphi_a > \psi_a > 0$ or $0 > \varphi_a > \psi_a$;

  (b) if $g(\theta) = 0$ on a zero measure set of $\mathbb{S}^1$, and

  i. if λ is relative small such that $\psi_a = 0$ at $a = 0$, then it is $\varphi_a > \psi_a \geq 0$ for $0 < a < a_0$;

  ii. if λ is big such that $\varphi_a = 0$ at $a = 0$, then $\psi_0 \leq 0$ is strange and nonchaotic who intersects $x = 0$ in a dense set of $\theta \in \mathbb{S}^1$, that is, it is an SNA.
  Moreover, $a_0 = 0$ for all such λ, hence there is no any invariant graph for $a > 0$.

In the case of $g(\theta) = 0$ on a positive measure set of $\theta \in \mathbb{S}^1$, no bifurcation happens. For any $\lambda > 0$ and a, there always exists a unique invariant graph, which is defined on a full measure set of $\theta \in \mathbb{S}^1$, and attracts almost all the points in the system.

Remark 3.4.1. The dynamics for α-concave case is totally analogous, with only the difference in appearance. Precisely, there is the same saddle-node bifurcation for a fixed λ with respect to the varying of parameter a. But in this case the critical value of a is $a_0 \leq 0$ and it is no invariant graph for $a < a_0$. When two invariant graphs exist for $a > a_0$, it is the attracting one above the repelling. The same as β-convex case, the repelling invariant graph partitions the phase space into two parts. All the points above it go to the attracting invariant graph monotonically; while the points below leave it monotonically decreasing to infinity. $\diamond$

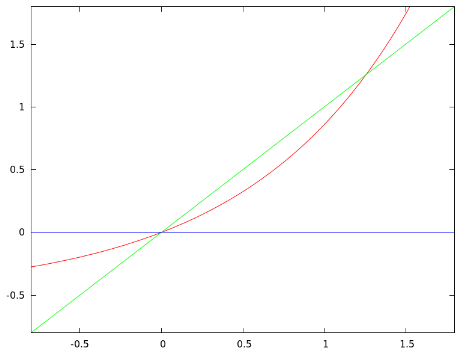
Once again as the previous model, in the non-pinched case, the dynamics of forced systems (3.16) can be viewed essentially the same with the family of unforced maps $\lambda f(x+a)$, so the forced terms do not affect the qualitative behaviours. We plot in Figure 3.5 and Figure 3.6 the attractors and repellors of some concrete examples of this kind of models, which correspond to the non-pinched and pinched systems respectively. Comparing with the dynamics of the unforced one-dimensional family $\lambda f(x + a)$ (refer to Figure 3.4), we can see this feature clearly. So before we start to prove Theorem C, we have a short look at the unforced interval maps $\lambda f(x + a)$, whose dynamics is the basis for analysing the corresponding forced systems.

Unforced interval maps $\lambda f(x + a)$

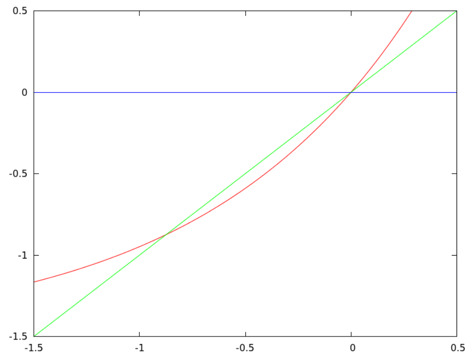
These convex increasing interval maps $\lambda f(x+a)$ are rather simple and well-understood already, which are usually used as typical models for the saddle-node bifurcation.

Generally, for a fixed λ, these exists a critical bifurcation value a_0. For $a > a_0$, there exists no any fixed point, and all the points go to $+\infty$ eventually. For $a < a_0$, there are two fixed points occurring in the system. The left (small) one is attracting, and the right (big) one is repelling. The attracting fixed point attracts all the points which are smaller than the repelling one, while all the points larger than the repelling one go eventually to $+\infty$.

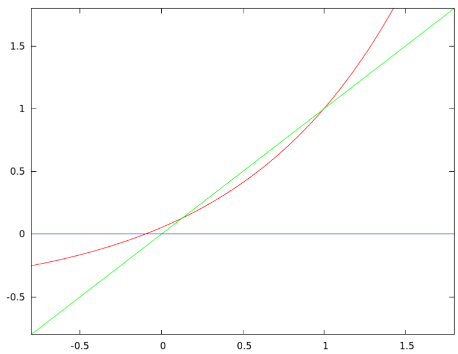
In Figure 3.4, we plot all the possible cases for such maps $\lambda f(x+a)$ with f increasing and $f(0) = 0$, except for those exactly at the bifurcations. Fix λ and let a increase from negative value, the behaviours of such maps go in the following way. When $a < 0$, since $\lambda f(x + a) < 0$ for point $x \leq 0$, so all its iterations remain to be negative. Particularly, if we iterate $x = 0$, we get a decreasing sequence. The contraction due to β-convexity at negative part implies that there is a unique limit point of this sequence, which must be a



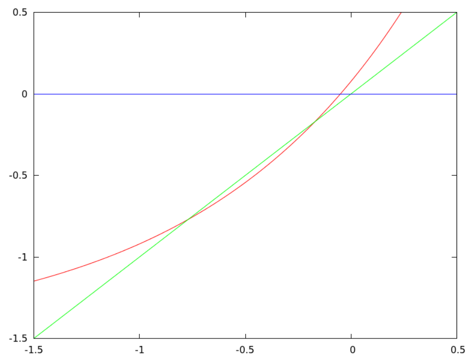
(a) $a = 0$ with λ small.



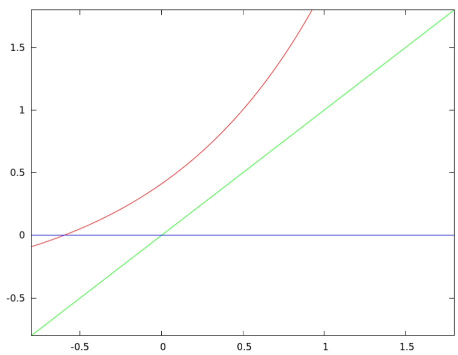
(b) $a = 0$ with λ big.



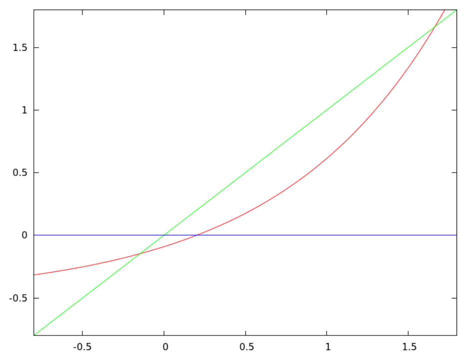
(c) $a > 0$ small with λ small.



(d) $a > 0$ small with λ big.



(e) $a > 0$ big enough.



(f) $a < 0$.

Figure 3.4: Graphs of $\lambda f(x + a)$ for different cases.

fixed point of the system. Consider the backward iterations of $x = 0$, which is equivalent to the forward iterations in a monotone increasing concave system, we have an increasing sequence whose limit point is another fixed point. So the attracting fixed point is negative, the repelling one is positive, they locate at two different sides of $x = 0$, which is shown in Figure 3.4(f).

With the increasing of a, the left fixed point moves to the right continuously, while the right one to left. When $a = 0$, $\lambda f(0 + a) = 0$, hence 0 is one of the fixed point. Whether $x = 0$ is attracting or repelling, it depends on the value of λ, which determines the derivative at $x = 0$ given by $\lambda f'(0)$. When λ is big enough such that $\lambda f'(0) > 1$, $x = 0$ is repelling, and the attracting fixed point at its left side is negative; for smaller λ with $\lambda f'(0) < 1$, $x = 0$ is attracting and the repelling fixed point is positive (see Figure 3.4(b) and Figure 3.4(a)).

When a goes to $a > 0$, both of the fixed points go to the same side of $x = 0$, which are positive for λ small or negative for λ big, depending on the cases at $a = 0$ above. At the last, they merge into one at the bifurcation point $a = a_0$. For $a > a_0$, all points in the system go to $+\infty$ as shown by Figure 3.4(e).

Proof of Theorem C

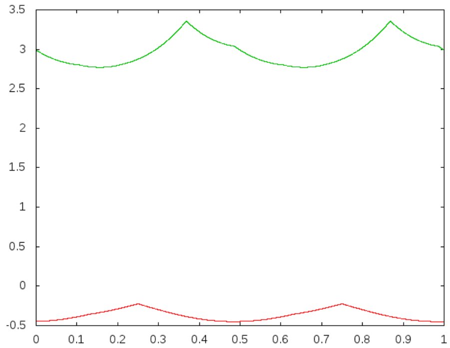
To prove Theorem C, we first discuss the conventional case of $g(\theta) \neq 0$ almost everywhere in $\mathbb{S}^1$, and leave the case of $g(\theta) = 0$ in positive measure set to the last of this section.

In this case, the behaviours are basically close to the unforced interval families $\lambda f(x + a)$, whatever the systems are pinched or non-pinched. So our arguments follow the similar way of the unforced interval systems above. In Figure 3.5, we take $\lambda g(\theta) f(x + a) = \lambda \left|1 + \cos(2\pi\theta)\right| (\exp(x + a) - 1)$ as an example, and plot the attractors and repellors for different cases of this model. The attractors are plotted in red, which are obtained by forward iterations with very small initial x value. The repellors are plotted in green, obtained by backward iterations with very big initial x value. From top to bottom, each row corresponds to a negative, zero and positive value of a before all points go to infinity. Concerning for the values of λ, it is taken as 0.3 for the left column, and 2.05 for the right. These pictures show intuitively the graph version of systems which are in same spirit of the unforced interval maps $\lambda f(x + a)$ (refer to Figure 3.4).

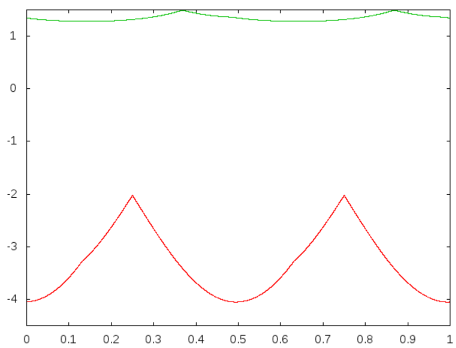
Next we present the details of this case.

Case of $g(\theta) \neq 0$ almost everywhere

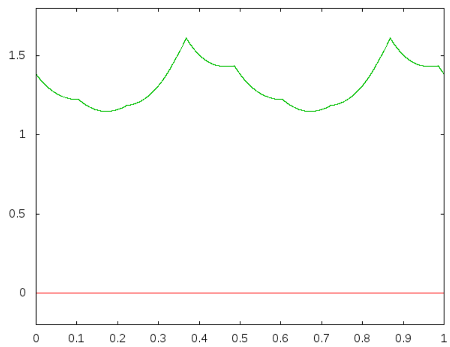
The properties due to the monotonicity of the systems are important ingredients of our arguments. Recall that the fibre maps are in form of



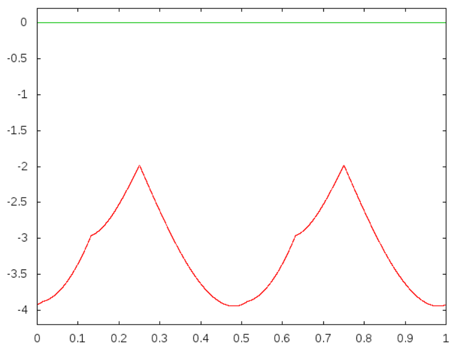
(a) $a = -1 < 0$ with $\lambda = 0.3$.



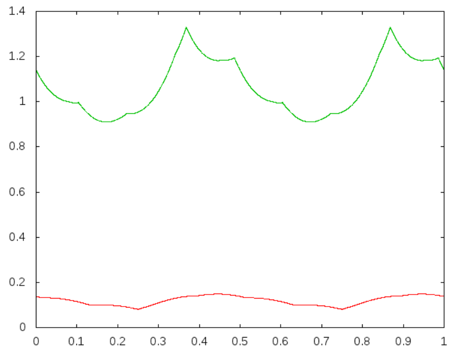
(b) $a = -1 < 0$ with $\lambda = 2.05$.



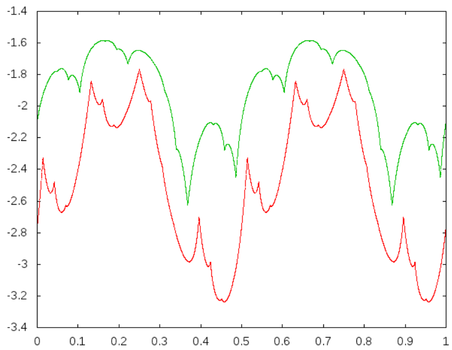
(c) $a = 0$ with $\lambda = 0.3$.



(d) $a = 0$ with $\lambda = 2.05$.



(e) $a = 0.1 > 0$ small with $\lambda = 0.3$.



(f) $a = 1 > 0$ small with $\lambda = 2.05$.

Figure 3.5: Graphs of different cases for the non-pinched system $\lambda g(\theta) f(x + a) = \lambda \left|1 + \cos(2\pi\theta)\right| (\exp(x + a) - 1)$. The red one is attracting and the green one is repelling.

$G(\theta, x) = \lambda f(x + a) \cdot g(\theta)$ with parameters λ and a. The transfer operator of such systems for a fixed a is given by

$$(\mathcal{T}_a\psi)(\theta) = \lambda f(\psi(\theta - \omega) + a) \cdot g(\theta - \omega), \tag{3.17}$$

where $\psi \in \mathcal{P}$ with $\mathcal{P}$ the space of all functions (not necessarily continuous) from $\mathbb{S}^1$ to $\mathbb{R}$. Let λ and a both be fixed, the monotonicity of the real map f yields the following results. For points (θ, x) and (θ, y) with $x \leq y$, denote $F^n(\theta, x)$ by (θ_n, x_n) and $F^n(\theta, y)$ by (θ_n, y_n). Trivially from the monotonicity of fibre maps, we have $G(\theta, x) \leq G(\theta, y)$, and hence the same order relation $x_n \leq y_n$ for all $n \in \mathbb{N}$. This is the same case for curves in $\mathcal{P}$, that is, for two graphs $\varphi \geq \psi$, $\mathcal{T}_a\varphi \geq \mathcal{T}_a\psi$ and also ${\mathcal{T}_a}^n\varphi \geq {\mathcal{T}_a}^n\psi$ for all $n \in \mathbb{N}$.

Moreover, the vertical Lyapunov exponent ν_φ of an invariant graph φ is given by

$$\begin{aligned} \nu_\varphi &= \int_{\mathbb{S}^1} \log\big(\lambda f'(\varphi(\theta) + a) \cdot g(\theta)\big)\mathrm{d}\theta \\ &= \int_{\mathbb{S}^1} \log g(\theta)\mathrm{d}\theta + \int_{\mathbb{S}^1} \log \lambda f'(\varphi(\theta))\mathrm{d}\theta. \end{aligned}$$

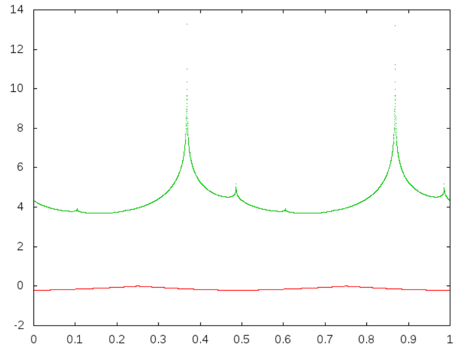
Therefore, if two invariant curves are $\varphi_a \geq \psi_a$, then $\nu_{\varphi_a} \geq \nu_{\psi_a}$ by the convexity of f.

Besides the monotonicity of the fibre maps themselves, there is also monotonicity with respect to the parameter a for such systems. Precisely, let λ be fixed, for $a_1 > a_2$, we have $\lambda f(x+a_1)g(\theta) \geq \lambda f(x+a_2)g(\theta)$ for any point (θ, x) by the monotonicity of f. This inequality follows directly from that f is strictly increasing and $\lambda g(\theta) \geq 0$. If we denote $(\theta_n, x_n^a) = F_a^n(\theta, x)$ for F given by $f(x+a)$ with parameter a, then it is easy to see that $x_n^{a_1} \geq x_n^{a_2}$ for all $n \in \mathbb{N}$ by induction. In terms of transfer operator, we also have $\mathcal{T}_{a_1}^n\psi \geq \mathcal{T}_{a_2}^n\psi$ for any graph $\psi \in \mathcal{P}$. The following lemmas are useful consequences of this monotonicity with respect to parameter a.

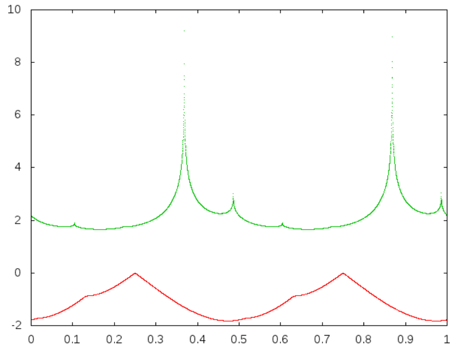
From now on in this section, we denote by ψ_a an attracting invariant graph which attracts (at least) all points below it, and by φ_a a repelling invariant graph such that all points above it go to $+\infty$, provided these graphs exist for a parameter a. Clearly, it must be $\varphi_a \geq \psi_a$ by assumption.

Lemma 3.4.2. *For parameters $a_1 > a_2$, assume that there exist invariant graphs ψ_{a_2}, φ_{a_2} and ψ_{a_1}, φ_{a_1} defined above, then we have $\psi_{a_1} \geq \psi_{a_2}$ and $\varphi_{a_1} \leq \varphi_{a_2}$. Actually, in the case of $g(\theta) > 0$, $\psi_{a_1}(\theta + \omega) > \psi_{a_2}(\theta + \omega)$ and $\varphi_{a_1}(\theta) < \varphi_{a_2}(\theta)$.*

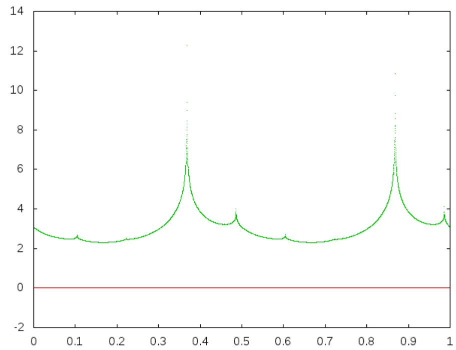
Proof. The proof is based on the fact that, for any $(\theta, x) \in \mathbb{S}^1 \times \mathbb{R}$, $x_n^{a_1} \geq x_n^{a_2}$ for all $n \in \mathbb{N}$.



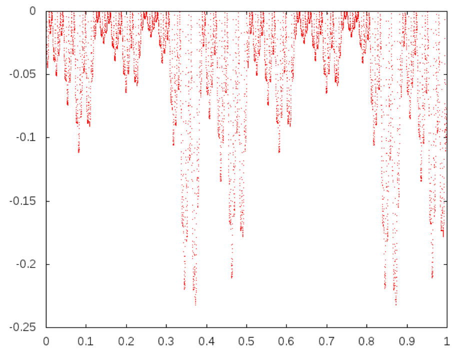
(a) $a = -1 < 0$ with $\lambda = 0.3$.



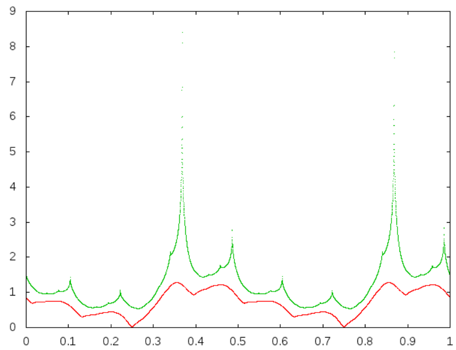
(b) $a = -1 < 0$ with $\lambda = 2.05$.



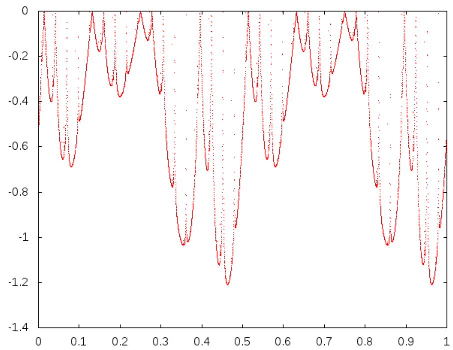
(c) $a = 0$ with $\lambda = 0.3$.



(d) $a = 0$ with $\lambda = 2.05$.



(e) $a = 0.9 > 0$ small with $\lambda = 0.3$.



(f) $a = 0$ small with $\lambda = 2.5$.

Figure 3.6: Graphs of different cases for the pinched system $\lambda g(\theta)f(x+a) = \lambda\,|\cos(2\pi\theta)|\,(\exp(x+a)-1)$. The red one is attracting and the green one is repelling.

To see that $\psi_{a_1} \geq \psi_{a_2}$, we consider an arbitrary point below ψ_{a_2}, that is, point (θ, x) such that $x < \psi_{a_2}(\theta)$. Because $x_n^{a_1} \geq x_n^{a_2}$ and $(\theta_n, x_n^{a_i})$ converges to ψ_{a_i} respectively, it must be $\psi_{a_1} \geq \psi_{a_2}$.

On the other hand, for any point (θ, x) above φ_{a_2}, we have $x_n^{a_1} \to +\infty$, because that $x_n^{a_1} \geq x_n^{a_2}$ and $x_n^{a_2} \to +\infty$. This implies $\varphi_{a_1} \leq \varphi_{a_2}$ by the definition of φ_a.

Next, we show that these inequalities are strict if $g(\theta) > 0$. Notice that $\psi_a(\theta+\omega) = \lambda f(\psi_a(\theta)+a) \cdot g(\theta)$. If $\psi_{a_1}(\theta) \geq \psi_{a_2}(\theta)$, then $f(\psi_{a_1}(\theta)+a_1) > f(\psi_{a_2}(\theta)+a_2)$ by $a_1 > a_2$. Hence, when $g(\theta) > 0$, it must be $\psi_{a_1}(\theta+\omega) > \psi_{a_2}(\theta+\omega)$.

Finally, if there is $\varphi_{a_1}(\theta) = \varphi_{a_2}(\theta)$ at some $\theta \in \mathbb{S}^1$, then we have $\varphi_{a_1}(\theta+\omega) > \varphi_{a_2}(\theta+\omega)$, which contradicts with $\varphi_{a_1} \leq \varphi_{a_2}$. Hence it can only be $\varphi_{a_1}(\theta) < \varphi_{a_2}(\theta)$. □

Lemma 3.4.3. *If there is an a_0 such that, ψ_{a_0} and φ_{a_0} are lower and upper semicontinuous graphs respectively, and $\psi_{a_0}(\theta_0) = \varphi_{a_0}(\theta_0)$ at some $\theta_0 \in \mathbb{S}^1$, then all points in the system F_a go to $+\infty$ for any $a > a_0$.*

Proof. By the definition of ψ_a and φ_a, there are two kinds of behaviours for all the points in the system F_{a_0} given by this parameter a_0. The first kind is for all the point (θ, x) above φ_{a_0}, who goes to $+\infty$. For any such point (θ, x), we know that $x_n^a \geq x_n^{a_0}$ for any $a > a_0$, hence it goes to $+\infty$ in any system F_a with parameter $a > a_0$.

Points of the second kind are those below φ_{a_0}, that is, (θ, x) with $x \leq \varphi_{a_0}(\theta)$. For such point, its limit set contains all the intersection points of ψ_{a_0} and φ_{a_0}, since ψ_{a_0} by definition is attracting at least from below. Next we show that, we can find a neighbourhood of some intersection point (which we denote also by $(\theta_0, \psi_{a_0}(\theta_0)) = (\theta_0, \varphi_{a_0}(\theta)))$, such that all points of the second kind enter this neighbourhood under F_{a_0}, and are mapped above φ_{a_0} by F_a for a given $a > a_0$. Then all points in system F_a must go to $+\infty$.

First, by Proposition 3.1.1, the existence of $\psi_{a_0}(\theta_0) = \varphi_{a_0}(\theta_0)$ implies that, the region enclosed by ψ_{a_0} and φ_{a_0} is invariant and pinched. Moreover, all the points $\theta \in \mathbb{S}^1$ such that $\psi_{a_0}(\theta) = \varphi_{a_0}(\theta)$ form a residual subset of $\mathbb{S}^1$, and both ψ_{a_0} and φ_{a_0} are continuous at these $\theta \in \mathbb{S}^1$. In this residual set, we can find a $\theta_0 \in \mathbb{S}^1$ at which $g(\theta_0) \neq 0$. This is because that, if $g(\theta) = 0$ at all this residual set, then the continuity of g implies that $g \equiv 0$, which contradicts with our assumption of $g \neq 0$ almost everywhere.

Now for any $a > a_0$, we know from the previous lemma that, at θ_0 such that $g(\theta_0) \neq 0$, $\psi_a(\theta_0+\omega) > \psi_{a_0}(\theta_0+\omega) = \varphi_{a_0}(\theta_0+\omega)$. The latter equality is due to the invariance of ψ_{a_0} and φ_{a_0}. Take $\varepsilon = \frac{1}{2}(\psi_a(\theta_0+\omega) - \varphi_{a_0}(\theta_0+\omega))$, since φ_{a_0} is upper semicontinuous, we can find a neighbourhood $\Delta \subseteq \mathbb{S}^1$ of $\theta_0 + \omega$, such that for every $\theta \in \Delta$, $\varphi_{a_0}(\theta) < \varphi_{a_0}(\theta_0+\omega) + \varepsilon$. Denote $\Delta_1 = \Delta \times (\psi_a(\theta_0+\omega) - \varepsilon, +\infty)$. All the points in Δ_1 are above φ_{a_0}, therefore they all go to $+\infty$ in the system F_{a_0}, and hence also go the same way in system F_a by $a > a_0$.

Finally we show that, all these second kind of points in F_{a_0} must enter Δ_1 under the action of F_a with $a > a_0$, then this lemme is proved. Notice that, the continuity of F_a implies that, there exist a neighbourhood Δ_2 of $(\theta_0, \psi_{a_0}(\theta_0))$ such that all points here are mapped to Δ_1. Since $(\theta_0, \psi_{a_0}(\theta_0))$ is a limit point of all the orbits of the second kind of point, which implies that all these points in system F_{a_0} enter Δ_2 at some iterates. Therefore, if $(\theta_n, x_n^{a_0}) \in \Delta_2$, it must be $(\theta_{n+1}, x_{n+1}^a) \in \Delta_1$, because $x_n^a \geq x_n^{a_0}$ for all $n \in \mathbb{N}$. $\square$

By the lemmas above we have proved that, if there exist two invariant graphs in the system with the attracting and repelling properties that we assume, then they must get close monotonically with the increasing of parameter a. Particularly, in the non-pinched case they do so strictly monotonically. The second lemma implies that, whenever these two invariant graphs intersect at some a_0, this a_0 is then the critical value of bifurcation. Both the invariant graphs disappear if a goes over it. Next we show that both this critical value a_0 and the two invariant graphs for $a < a_0$ do exist, which are as we claim in Theorem C.

To avoid duplication, we try to show the existence of invariant graphs with the results of the previous model, whenever we can do so.

Recall in Proposition 3.3.4, the key point for proving the existence of a unique invariant graph is that, the region $\mathbb{S}^1 \times [0, +\infty)$ is invariant for the system $F = \lambda f(x)g(\theta)$ with map f increasing, α-concave for some $\alpha > 0$, and $f(0) > 0$. In this case, Lemma 3.3.3 ensures that

$$\lim_{n\to\infty} |x_n - y_n| = 0,$$

for any two points (θ_0, x_0) and (θ_0, y_0) in the same fibre with $x_0 \geq 0$ and $y_0 \geq 0$. We refer this case as *Case A* below.

Analogously, there exists also a unique invariant graph which is attracting in *Case B*, which is that the region $\mathbb{S}^1 \times (-\infty, 0]$ is invariant for the system $F = \lambda f(x)g(\theta)$ with map f increasing, β-convex for some $\beta > 0$, and $f(0) < 0$.

Notice that, the backward iteration of the current model (3.16) is equivalent to the forward iteration of a monotone increasing concave one, and the attracting invariant graph of backward iteration is just the repelling one under forward iteration. We will frequently use these facts in the following discussions.

Case $a < 0$ The pictures of the attractors and repellors for this case are plotted in Figure 3.5(a), 3.5(b), 3.6(a) and 3.6(b), which correspond the non-pinched and pinched cases at different values of parameter λ. In all these situations, there is a unique repelling invariant graph $\varphi_a > 0$, together with a unique attracting graph $\psi_a \leq 0$.

The dynamics in this case can be converted into two parts, one is just in Case A we give above, the other is in Case B. This is done by considering the forward iterations at the negative part and the backward iterations at positive part respectively. Notice that, all the graphs of fibre maps in this case have the same shape shown in Figure 3.4(f).

Precisely, $a < 0$ implies $\lambda g(\theta) f(0 + a) < 0$ for any $(\theta, 0)$, hence all the points (θ, x) with $x \leq 0$ always remain at the part of $\mathbb{S}^1 \times (-\infty, 0]$. The maps $f(x+a)$ at the part $\mathbb{S}^1 \times (-\infty, 0]$ are increasing, β-convex, and $f(0+a) < 0$. Thus for this region it is in Case B, there exists a unique invariant graph $\psi_a \leq 0$ which attracts all points of $\mathbb{S}^1 \times (-\infty, 0]$.

If we consider the backward iteration F^{-1} starting from $x = 0$, then all the iterates stay inside the part of $\mathbb{S}^1 \times (0, +\infty)$, that is, $\mathbb{S}^1 \times (0, +\infty)$ is invariant for F^{-1}. Notice that, in this case f^{-1} is increasing, β-concave, and $f^{-1}(0) > 0$. Thus this is in Case A for F^{-1}, there is a unique invariant graph $\varphi_a > 0$ which is attracting for F^{-1}. Therefore in system (3.16) given by F, this $\varphi_a > 0$ is repelling and the unique invariant graph in the positive part , $\mathbb{S}^1 \times (0, +\infty)$.

For the backward iteration F^{-1}, a particularly situation occurs for pinched systems. In this case, we cannot define such operation on all the fibres. For those fibres which are preimages of the system pinched points, all their points are mapped to the attracting graph ψ_a and the repelling graph φ_a has no values there. Because that $g(\theta) = 0$ only in a zero measure set, the set of its preimages also has zero measure. On the fibres which is not in this set, the existence of the values of the repelling graph is guaranteed by the contraction from the concavity of every iterate, which is proved in Lemma 3.3.3. Therefore, for the pinched systems, their repelling graphs do have values in a full measure set. This is also the case when $a \geq 0$, unless the repelling graph is $\varphi_0 = 0$.

Case $a = 0$ If $a = 0$, $x = 0$ must be an invariant graph since $f(0 + a) = f(0) = 0$. Whether it is attracting or repelling depends on the value of its vertical Lyapunov exponent, or to say, the value of the parameter λ. Since the vertical Lyapunov exponent ν_φ of an invariant graph φ is defined by

$$\nu_\varphi = \int_{\mathbb{S}^1} \log\big(\lambda f'(\varphi(\theta) + a) \cdot g(\theta)\big) \mathrm{d}\theta.$$

At invariant graph $x = 0$, it is

$$\nu_{x=0} = \int_{\mathbb{S}^1} \log\big(\lambda f'(0) \cdot g(\theta)\big) \mathrm{d}\theta.$$

For any g given, the value of $\nu_{x=0}$ is decided by the value of λ. If λ is small enough such that $\nu_{x=0} < 0$, then $x = 0$ is attracting. It is repelling if

$\nu_{x=0} > 0$. However, to see more complete details of the dynamics of whole system in each these two cases, we still need to refer to the fibre maps, which we plot in Figure 3.4(a) and Figure 3.4(b).

Similar with the previous case $a < 0$, we also consider the forward iteration of F at negative part $\mathbb{S}^1 \times (-\infty, 0]$ and the backward iteration of F^{-1} at positive part $\mathbb{S}^1 \times [0, +\infty)$. In this case of $a = 0$, each part forms a subsystems itself, which is Keller's model.

We know that a negative vertical Lyapunov exponent of forward iteration is positive for backward, and vice versa. Hence, if $\nu_{x=0} < 0$, $x = 0$ is the unique invariant graph in $\mathbb{S}^1 \times (-\infty, 0]$, which attracts all the points in this region; meanwhile under backward iteration in $\mathbb{S}^1 \times [0, +\infty)$, $x = 0$ must be the repelling one. Furthermore, by Keller's Theorem, there is uniquely another graph which is attracting in $\mathbb{S}^1 \times [0, +\infty)$ under the backward iteration. Therefore, for forward iteration in this part $\mathbb{S}^1 \times [0, +\infty)$, there exist two invariant graphs that $x = 0$ is attracting and $\varphi_0 > 0$ is repelling.

When $\nu_{x=0} > 0$ with large enough λ, the situations of negative and positive parts just exchange as in case of $\nu_{x=0} < 0$ above. See the first row of Figure 3.4 for the fibre maps and the pictures in the middle row of Figure 3.5 and Figure 3.6 for invariant graphs in forced systems. So in this case, we have that $x = 0$ is the unique repelling invariant graph and there is another one in $\mathbb{S}^1 \times (-\infty, 0]$ which is attracting.

Notice that, if the system is pinched, the lower attracting invariant graph is strange in this case. We plot in Figure 3.6(f) with more large λ for a better view. Moreover for any $a > 0$, Lemma 3.4.3 implies that there exists no any invariant graph and all points in the system go to $+\infty$ eventually. So the critical value a_0 of bifurcation in this case are all at $a_0 = 0$.

The particular case is $\nu_{x=0} = 0$ at a λ_0. Keller's Theorem says $x = 0$ is the unique and attracting invariant graph of both the forward and backward iteration in each of the two parts respectively. This means that it is the only invariant graph in the system, which attracts all the points below it and repels the points above it to $+\infty$. We view this case as that the attracting and repelling graphs coincide, by Lemma 3.4.3 we know that $a = 0$ is also the critical bifurcation value for this λ_0.

Case $a > 0$ The monotonicity with respect to the parameter a implies that all the invariant graphs are now at the same side of $x = 0$ when $a > 0$, except for the situations that the critical value of bifurcation is $a_0 = 0$ which we point out above.

For each part that the invariant graphs locate in, its dynamics now goes into either the case of Proposition 3.3.8 or its analogous convex situation, where we prove that there are two invariant graphs. Lemma 3.4.2 shows that the two invariant graphs get more and more closer with the increasing of a. We also know from Lemma 3.4.3 that, when two invariant graphs intersect at some $a_0 > 0$, there exists no any invariant graph for $a > a_0$. All these facts lead to the similar arguments of Proposition 3.3.10, which shows that a saddle-node bifurcations take place at some critical value a_0.

Positive measure for zeros of g

This last case is rather special which owns the distinctive behaviour of forced systems, that is, there exists a unique invariant graph which is attracting for any parameters value. So there is no bifurcation like the unforced interval maps f and the common forced systems (3.16) in the previous case.

Let the set $P = \{\theta \in \mathbb{S}^1 : g(\theta) = 0\}$, and denote its Lebesgue measure by $m(P)$. For the union set of preimages of P under the irrational rotation $R(\theta)$, we have known that the repelling invariant graph is not defined on it. When $m(P) = 0$, this union set is also a zero measure set, so it is possible for the existence of a repelling invariant graph almost everywhere, as we have seen in the previous case. If $m(P) > 0$, this union of preimages of P is a full measure set, therefore there is no repelling invariant graph in the system.

As for the attracting invariant graph, we have the following.

Proposition 3.4.4. *In case of $m(P) > 0$, for any λ and a, there always exists a unique invariant graph in the system, which attracts almost all the points in $\mathbb{S}^1 \times \mathbb{R}$.*

Proof. Let $A = \cup_{i=1}^{+\infty} R^i(P)$, we know that $m(A) = 1$ due to the irrational rotation $R(\theta) = \theta + \omega$. For any $\theta \in A$, it must be $\theta = R^n(\bar{\theta})$ with some $\bar{\theta} \in P$ and $n \geq 1$. Denote $(\bar{\theta}_n, x_n) = F^n(\bar{\theta}, x) = F^{n-1}(\bar{\theta}_1, 0)$, then we can define a function ψ on this full measure set A by $\psi(\theta) = x_n$, the graph of this function ψ is just the unique invariant graph that we claim.

First we show that ψ is well-defined in this way, that is, for any $\theta \in A$, the value $\psi(\theta)$ is unique. Assume that there exist $k \geq 1$ and $\tilde{\theta} \in P$ such that $\theta = R^k(\tilde{\theta})$, at the same time with some $n \neq k$ that, $\theta = R^n(\bar{\theta})$ with some $\bar{\theta} \in P$ and $n \geq 1$ too. We show that for any $(\tilde{\theta}, y)$ and $(\bar{\theta}, x)$, it must be $y_k = x_n$, which means that ψ is well-defined.

Without loss of generality, assume that $k > n$, then it must be $\bar{\theta} = R^{k-n}(\tilde{\theta})$. Hence we have $F^k(\tilde{\theta}, y) = F^n(\tilde{\theta}_{k-n}, y_{k-n}) = F^{n-1}(\bar{\theta}_1, 0)$ for any y and x.

Clearly from the definition of ψ, it is an invariant graph. Moreover, all the points in fibres of $\theta \in \cup_{i=0}^{+\infty} R^{-i}(P)$ are mapped into it, they form a full measure set of the system since $m\big(\cup_{i=0}^{+\infty} R^{-i}(P)\big) = 1$. We also notice that $\psi(\theta) = 0$ for any $\theta \in R(P)$, so its vertical Lyapunov exponent must be negative since $m(R(P)) = m(P) > 0$. ☐

Finally, we point out one special fact on the problem of finding SNAs.

For the case of $g(\theta) = 0$ on a zero measure set in this model, if we regard a as the varying parameter with λ being fixed, then the bifurcations are all of saddle-node type. They do result in SNAs when λ is big. On the other hand, when we treat this example as λ increasing with $a = 0$ fixed, the bifurcation is a transcritical one. So we have all the examples of these regular types of bifurcations in one-dimensional systems, namely, the pitchfork, saddle-node, transcritical, and period-doubling ones, each one can give the birth of an SNA in a pinched system.

However, although this is possible for any one of these types of bifurcations, one cannot just expect such a result by resembling simply a bifurcation of the real map to the pinched system. We have to consider that, how the pinched condition is acted on the system. A bifurcation can happen only when the pinched orbits locate appropriately in a relative special position. The Keller's model is one example. Also the bifurcations result in SNAs in this model. They all happen at some $a \neq 0$ in both the one-dimensional case and the non-pinched ones, but are moved to $a = 0$ with the intersection of two graphs, which is due to the pinched condition.

Chapter 4

Attractors of quasi-periodically forced S-unimodal maps

We discuss two subjects in this chapter. One is the reverse bifurcations of S-unimodal maps, which take place on the attractors of cycles of chaotic intervals. We explain their detailed reason and show their correspondence to the bifurcations of periodic orbits. The bifurcation theories of periodic orbits are well developed and very classic, however the bifurcations of cycles of chaotic intervals can only be found in physical context up to now. In Theorem D, the first main result of this chapter, we give the mathematical description of such phenomena, which shows that each of them is the reverse of a bifurcation of periodic orbit. The crucial concept for this theorem is the block structures of restrictive intervals of S-unimodal maps. We display that a set of restrictive intervals occurs together with a periodic orbit, and the reverse bifurcation takes place when it changes to be non-restrictive.

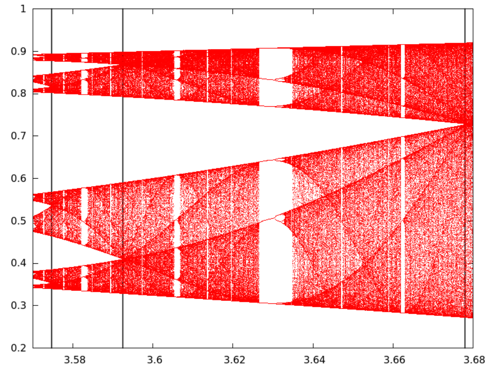
The other subject is the transition of attractors in quasi-periodically forced S-unimodal systems, which is also based on the block structures of restrictive intervals of S-unimodal maps. We propose the general mechanism of the transition of attractors with respect to a increasing parameter of forcing term on a fixed S-unimodal map, which is well verified with numerical evidences.

This chapter has four sections. In the first one we give a short introduction of the reverse bifurcation phenomena to make our statements above more intuitively and more definitely. We display two concrete examples first, and then exhibit briefly the concept of restrictive intervals and the mechanism of reverse bifurcations. Formal explorations start from the second section, which is devoted to the definition and properties of restrictive intervals of S-unimodal maps, their representation via extension pattern, and the characterization of the topological attractors by these intervals. With these knowledge, in the third section we discuss precisely the reverse bifurcations of S-unimodal maps. Theorem D is presented and proved in the first subsection. By viewing each pair of corresponding bifurcations, we can also obtain an integrated perspective on the transition and self-similarity of attractors in a full S-unimodal family, that we shortly explore in the second subsection. The other main result is given in the fourth section, where we demonstrate the general mechanism of the transitions of attractors in quasi-periodically forced S-unimodal maps. This mechanism displays how the periodicity changes with the increasing of the perturbation on a fixed S-unimodal map.

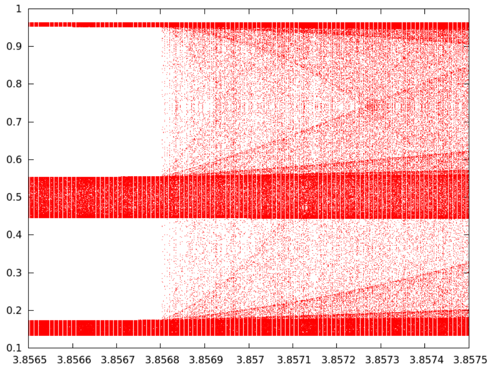
4.1 Introduction of reverse bifurcations

In this section we exhibit the reverse bifurcations of S-unimodal maps, and introduce briefly the idea of their treatments and the concepts involved. Such bifurcations happen on the attractors in form of cycles of intervals, that we first get acquainted with their phenomena intuitively below. Then we summarize the formal arguments that we develop in the following sections, they can show that how each bifurcation of this type corresponds to a classical bifurcation of periodic orbit and hence can be regarded as its reverse. Furthermore, they are also the basis for understanding the corresponding mechanism of the periodicity of attractors in quasi-periodically forced systems.

The reverse bifurcations of S-unimodal maps are reported for decades, but few mathematical treatments are known yet. The S-unimodal maps are unimodal maps with negative Schwarzian derivative in every point except for the critical point c. By unimodal maps we mean continuous interval maps $f : [a, b] \to [a, b]$ who have only one extreme c on $[a, b]$ and map the endpoints to one of them. A prototype of family of S-unimodal maps is given by the popular logistic family $f_\mu(x) = \mu x(1 - x)$ where μ is a parameter. The classification of topological attractors of S-unimodal maps and the bifurcations of periodic orbits are all very classic results in the field of dynamical systems on intervals. It is known that (see [32, 52]), for an S-unimodal map, there exists only one topological attractor which belongs to one of the following three types: an attracting periodic orbit, a solenoidal (Feigenbaum-like) attractor, or a cycle of intervals. In the transition of an S-unimodal family like f_μ, attractors of these three types alternate in a certain order. The solenoidal attractors are not generic in the sense that they occur only for a zero measure set of parameter value μ. For the other two types, it is well-known that a new attracting periodic orbit comes out of a bifurcation. There are two generic types of such bifurcations which are period-doubling and saddle-node one respectively. The period-doubling cascade to chaos is one of the most important result in chaotic dynamical



(a) Attractor diagram of $f(x) = \mu x(1-x)$ for $\mu \in [3.57, 3.68]$. The attractor changes from 8 bands to 4, 4 bands to 2, and 2 bands into 1 at about 3.5747, 3.5924 and 3.678 respectively, where we marked by vertical lines.



(b) Attractor diagram of $f(x) = \mu x(1-x)$ for $\mu \in [3.8565, 3.8575]$. At the place a little larger than 3.8658, the attractor changes from 3 bands directly into a one piece band. It is the reserve of saddle-node bifurcation.

Figure 4.1: Examples of band merging by reverse bifurcations.

systems. After the cascade, chaotic attractors in form of cycles of intervals appear. When the attractor is a cycle of n intervals, each of the interval is a chaotic subsystem under f^n with sensitive dependence on initial conditions, hence the attractor appears as n bands. Relative to the developed theory on bifurcations of periodic orbits, the bifurcations of cycles of intervals can only be found in physical context, we review shortly now.

There are two types of bifurcations of chaotic interval attractors. In 1980, Lorenz published a paper [51], where he found a series of procedure which he thought to be reverse of the period-doubling cascade in a one-parameter unimodal family. The phenomena are: after the period-doubling cascade to the Feigenbaum attractor, there exists a series of μ_i values for $i = \dots n, n-1, \dots 1$, such that 2^i bands merge into 2^{i-1} bands at μ_i, with two adjacent bands meeting at a common endpoint and then becoming one. See Figure 4.1(a) for these reverse bifurcations of the logistic family.

Another type of bands merging was found by Grebogi et al [30], which happens when the chaotic three-bands attractor touches the unstable period 3 orbit born from a saddle-node bifurcation. It results the chaotic attractor in three narrow bands into a chaotic attractor filling a entire interval, see Figure 4.1(b). This bifurcation is called an interior crisis in physical context, which occurs at a precise parameter value that marks a discontinuous jump in size of the chaotic attractor as in Figure 4.1(b). Such jumps or explosions are typical and common in nonlinear dynamics, for example, phenomenon in the forced Duffing oscillator by Ueda [75], and also the quasi-periodically forced S-unimodal maps we will study later in this chapter. It is observed that these explosions always involve collisions between attractors and unstable periodic motions or their insets, which are basin boundaries.

Actually, these two types of reverse bifurcations are treated totally in consentaneous way for the S-unimodal maps by us. In Theorem D we explain mathematically the mechanism of these reverse bifurcations, which shows that each of them just corresponds to one bifurcation of periodic orbit in an S-unimodal family. Their correspondence are displayed by the restrictive intervals of unimodal maps.

Briefly saying, for any point $p \in [a, b]$, denote by p' the unique point such that $f(p) = f(p')$ for a unimodal map f. For $\{p_0, p_1, \ldots, p_{n-1}\}$ a periodic orbit of period n of f, its central point p_0 is the only one that no any other p_i of this orbit can belong to $[p_0, p_0']$, the central interval. Associated to this orbit, there exists a unique set of K intervals $[p_i, q_i]$ (or $[q_i, p_i]$) for $0 \leq i < K$ with $K = n$ or $2n$, which satisfies the following properties: $q_0 = p_0'$; $f^K(p_i) = f^K(q_i)$; $(p_j, q_j) \cap (p_i, q_i) = \emptyset$ if $i \neq j$; and $f([p_i, q_i]) = [p_{i+1}, q_{i+1}]$ for all $0 < i < K$. These K intervals are called to be restrictive if $f^K([p_i, q_i]) \subseteq [p_1, q_1]$, which is equivalent to that $\cup_{i=0}^{K}[p_i, q_i]$ is forward invariant under f, and also that every $[p_i, q_i]$ is periodic of period K (that is, $f^K([p_i, q_i]) \subseteq [p_i, q_i]$ for all $0 \leq i < K$).

If p_0 and q_0 are two central points of different periodic orbits, and both sets of the intervals linked to their orbits are restrictive, then we have either $p_0 \in [q_0, q_0']$ or $q_0 \in [p_0, p_0']$. Trivially by invariance, the restrictive intervals of the inner one are also all contained in the restrictive intervals of the outer, this means that the intersection of all the restrictive intervals of a unimodal map f forms a nested and invariant set. If f is in addition S-unimodal, then all these intersection contains the only attractor of system. Moreover, the situations of restrictive intervals also characterize the types of attractors of S-unimodal maps: the restrictive intervals of an S-unimodal map f are infinitely many, if and only if its attractor is solenoidal; the restrictive intervals are finitely many, and there is no other periodic point inside their intersection, if and only if the attractor is a periodic orbit; otherwise, there are other periodic points inside the intersection of all the finitely many restrictive intervals, if and only if the attractor is a cycle of chaotic intervals.

To describe the structures of these restrictive intervals and the attractor contained inside them, extension pattern is a natural language.

A pattern θ represented by a single periodic orbit $\mathcal{O}(p)$ is a cyclic permutation of $\mathcal{O}(p)$. If $(\mathcal{O}(p), \theta)$ is a pattern of cycle of period n and $(\mathcal{B}, \gamma)$ is a pattern of period m. Let $\mathcal{O}(p) = \{p_1 < p_2 < \cdots < p_n\}$ and $\mathcal{B} = \{i_1, i_2, \ldots, i_m\}$. Then $(\mathcal{O}(p), \theta)$ has a block structure over $\mathcal{B}$ provided that, $n = sm$, $\mathcal{O}(p) = P_1 \cup P_2 \cup \ldots \cup P_m$ with $P_i = \{p_{(i-1)s+1}, p_{(i-1)s+2}, \ldots, p_{(i-1)s+s}\}$ for all $i = 1, 2, \ldots, m$; $\theta(P_{i_j}) = P_{i_{j+1}}$ for all $j = 1, 2, \ldots, m-1$ and $\theta(P_{i_m}) = P_{i_1}$.

Thus each of the sets P_i is a block of $\mathcal{O}(p)$. We can view each block as a "fat" point and $\mathcal{O}(p)$ as a "fat" cycle with the pattern $(\mathcal{B}, \gamma)$. In this case, we say, by an abuse of notation that, $\mathcal{O}(p)$ is an extension pattern of $\mathcal{B}$ which has a block structure over it.

For p and q two central points with $q \in (p, p')$, denote by $\mathcal{O}([p, p'])$ and $\mathcal{O}([q, q'])$ the patterns of the orbits of their restrictive intervals respectively. Both $\mathcal{O}(q)$ and $\mathcal{O}([q, q'])$ are extension patterns of $\mathcal{O}([q, q'])$. Moreover, they are also uniquely decided by the pattern $\mathcal{O}([q, q'])$ and their patterns $\mathcal{O}_{[p,p']}(q)$ and $\mathcal{O}_{[p,p']}([q, q'])$ of the central block. Here by $\mathcal{O}_{[p,p']}(q)$ and $\mathcal{O}_{[p,p']}([q, q'])$ we denote the patterns of orbits of q and $[q, q']$ under unimodal map $G = f^{K_p}|_{[p,p']}$, with K_p the period of periodic interval $[p, p']$, that is, their patterns limited in the central block $[p, p']$. So we can define an operation on the patterns in these cases, which we write as,

$$\mathcal{O}(q) = \mathcal{O}([p, p']) \ltimes \mathcal{O}_{[p,p']}(q),$$

and

$$\mathcal{O}([q, q']) = \mathcal{O}([p, p']) \ltimes \mathcal{O}_{[p,p']}([q, q']).$$

Concerning the case of a series of nested restrictive intervals, if we have restrictive central points p_i for $0 \leq i \leq k+1$ and $p_{i+1} \in (p_i, p_i')$, we have

$$\mathcal{O}([p_{k+1}, p_{k+1}']) = \mathcal{O}([p_0, p_0']) \ltimes \mathcal{O}_{[p_0,p_0']}([p_1, p_1']) \ltimes \ldots \ltimes \mathcal{O}_{[p_k,p_k']}([p_{k+1}, p_{k+1}']).$$

We call $\mathcal{O}([p_i, p_i'])$, the pattern of the orbit of intervals $[p_i, p_i']$, also the layer of $[p_i, p_i']$. For different layers of a given unimodal map, the one who has block structure over another is said an upper (or inner) layer over its reduction.

A reverse bifurcation happens whenever the innermost layer of restrictive intervals change from restrictive to be non-restrictive. Namely, in an S-unimodal families $f_\mu(x)$ with a parameter μ, at μ_0 such that $f_{\mu_0}^{K_p}(c) = p_{\mu_0}'$ for a central periodic point p_{μ_0}, if at the two sides of μ_0, it is $f_\mu^{K_p}(c) \in [p_\mu', p_\mu]$ and $f_\mu^{K_p}(c) \notin [p_\mu', p_\mu]$ respectively, then μ_0 is a critical value of reverse bifurcation.

Theorem 4.1.1 (Theorem D). *Suppose that μ_0 is a critical value such that $f_{\mu_0}^{K_p}(c) = p_{\mu_0}'$ for a central periodic point p_{μ_0}. Denote by q_{μ_0} the restrictive central period point who is the second closest to c, that is, with p_{μ_0} the only restrictive central point in (q_{μ_0}', q_{μ_0}). For value of μ (in Lebesgue measure sense) arbitrarily close to μ_0 with $f_\mu^{K_p}(c) \notin [p_\mu', p_\mu]$, the attractor changes from a cycle of period $K_{p_{\mu_0}}$ intervals of $[p_{\mu_0}', p_{\mu_0}]$, to a cycle of intervals of period $K_{q_{\mu_0}}$, contained in the restrictive intervals of orbit $[q_\mu', q_\mu]$. Precisely for the patterns, as $\mu_0 \to \mu$, we have*

$$\mathcal{O}(p_{\mu_0}, p_{\mu_0}']) = \mathcal{O}([q_{\mu_0}', q_{\mu_0}]) \ltimes \mathcal{O}_{[q_{\mu_0}', q_{\mu_0}]}([p_{\mu_0}', p_{\mu_0}]) \to \mathcal{O}([q_{\mu_0}, q_{\mu_0}']).$$

The similar bifurcations that the attractor goes from the innermost layer to the second innermost can also be observed in quasi-periodically forced S-unimodal maps, which are in form of

$$\begin{cases} \theta_{n+1} &= R(\theta_n) = \theta_n + \omega \pmod 1, \\ x_{n+1} &= \psi(\theta_n, x_n), \end{cases} \tag{4.1}$$

where $(\theta, x) \in \mathbb{S}^1 \times I$. Here $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is the unit circle and $\omega \in \mathbb{R} \setminus \mathbb{Q}$ is a fixed irrational number. The function $R\colon \mathbb{S}^1 \to \mathbb{S}^1$ denotes an irrational rotation of the circle $\mathbb{S}^1$ by the fixed angle ω as usual. Furthermore, the function $\psi(\theta_n, x_n)$ is continuous on both x and θ, which is in form of $\psi_\theta(x) = f(x) \cdot g_\epsilon(\theta)$ or $\psi_\theta(x) = f(x) + g_\epsilon(\theta)$ with f an S-unimodal map defined on I. Here $\epsilon \geq 0$ is used as the parameter to control the perturbation given by the forcing function $g(\theta)$. Moreover, we suppose that $\psi_\theta(x) = f(x)$ for all $\theta \in \mathbb{S}^1$ if $\epsilon = 0$, and in case of $\psi_\theta(x) = f(x) \cdot g_\epsilon(\theta)$, $g_\epsilon(\theta) \geq 0$ so that the S-unimodal structure can be maintained in fibre maps.

For the attractors of quasi-periodically forced S-unimodal maps, if we fix an S-unimodal map and let a control parameter of the forcing term increase, we can see that there are also the similar bifurcations of bands merging. Like the unforced maps, it happens when the attractor goes beyond the boundary of the invariant region and spreads to the larger invariant region outside. With the help of numerical evidences, the procedure of a series of bifurcations of band merging is shown to be coincident with the structure of the restrictive intervals of the unforced S-unimodal map f. This gives a general mechanism on the change of the periodicity of the attractors for the forced systems. Precisely, this general mechanism with respect to the increasing of ϵ is as below.

Theorem 4.1.2 (**Claim E**). *Suppose the attractor of the unforced S-unimodal map $f(x)$ is contained in restrictive intervals of pattern*

$$\mathcal{O}([p_k, p_k']) = \mathcal{O}([p_0, p_0']) \ltimes \mathcal{O}_{[p_0,p_0']}([p_1, p_1']) \ltimes \ldots \ltimes \mathcal{O}_{[p_{k-1},p_{k-1}']}([p_k, p_k']).$$

With f being fixed, let ϵ increase from 0, we can see a process that the attractor of quasi-periodically forced systems (4.2) *becomes stripes on the cylinder with patterns step by step (maybe not monotonically)*

$$\mathcal{O}([p_k, p_k']) \to \mathcal{O}([p_{k-1}, p_{k-1}']) \to \ldots \to \mathcal{O}([p_1, p_1']) \to \mathcal{O}([p_0, p_0']).$$

That is, the attractor merges into stripes with bigger size and less period, according to the block structure of $f(x)$ in general order.

4.2 Restrictive intervals and structures of attractors

In this section we define the basic terminologies and notions of this chapter. The key concept is restrictive intervals of unimodal maps. A set of restrictive intervals comes together with a periodic orbit, and the union of every set of restrictive intervals is forward invariant and contains the only attractor of a generic S-unimodal map. So they are the major tool for characterizing the periodicity and states of topological attractors of S-unimodal maps.

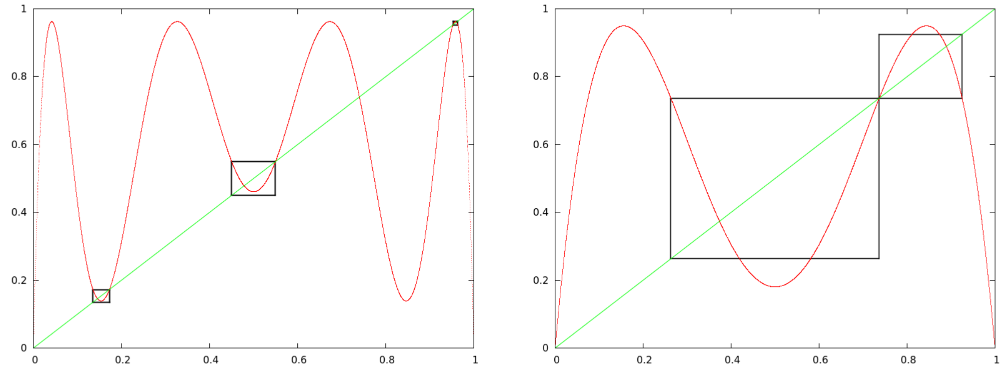
The content of this section consists of three parts, each is given in one subsection. The first one is devoted to the definition and basic properties of restrictive intervals. In the second one, we introduce extension pattern, a combinatorial tool which is a natural and convenient tool to describe the dynamical structure of restrictive intervals. Finally, we introduce the topological attractors of S-unimodal maps, and characterize them by restrictive intervals in the last subsection.

4.2.1 Definition and properties

This subsection is devoted to set up the concept of restrictive intervals of unimodal maps and to exhibit their properties. It can be seen finally that the intersection of all the restrictive intervals of a unimodal map forms a forward invariant nested set.

More precise, we start by construction to show that, to every periodic orbit of a unimodal map, there exists a set of intervals associated naturally. These intervals are selected preimages of the central interval (see definition below), and are uniquely determined by this periodic orbit. The construction of them also shows that, it is equivalent to be restrictive, periodic and invariant for every set of such intervals. We list all those equivalent conditions by Proposition 4.2.4, thereafter we give our definition of the restrictive intervals based on this proposition. Due to the invariance of these intervals, the periodicity of any orbit inside them must be a multiple of the periodicity of these intervals, such properties on periodicity are given by Proposition 4.2.7. Notice that, the invariance also implies that any a series of restrictive intervals forms a nested set, with the critical point inside their intersection. This is the crucial fact for us to study the attractors of S-unimodal maps with restrictive intervals in the third subsection.

First we introduce some definitions and notations. By saying "an interval" we mean a "closed interval". The notation $I = [a, b]$ means that a and b are the endpoints of I but does not necessarily mean that $a < b$ unless this is specified. This is also the case when we discuss other kinds of intervals. The n-fold iterates of a map f is denoted by f^n. Assume $f^n(p) = p$, one says then that p is periodic with *period* n. Moreover, if n is the smallest



(a) Intervals of a periodic orbit of period 3. The red line is graph of f^3 with $f = 3.85x(1-x)$ (in green).

(b) Intervals of $2n$ type of a repelling fixed point. The red line is graph of f^2 with $f = 3.8x(1-x)$ (in green).

Figure 4.2: Examples of periodic intervals associated with periodic orbits.

integer for which this is true, then we say that it is the *prime period*. In most of situations, when we say "period" we refer to the prime period. If $n = 1$, p is called a fixed point. The set $\mathcal{O}(x) = \{f^n(x)\}_{n=0}^{\infty}$ is called the *orbit* of $x \in I$. The limit set of $\mathcal{O}(x)$ is denoted by $\omega(x)$.

A continuous interval map $f : I \to I$ is called *unimodal* if it has only one extreme c on I, and if it maps the endpoints of I to one of them. Moreover, we assume f is strictly monotone on each of intervals $[a, c]$ and $[c, b]$. For simplicity reason, we often prove our results for the case that c is the maximum. It is easy to see that the other case of c being minimum can be treated analogously, since they can be transformed into each other just by a simple change of coordinates. Points p and p' are called *"c-symmetric"* if $f(p) = f(p')$, and *"c-symmetric" interval* refers to an interval whose endpoints are just "c-symmetric". For two points $p, q \in I$, if $p \in (q, q')$, we say p is *"closer" to* c than q. If p is a periodic point of period n, we say p is *central* if it is closer to c than any other periodic points of its orbit, and call $[p, p']$ a *center interval*. Thus in the central interval, there is no any other periodic points of the orbit of p.

Like the case of periodic points, we are also interested in whether the central intervals can exhibit some periodicity. For this we first show with the lemma below that, there are a specific set of intervals naturally linked to every periodic orbit.

Lemma 4.2.1. *If p_0 the central point of a periodic orbit of period n and $p_i = f^i(p_0)$, then there is a set of intervals $J_i = [p_i, q_i]$ for $0 \leq i \leq K$ with $K = n$ or $2n$, such that $f(J_i) = J_{i+1}$ monotonically for each $1 \leq i < K$, where $J_K = J_0 = [p_0, p_0']$.*

Proof. Assume that $p_0 \neq c$, otherwise we just take this set of intervals to be degenerated, which is exact the periodic orbit itself. Now we construct the set of intervals $J_i = [p_i, q_i]$ as follows.

If $n = 1$ and $f(c) \in [p_0, p_0']$, we are done with $J_0 = [p_0, p_0']$. Otherwise, let $p_2 = p_1 = p_0$, $q_2 = p_0'$, choose q_1 such that $f(q_1) = q_2$ and q_1 is at the same side of c with p_0 (see Figure 4.2(b)). This is for the case $n = 1$.

Now assume that $n > 1$. Let $q_n = q_0 = p_0'$ first, then choose q_i such that $f(q_i) = q_{i+1}$ and $f|_{[p_i,q_i]}$ is monotone for $0 < i < n$. This means that q_i is one of the preimages of q_{i+1} which belongs to the same side of c as p_i. So the construction can be completed if this preimage of each q_{i+1} exists.

The existence of the preimages of point q_{i+1} is equivalent to $q_{i+1} \leq f(c)$, and we know that $p_1 = f(p_0) < f(c)$, so we just need to prove $q_i < p_1$ for all $1 < i \leq n$. First we start from q_n. By assumption, $q_n = p_0'$, thus it must be $q_n < p_1$. This is because that c is the maximum, and $p_i \notin (p_0, p_0')$ by the definition of the central point, so $f(p_i) < f(p_0) = p_1$ for all $i \neq 0$, that is $p_1 = f(p_0)$ is the largest one in its orbit, which also means $p_0 < p_1$. Hence we can get q_{n-1} as the required preimage of q_n. Using induction next, assume that we have obtained q_k for some $1 < k < n$. Notice that $p_i < p_1$ for all $i \neq 1$, thus we can obtain $q_k < p_1$. Otherwise, $q_k > p_1$ and $p_k < p_1$ implies $p_1 \in (p_k, q_k)$, and then $f^{n-k}(p_1) = p_{n-k+1} \in (p_n, q_n) = (p_0, q_0)$, which is a contradicts to p_0 being central.

When we get q_1 and if $q_1 > p_1$, the claimed intervals are all constructed completely, hence we stop with $K = n$ (see Figure 4.2(a)). If $q_1 < p_1$, we recode the subindex by replacing n with $2n$ and continue the same construction, which means that q_1 is reindexed as q_{n+1} now. The same arguments above guarantee the construction process until we get the new q_1 reindexed, and finally $q_1 > p_1$ for this time. Notice that if $K = 2n$, then $p_{n+k} = p_k$, q_{n+k} and q_k are at different side of p_k (see the periodic orbit of period 3 in Figure 4.4(b)). □

Thus if $K = n$, $J_i \cap J_j = \emptyset$ for any $0 \leq i < j < K$; and for $K = 2n$, $J_i \cap J_j = \emptyset$ if $|i - j| \neq n$, and $J_i \cap J_j = \{p_i = p_j\}$ if $|i - j| = n$. In fact, these intervals linked to periodic orbit can exhibit many useful dynamical structures of system of unimodal map, one is as the corollary below.

Corollary 4.2.2. *Let p be a periodic point of period n of a differentiable unimodal map f, we have $(f^n)'(p) = 0$ if and only if $c = f^i(p)$ for some $0 \leq i < n$; $(f^n)'(p) > 0$ if and only if $K = n$; and $(f^n)'(p) < 0$ if and only if $K = 2n$.*

Proof. By the chain rule, $(f^n)'(p) = \prod_{i=0}^{n-1} f'(f^i(p))$. Hence it is 0 if and only if the only extreme c belongs to this orbit. For the other two cases, now we just need to show their "if" parts.

Notice that $(f^n)'(p)$ is a constant for all the points of the orbit of p, so we consider only the largest point of the orbit, which is p_1 in the previous lemma. If $K = n$, we have $[p_0, p_0'] = [p_0, q_0] = f^{n-1}([p_1, q_1])$, so $f^n([p_1, q_1]) = f([p_0, p_0']) \subseteq [p_1, f(c)]$. This implies that, for any $x \in (p_1, p_1 + \delta)$ with $\delta > 0$, $f^n(x) > p_1 = f^n(p_1)$, hence $(f^n)'(p) = (f^n)'(p_1) > 0$. The same arguments show that, if $K = 2n$, $f^n([p_1, q_1]) \subseteq [c, p_1]$, so $(f^n)'(p) < 0$ (notice in this case that, instead of f^n, the derivative for f^K at p is also $(f^K)'(p) = (f^{2n})'(p) > 0$). □

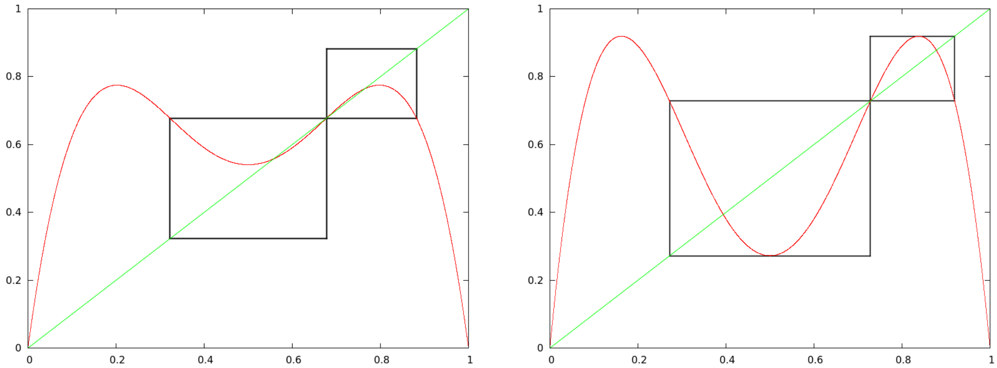
Remark 4.2.3. For a one parameter family of unimodal maps who is continuous with respect to the parameter, it is common that its periodic orbits also move continuously with respect to the parameter. The arguments above imply that, these intervals linked to a periodic orbit points change their style of $K = n$ or $K = 2n$ whenever the central point moves cross the extreme c. This is because that, for points of a periodic orbit, only the derivative of the central point changes its sign during this process. For example, the fixed point $x = 1 - 1/\mu$ of the logistic family $f_\mu(x) = \mu x(1 - x)$ is of type $K = n$ when $\mu \in [1, 2)$, and changes to be type $K = 2n$ with its value $1 - 1/\mu > c = 1/2$ at $\mu \in (2, 4]$. ◇

Now, we discuss when a central interval can be periodic. An interval $J \subseteq I$ is called to be *periodic* of period n if $f^n(J) \subseteq J$ and $\mathrm{Int}(f^k(J)) \cap \mathrm{Int}(f^j(J)) = \emptyset$ for all $0 \le k < j \le n - 1$. In such a case the set $\mathcal{O}(J) := \cup_{k=0}^{n-1} f^k(J)$ is called a *cycle* of intervals. Any cycle of intervals is necessarily forward invariant. A set I is *forward invariant* if $f(I) \subseteq I$. Thus a point p stays eventually inside a forward invariant set after it enters. For the well-known S-unimodal maps, the cycles of intervals containing the critical point c are of particular interest for their attractors. A simple and direct condition which ensures that a central interval is periodic is that, the central point p of period n is restrictive, which means that $f^n(c) \in [p, p']$ by the original idea of Guckenheimer in [32]. However, this condition is only suitable for those central intervals which are of type $K = n$ in Lemma 4.2.1. We give new definitions of this concept as follows.

By *intervals of a periodic orbit of* p, we mean the set of those intervals constructed by Lemma 4.2.1. We also denote by $p(R)$ the point q_1 in the lemma, which is the most right point of all those intervals. For p a central periodic point of period n, both the points and intervals of the orbit of p are called to be *restrictive* if $f^K(c) \in [p, p']$. Notice that the period of such intervals is K with $K = n$ or $2n$, not necessarily the same as the prime period of point p. If $K = 2n$, we call p a periodic point of *period-doubling (or $2n$) type*. Thus a periodic point of $2n$ type is itself of prime period n, but the set of intervals of its orbit has $2n$ elements.

Some clear and handy equivalent conditions of intervals being restrictive are given in the following proposition.

Proposition 4.2.4. *Let p_0 be a central periodic point of period n and p_i, J_i, K be defined as Lemma 4.2.1 above, then the following conditions are*



(a) Restrictive intervals and graph of f^2 (red line) for $f = 3.1x(1-x)$.

(b) Restrictive intervals and graph of f^2 (red line) for $f = 3.6785x(1-x)$.

Figure 4.3: Examples of restrictive intervals of a repelling fixed point of $2n$ type.

equivalent:

(1) $f^K(c) \in [p_0, p_0']$, *that is,* p_0 *is restrictive;*

(2) $f(c) \leq q_1$*;*

(3) $J_0 \ldots J_{K-1}$ *are all periodic;*

(4) each $f^K|_{J_i}$ *is unimodal for* $0 \leq i < K$*;*

(5) $\cup_{i=0}^{K-1} J_i$ *is invariant.*

Proof. By Lemma 4.2.1, we know that $f(J_i) = J_{i+1}$ and f is monotone on each J_i for $1 \leq i < K$. This implies that, we only need to verify whether $[p_1, f(c)] = f(J_0) \subseteq J_1 = [p_1, q_1]$ or not. All of the above follow straightforward by it. □

We can see in Figure 4.2(a) that, the intervals of a orbit of period 3 are restrictive; while for the example in Figure 4.2(b), the repelling fixed point is of $2n$ type, but the intervals of its orbit are not restrictive. More examples of restrictive intervals of this type can be seen in Figure 4.3. All these pictures display clearly the properties given in the above proposition.

Remark 4.2.5. Notice that the $2n$ type is not viewed as being restrictive by Guckenheimer in [32]. Furthermore, he requires the restrictive central points to be repelling, but we do not. Our definition has better consistency for our later arguments. The reader who is familiar with the renormalization theory of interval maps knows that, some authors use also this name refer to the commonly called renormalization intervals. Although a little similar, but our definition is different with the renormalization intervals either. Their definition in fact implies the endpoints being also repelling. Here we will not dwell on the details of this field of renormalization, which consider the forward iterations of these intervals under the renormalization operator. We only discuss the topological aspect of dynamics exhibited by these intervals instead. $\diamond$

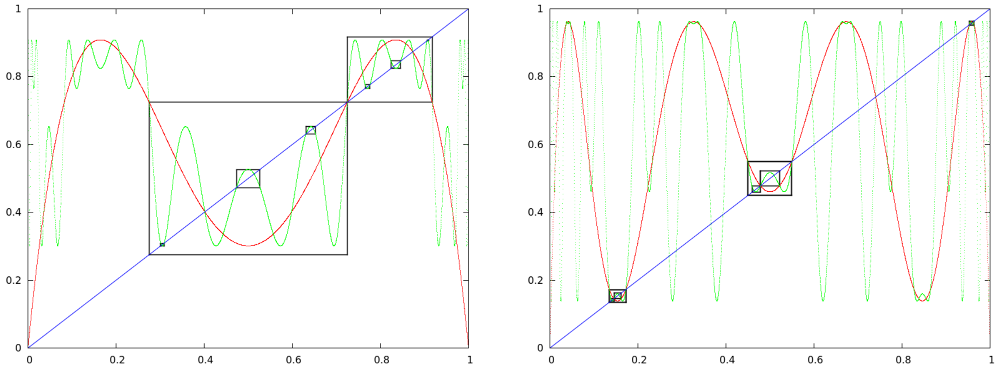
Some other useful and simple facts about the restrictive intervals can be easily concluded. First, due to the construction, it can be seen that $f^K(q_i) = f^{K-n+i}(p_0') = p_i = f^K(p_i)$, thus we can view q_i and p_i as being "$f^i(c)$ central symmetric" for f^K (see Figure 4.2).

Second, if $K = 2n$, then $f^n(c) \in f^n([p_0, p_0']) \subseteq J_n$. Notice that $J_n = [q_n, p_0]$ and q_n, p_0' are at different side of p_0. This is the case for all the points in the orbit of periodic point of $2n$ type. That is, there are two associated intervals each at one side of every point of the orbit, and every point of the orbit is the common endpoint of both two intervals. Therefore, when there are new periodic points occur in such restrictive intervals, it must be simultaneously in two intervals at both sides (see Figure 4.3(a)). For this reason, we call them as period-doubling type.

Example 4.2.6. In Figure 4.3(a), we plot the picture of f^2 for $f = \mu x(1-x)$ in $x \in [0,1]$ with $\mu = 3.1$. In this case, f has a fixed point $p = 1 - 1/\mu \approx 0.67742$, its central interval is $[p', p] = [1/\mu, 1 - 1/\mu]$. For the fixed point p, its central interval $J_0 = [p', p]$ is not restrictive under f, since $f(x) \geq p$ for all $x \in [p', p]$. However, if we consider for map f^2, then it is restrictive. Moreover, both this interval J_0 and its preimage J_1 of f, are invariant under f^2.

If we consider this example from the point of view of family $f_\mu(x) = \mu x(1-x)$, the following facts are worth noting. When $\mu > 1$, there occurs the second fixed point $p = 1 - 1/\mu$ other than $x = 0$. This fixed point p is restrictive of n type for $\mu \in (1,2)$ with $p < f_\mu(c) < c < p'$; and becomes $2n$ type when $\mu > 2$, for which $p' < c < p < f_\mu(c)$. Then it keeps to be restrictive (of $2n$ type) until $\mu_1 \approx 3.6785735$, which is the root of $f_\mu^2(c) = f_\mu^2(1/2) = 1/\mu = p'$, equivalently, of $\mu^3 - 2\mu^2 - 4\mu - 8 = 0$.

It is clear that, when $2 < \mu < 3.6785735$, for any $q \in J_0$, it must be $f^{2k+i}(q) \in J_i$ for $k \in \mathbb{N}$ and $i = 0, 1$. Particularly, when $2 < \mu < 3$, we know that all points inside these two intervals converge to p oscillatingly from its two sides. At $\mu = 3$, a period-doubling bifurcation takes place with p becoming repelling and two new attracting periodic points of period two occurring at both its sides (refer to Figure 4.3(a)). Thus for $3 < \mu < 3.6785735$, the attractors all stay in $J_0 \cup J_1$ with p repelling. Figure 4.2(b) shows that, at $\mu = 3.6785$, the period-doubling intervals of p is coming to the end of being restrictive. In fact, a reverse bifurcation just happens when



(a) Nested restrictive intervals for $f(x) = 3.635x(1-x)$. The red line is graph of f^2, The green one is of f^6.

(b) Nested restrictive intervals for $f(x) = 3.85x(1-x)$. The red line is graph of f^3, The green one is of f^6.

Figure 4.4: Examples of nested restrictive intervals.

these intervals become non-restrictive, the two pieces of attractor merge into one as shown in Figure 4.1(a).

This is an example of the correspondence of the bifurcations at two sides of the restrictive intervals J_0 and J_1 of p: which starts when $\mu > 2$ with p becoming $2n$ type; becomes apparent after $\mu = 3$ by the occurring of new period-doubling attracting orbit; and finally ends at this reverse bifurcation with these two intervals of p becoming non-restrictive. In fact, this is the universal mechanism for all the restrictive intervals of period-doubling type.$\diamond$

Finally, let us show the properties on the periodicity of periodic orbits which are inside restrictive intervals. Let $\{J_i\}$ be a set of restrictive intervals for $0 \le i < K$. Since J_i is invariant under f^K for each $0 \le i < K$, denote $f^K|_{J_i}$ by G_i, we have $f^{mK}|_{J_i} = G_i^m$. This implies that, if $s \in J_i$ is a periodic point of period mK of f, then it is a periodic point of period m of G_i (see Figure 4.4).

On the other hand, given p_0 a periodic point of period n with $\{J_i\}$ the set of restrictive intervals of its orbit, then for any point $p \in J_0$, it must be $f^{K+k}(p) \in J_k$ due to the invariance of the union of these intervals J_i. Hence, if p is also periodic and it is not an endpoint of J_0, its period must be a multiple of K. In this case, each J_k can be viewed as a block who contains some points of the orbit of p with the same number. The same arguments also work if this point p is replaced by a subinterval $J \subseteq J_0$. Hence we have the following proposition.

Proposition 4.2.7. *All the restrictive intervals of a unimodal map f form a nested set of invariant intervals. Particularly, if p is a restrictive central* *periodic point, and the set of intervals of its orbit are* $\{P_i\}$ *for* $0 \leq i < K_p$, *then*

(1) if r *is a central periodic point of period* n *who is closer to* c *than* p*, that is,* $r \in (p, p')$*, then* $n = k \times K_p$ *for some* $k \in \mathbb{Z}^+$*, and* $f^{mK_p+j}(r) \in P_j$ *for all* $0 \leq j < K_p$*,* $m \geq 0$*;*

(2) if r *is also restrictive and the periodic intervals of its orbit are* R_i *for* $0 \leq i < K_r$*, then* $R_{mK_p+j} \subset P_j$ *for all* $0 \leq j < K_p$*,* $m \geq 0$*;*

(3) for any central periodic point s*, if* s *is closer to* c *than* p *and the restrictive central point* r *is closer to* c *than* s*, namely,* $s \in (p, p') \setminus [r, r']$*, then its orbit* $\mathcal{O}(s) \subset \cup_{i=0}^{K_p} P_i \setminus \cup_{i=0}^{K_r} R_i$*.*

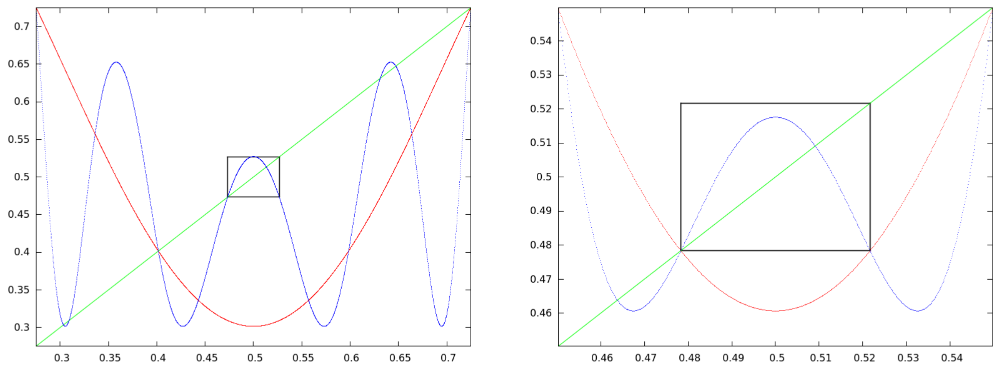
The proof of this proposition is straightforward from the invariance of the union of restrictive intervals. Just notice that, for any two different central restrictive periodic points, it must be one is closer to c than the other, hence all the restrictive intervals are nested. In Figure 4.4 we plot two examples of restrictive periodic intervals of period 6, they can show this proposition intuitively. Note that, Figure 4.4(a) shows three intervals inside each of two intervals of a fixed point of period-doubling type; while in Figure 4.4(b), it is two period-doubling intervals contained in each of a periodic three interval. Clearly, although there are orbits of period 6 in both of the inner restrictive intervals, but the dynamics of these two period orbits are not same. Next we introduce extension pattern, which can specify the dynamical structures of these restrictive intervals and periodic orbits.

4.2.2 Extension patterns

This subsection is devoted to extension pattern, a concept from combinatorial dynamics, which is the most natural and convenient language for the dynamics and structures of restrictive intervals and those periodic orbits inside them. The reader interested more on combinatorial dynamics can refer to [2] for a comprehensive discussion on this concept. Here we introduce only knowledges fitting our purpose: first the necessary definitions, and then a special property for restrictive intervals of unimodal maps. With this property, we define an operation on the patterns of these intervals, which we use to describe of the dynamics of attractors of S-unimodal maps in the next subsection.

The combinatorial structure we wish to study can be set up as follows. Given $f\colon I \to I$ a continuous map of a closed interval I to itself and $\mathcal{P}$ a f-invariant set (i.e., $f(\mathcal{P}) \subset \mathcal{P}$) with finite elements which are intervals (may be degenerated, that is, a single point), label the elements of $\mathcal{P}$ by

$$p_1 < p_2 < \cdots < p_n$$



(a) Enlarged central restrictive intervals for $f(x) = 3.635x(1-x)$. The red line is graph of f^2, The blue one is of f^6.

(b) Enlarged central restrictive intervals for $f(x) = 3.85x(1-x)$. The red line is graph of f^3, The blue one is of f^6.

Figure 4.5: Structures of nested restrictive intervals inside the central ones.

(where $p_i < p_{i+1}$ means the right endpoint of p_i is no larger than the left of p_{i+1}). Then the action of f on $\mathcal{P}$ can be codified in the map

$$\theta : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

defined by

$$f(p_i) = p_\theta(i) \quad i = 1, \ldots, n.$$

The map θ encodes the combinatorial structure of each orbit and the way these orbits intertwine. To stress the combinatorial role of θ, we refer to any map of $\{1, \ldots, n\}$ to itself as a *combinatorial pattern* on n elements, or a *pattern* for short. The *degree* of θ, denoted by $|\theta|$, is the number n. We say that the map f exhibits the combinatorial pattern θ on $\mathcal{P}$, and call $\mathcal{P}$ a *representative* of θ in f. A given finite invariant set $\mathcal{P}$ represents a unique combinatorial pattern θ, but a given combinatorial pattern θ may have many representatives in f. To make the situation clear, we may denote a pattern θ and a representative $\mathcal{P}$ by $(\mathcal{P}, \theta)$, and may also call it a pattern for short.

Let $\mathcal{P} = \{1, \ldots, n\}$. A *block* in $\mathcal{P}$ is defined as a set of the form $B = \{i \in \mathcal{P} | a \leq i \leq b\}$, where $a \leq b \in \mathcal{P}$. By a *block structure* for a pattern θ represented by $\mathcal{P}$, we mean a partition $\mathcal{B} = \{B_1, \ldots, B_k\}$ of $\mathcal{P}$ into disjoint blocks such that if $x, y \in \mathcal{P}$ belong to the same block, their images under θ belong to a single block. We number the blocks B_j so that $x \in B_i, y \in B_j$ and $i < j$ implies that $x < y$; then there is a unique pattern γ defined by

$$\theta[B_j] \subset B_{\gamma(j)}, \quad i = 1, \ldots, k.$$

We call γ a *reduction* of θ, θ an *extension* of γ, and refer to $\mathcal{B}$ as a *block structure for θ over γ*.

Particularly, if $\mathcal{P}$ is a single periodic orbit, the combinatorial pattern θ represented by $\mathcal{P}$ is a cyclic permutation. Let $(\mathcal{P}, \theta)$ be a cycle of period n and let $(\mathcal{B}, \gamma)$ be a pattern of period m. Let $\mathcal{P} = \{p_1 < p_2 < \cdots < p_n\}$ and $\mathcal{B} = \{i_1, i_2, \ldots, i_m\}$. Then $(\mathcal{P}, \theta)$ has a block structure over $\mathcal{B}$ provided that, $n = sm$, $\mathcal{P} = P_1 \cup P_2 \cup \ldots \cup P_m$ with $P_i = \{p_{(i-1)s+1}, p_{(i-1)s+2}, \cdots, p_{(i-1)s+s}\}$ for all $i = 1, 2, \ldots, m$; $\theta(P_{i_j}) = P_{i_{j+1}}$ for all $j = 1, 2, \ldots, m-1$ and $\theta(P_{i_m}) = P_{i_1}$. Thus each of the sets P_i is a block of $\mathcal{P}$. We can view each block as a "fat" point and $\mathcal{P}$ as a "fat" cycle with the pattern $(\mathcal{B}, \gamma)$.

Example 4.2.8. Both the periodic orbits of period six in Figure 4.4 have typical block structures linked to their restrictive intervals. They are of maps $f(x) = \mu x(1-x)$ for two different values μ respectively, and their pattens and block structures are not the same either.

The map $f(x) = 3.635x(1-x)$ in Figure 4.4(a) has two orbits of period six, both belong to the two restrictive intervals of the repelling fixed point which is of $2n$ type, with each of these two intervals containing three points of each orbit as shown in the plotted boxes. That is, for these two period six orbits, each of their three points inside a same restrictive interval form a block, thus the pattern of both two period six orbits have the same block structure over the pattern of two restrictive intervals.

In the case of the periodic orbit of period six for the map $f(x) = 3.85x(1-x)$ in Figure 4.4(b), the six points in the plotted boxes belong to three blocks respectively, the blocks are given by restrictive intervals of a period three orbit who contain this period six orbit. $\diamond$

Remark 4.2.9. Here we'd like to point out that, the two orbits of $\mu = 3.635$ come together continuously from a saddle-node bifurcation at a smaller μ. They exist thereafter for all value of μ larger than this bifurcation value, even after they both become unstable. For instance, in Figure 4.4(b) for $\mu = 3.85$, it can been seen that, outside the boxes we plot, there are also two other orbits which given by the intersection of f^6 and the diagonal. They are just the continuous successors of these two orbits of $\mu = 3.635$. However, this cannot be shown in the bifurcation diagram for the logistic family, which is given only by attractors who must be contained in all restrictive intervals as we will see in the next subsection. For this case $\mu = 3.635$, refer to our discussion of Example 4.2.6 and Figure 4.1(a), the period two intervals created by the first period-doubling bifurcation are still restrictive, hence new restrictive intervals and orbits brought by bifurcations can only lie in these two intervals.

While for $\mu = 3.85$, the orbit inside the plotted boxes originate from period-doubling bifurcation of a period three orbit (see Figure 4.5(b)). It is the first bifurcation in the big period three window of the transition diagram of logistic maps. Thus the extension pattern represented by this orbit is over the three blocks given by restrictive intervals. $\diamond$

The examples above demonstrate concretely extension patterns of restrictive intervals. The block structure is exact the combinatorial aspect of Proposition 4.2.7, and therefore is valid for all periodic orbits of points or restrictive intervals, who are inside another set of restrictive intervals.

Furthermore, there is another special property of this structure for those unimodal maps: if the extension pattern of an orbit has block structure over a set of restrictive intervals, then it is uniquely determined by the pattern of blocks of restrictive intervals, and the pattern of the orbit in the central block. The details are as follows.

For simplicity reason, when the unimodal map f is known, we abuse the notation of an orbit and the pattern that it represents. That is, if p is a periodic point, we use $\mathcal{O}(p)$ to denote also the pattern represented by this orbit, and call it the pattern of (the orbit of) p. If p is a restrictive central point, we denote by $\mathcal{O}([p,p'])$ the pattern represented by the restrictive intervals of its orbit. Furthermore, let $G = f^K|_{[p,p']}$ with K the period of interval $[p,p']$, we know this restriction function G itself is also a unimodal map on $[p,p']$. If $q \in [p,p']$ is a central restrictive point of f, then q is a periodic point of G on $[p,p']$. We denote by $\mathcal{O}_{[p,p']}(q)$ and $\mathcal{O}_{[p,p']}([q,q'])$ the patterns that G exhibits on the orbit and the restrictive intervals of q respectively. Thus we have the following property.

Proposition 4.2.10. *Given p and q two central periodic points of a unimodal map f with $q \in [p,p']$. If p is restrictive, then $\mathcal{O}(q)$, the pattern of point q, is uniquely decided by its reduction $\mathcal{O}([p,p'])$ and the pattern $\mathcal{O}_{[p,p']}(q)$ of the central block. Similarly, if q is also restrictive, $\mathcal{O}([q,q'])$ is also uniquely decided by $\mathcal{O}([p,p'])$ and $\mathcal{O}_{[p,p']}([q,q'])$.*

Proof. This proposition is a direct consequence of the monotonicity of unimodal map on each side of c. Recall the construction of the orbit of intervals of p, $[p,p']$ is set as $J_K = J_0$, and for $0 < i < K$, $f(J_i) = J_{i+1}$ strictly monotonically. Hence, when the pattern $\mathcal{O}_{[p,p']}(q)$ on $[p,p']$ is known, every pattern that $f^K|_{J_i}$ exhibits on the orbit of q in J_i is uniquely given by the preimages of $\mathcal{O}_{[p,p']}(q)$. Finally, from the pattern that f^K exhibits on J_1 to J_0, it is uniquely given by the order of $f^K|_{J_1}(q)$, and how those points of $\mathcal{O}_{[p,p']}(q) = f^K|_{J_0}(q)$ close to c. The same is for $\mathcal{O}_{[p,p']}([q,q'])$ with those preimages of q'. $\square$

To illustrate this proposition, in Figure 4.5 we plot the pictures of the central restrictive intervals of Examples 4.2.8, which are shown by the outer big boxes plotted in Figure 4.4. Denote by $[p,p']$ the central restrictive intervals that we enlarge by Figure 4.5, in their pictures we plot the graphs of f^6 and f^K with K the period of $[p,p']$ under f, which is $K = 3$ for Figure 4.5(a) and $K = 2$ for Figure 4.5(b) respectively. Clearly, both $G = f^K|_{[p,p']}$ are unimodal, and $f^6|_{[p,p']} = G^i$ where $i = 6/K$. In each case, there is a $q \in [p,p']$, which is central for both maps G and f. The pattern of q that

G exhibits is $\mathcal{O}_{[p,p']}(q)$, represented by the orbit of those i points inside $[p, p']$ in Figure 4.5. $\mathcal{O}(q)$ is the pattern of q that f exhibits, represented by the period six orbit in Figure 4.4. The pattern $\mathcal{O}([p, p'])$ is represented by the orbit of interval $[p, p']$ in f, which is shown by those plotted outer big boxes in Figure 4.4. Clearly with these boxes, $\mathcal{O}([p, p'])$ is the reduction pattern of $\mathcal{O}(q)$, whose elements are blocks for $\mathcal{O}(q)$. If we embed the central blocks in Figure 4.5 back into Figure 4.4 and take its preimages, then what obtain are exact those other boxes plotted.

Finally by Proposition 4.2.10, we can define an operation on extension patterns of these central points and intervals with their block structures, which we write as

$$\mathcal{O}(q) = \mathcal{O}([p, p']) \ltimes \mathcal{O}_{[p,p']}(q)$$

for $q \in [p, p']$ and p restrictive. Concerning a series of restrictive central points, an easy induction yields that: if $p_{i+1} \in [p_i, p'_i]$ for every restrictive central point p_i of $0 \leq i \leq k$, then

$$\mathcal{O}(p_{k+1}) = \mathcal{O}([p_0, p'_0]) \ltimes \mathcal{O}_{[p_0,p'_0]}([p_1, p'_1]) \ltimes \ldots \ltimes \mathcal{O}_{[p_k,p'_k]}(p_{k+1}).$$

In this case, we call $\mathcal{O}([p_i, p'_i])$, the pattern of the orbit of intervals $[p_i, p'_i]$, *the layer of* p_i. For two layers of a given unimodal map f, the one who has block structure over the other is refer to *an upper (or inner) layer* over its reduction.

We point out that, if an upper layer has only one element in the central block of a lower one, these two layers have the same combinatorial pattern in the sense of the very original definition of pattern. For instance, this is the case of the repelling and attracting orbits of a saddle-node bifurcation when they are just separated. But we regard them as different patterns to emphasis the fact that they are representatives with different structures of restrictive intervals, that is, the attracting orbit has an upper layer since each of its points is inside one interval of the repelling.

This operation provides us a convenient tool to describe the dynamics of topological attractors of generic S-unimodal maps, whose unique attractor must be contained inside all the layers.

4.2.3 Criteria for topological attractors of S-unimodal maps

In this subsection we demonstrate how the structures of topological attractors of S-unimodal maps can be characterized by restrictive intervals.

It is well-known that, a generic S-unimodal map f has at most one topological attractor who attracts all the points but a meagre set of the system. Moreover, this attractor can only be one of the following three types: a periodic orbit, a solenoidal set, or a finite cycle of intervals that f is chaotic on their union. Here we give a brief introduction of this classification first, then show that each type of the attractors just corresponds to a situation of those restrictive intervals: a finite number of restrictive central intervals without any periodic point within the innermost one; infinite many of restrictive central intervals; and a finite number of restrictive central intervals with all periodic points within the innermost one being non-restrictive.

First we review the classic results on topological attractors of S-unimodal maps. By attractor we mean only the topological attractor, which is a set in the topological space I with dynamical structure similar to the metric one in the sense of Milnor. More precisely, a closed invariant set $A \subseteq I$ is called a *topological attractor* of f if
(i) $rl(A)$ is a set of second Baire category;
(ii) for any proper closed invariant subset $A' \subset A$, the set $rl(A) \setminus rl(A')$ is of second Baire category as well.
Here $rl(A) = \{x : \omega(x) \subseteq A\}$ is its"realm of attraction".

A unimodal map is called *S-unimodal* if it is three times differentiable with negative Schwarzian derivative outside c:

$$Sf = \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2 \leq 0.$$

A primary reason for working with negative Schwarzian derivative is *Singer's theorem.* This theorem ensures that there is only one attractor of a generic S-unimodal map that we specify later. To state this theorem we recall some of definitions. A periodic point x of period n is *stable* if there is a non-trivial interval U of x with $f^n(y) \to x$ for all $y \in U$. When x is an endpoint of U, it is one-side stable. A necessary condition for x to be stable is that $|Df^n(x)| \leq 1$. A sufficient condition for x to be stable is that $|Df^n(x)| < 1$. We denote the derivative of f by either f' or Df as convenient.

Theorem 4.2.11 (Singer[72]). *Let $f : I \to I$ has negative Schwarzian derivative. For every stable periodic point x of period n, there is an $i < n$ and a critical point c or endpoint of I such that $y \in [c, f^i(x)]$ implies $f^{kn}(y) \to f^i(x)$ as $k \to \infty$.*

The proof of Singer's theorem is based upon several facts about Schwarzian derivatives which we list here and use later:

(1) If Sf is negative, then Sf^n is negative for all $n > 0$.

(2) If Sf is negative, then $|f'|$ has no positive local minimum. If $J = [a, b]$ is an interval on which f is monotone and $x \in J$, then $|f'(x)| \geq \min(|f'(a)|, |f'(b)|)$.

(3) If $x_1 < x_2 < x_3$ are consecutive fixed points of $G = f^n$ and $[x_1, x_3]$ contains no critical point of G, then $G'(x_2) > 1$.

We give an assumption on the S-unimodal maps which excludes an insignificant situation, which we think it is not generic. Due to this theorem, except for the fixed endpoint of I, an S-unimodal map has at most one stable orbit. Furthermore, if this stable orbit exists, it must attract the critical point c. This implies that, if the fixed endpoint is stable, either it is the only one which attracts all points of I; or there is some other point between it and c, say p, who is fixed and repelling.

For the latter case, if p is not restrictive, then almost all points in $[p, p']$ go outside this interval and are attracted by the stable fixed endpoint finally. Thus the endpoint is also the only attractor, the dynamics of the system is trivial. This is not an interesting case for us. On the other hand, if p is restrictive, $[p, p']$ is invariant under f, while all the other points in $I \setminus [p, p']$ are attracted by the endpoint. In this case what we really need to consider is just $f|_{[p,p']}$. So without loss of generality, we assume that, for an S-unimodal map, if its fixed endpoint is no longer a stable one which attracts all points of I, then it changes to be unstable. This implies that there can be at most one stable orbit in the system, which must attracts the critical point c if it exists. In fact, with this assumption, it can also have only one attractor, which is given by the following theorem.

Theorem 4.2.12. *Let $f : I \to I$ be an S-unimodal map. There exists a set $A \subseteq I$ of second Baire category, so that for each $x \in A$ the set $\omega(x)$ has to be of one of the following three types:*

(1) $\omega(x)$ is a stable periodic orbit;

(2) $\omega(x) = \omega(c)$ with $\omega(c)$ a minimal, solenoidal set of zero Lebesgue measure;

(3) $\omega(x)$ is a cycle of intervals, that is, it is a finite union of intervals containing c, and f acts as a topologically transitive map on this union of intervals.

Remark 4.2.13. The above theorem first dated back to Guckenheimer, who proved in [32] that there is sensitive dependence on initial conditions in the cycle of intervals, while it must not for the case of stable periodic orbits. Later it was proved in increasing integrity (see [52] for transitivity) and generality, refer to [55] for more details. Now we have known that, the negative Schwarzian derivative is just one of the sufficient conditions such that a family of unimodal maps have topological attractors of these types. One can obtain the same kind of results by estimating distortion of cross-ratios, thus the assumption of negative Schwarzian derivative is rather restrictive and turns out to be unnecessary. Here in this work we choose such families just as typical examples to study the transition of their attractors, but note that our results also work for any unimodal families whose topological attractors are of the three types as the theorem above, for instance, the real analytic unimodal families. ◇

The detailed definitions of the other two types besides the stable periodic orbit are as below.

An invariant set S is called a *solenoidal attractor* or a *Feigenbaum-like attractor* if it has the following structure:

$$S = \cap_{n=1}^{\infty} \cup_{k=0}^{p_n-1} f^k(I_n),$$

where each I_n is a periodic interval of period p_n, with $I_1 \supset I_2 \ldots$ and $p_n \to \infty$. Notice that a solenoidal attractor is a minimal invariant set, which means that it is the limit set for any point of it.

For the attractors of the third type, clearly the intervals whose union forms such an attractor must be periodic, so an attractor of this type is called as *cycle of intervals*. We point out here that, actually such an attractor is given by $A = \cup_{i=0}^{K-1}[f^K(c_i), f^{2K}(c_i)]$, where these J_i's are intervals of the periodic orbit whose central point is the closest one to c among all the restrictive periodic points of f, and each c_i is the extreme of f^K in the restrictive interval J_i. In this case, $f|_A$ is chaotic in the sense of Devaney ([15]).

Concerning on a family of unimodal maps, a natural and fundamental problem is to describe the set of parameters corresponding to these three different types of attractors. In the quadratic case we have known the following (see [28, 47]).

- for the case of periodic orbits, the parameters form a set dense in parameter space, which consists of countably infinitely many nontrivial intervals. Moving the parameter inside one connected component of this set, we see the period-doubling scenario, with universal scaling in parameter space.
- for the case of solenoidal attractors, the parameters set is a completely disconnected set of Lebesgue measure zero.
- for the case of cycles of intervals, the parameters set is a completely disconnected set of positive Lebesgue measure.

So any nontrivial parameter interval contains maps with stable periodic orbits, and we cannot find cycles of intervals in a whole connected components of the parameter space. But close some parameter with interval attractor, we are likely to find also interval attractors. Recall that this is a general property of a set of positive Lebesgue measure: almost all points of the set are so-called Lebesgue density points, where measure accumulates. We will discuss this a little bit when we later review the intermittency with saddle-node bifurcation in the next section.

Next we introduce criteria for the classification of these three types of attractors. Both the definition and Theorem 4.2.12 are given in terms of limit sets, which is not so easy to use in practice. However, these three types of attractors can be characterized simply by the exact three cases of the sets of central periodic points of the maps, so we can use this fact as criteria of the classification.

Theorem 4.2.14. *The attractor of an S-unimodal map is contained in the intersection of all the restrictive intervals, and the sets of central periodic points and the attractors of S-unimodal maps correspond with each other as follows:*

(1) if there are infinite many of restrictive central periodic points of f, then its attractor is a solenoid;

(2) if there are a finite number of restrictive central periodic points of f, and no any other (central) periodic point within the smallest restrictive (central) interval, then the orbit of the restrictive central periodic point closest to c is stable, and hence is the attractor;

(3) if there are a finite number of restrictive central periodic points of f, with other periodic points (hence not restrictive) inside the smallest restrictive central interval, then its attractor is a cycle of intervals, who is contained inside the cycle of the smallest restrictive central interval.

Proof. We discuss these three cases one by one as follows.

(1) The case that infinite many of restrictive central periodic points means the solenoidal attractor, is just from the definition of solenoid and Proposition 4.2.7.

(2) Now for the case that there are a finite number of restrictive central periodic points without any periodic point inside. The equivalence of this case to the existence of a stable periodic orbit follows easily by the simple lemma below.

    Lemma 4.2.15. *If $f : [a, b] \to [a, b]$ $(a < b)$ is a unimodal map with $f(a) = f(b) = a$, then it has only one periodic point a if and only if $f(x) < x$ for all $x \in (a, b]$.*

    Proof. $f(x) < x$ for all $x \in (a, b]$ means trivially that a is the only fixed point of f. On the other hand, if there is some $x_0 \in (a, b)$ such that $f(x_0) \geq x_0$ then, either x_0 is another fixed point when $f(x_0) = x_0$, or there must be some other fixed point at (x_0, b) when $f(x_0) > x_0$, which is the consequence of the Mean Value Theorem and the fact $f(b) = a < b$. □

Now suppose that p is a restrictive central periodic point of period n, and it is the closest one to c among all those restrictive central periodic points of f. We have that $f^n : [p,p'] \to [p,p']$ is unimodal by Proposition 4.2.4. Then there is no any other periodic point inside $(p,p']$ means that p is the only periodic point of $f^n|_{[p,p']}$. So p attracts all the points in this interval due to the lemma above, its orbit must be a stable one.

On the other hand, if there is a stable orbit of f, Singer's Theorem says that, it must be some point p of this orbit such that any $y \in [p,c]$ is attracted to p under f^n, where n is the period of p. We prove only for the case of $p < c$, $p > c$ can be treated similarly. Of course, we can let p be the central one, and clearly its attracted interval is at least $[p,p']$, which means that there is no any fixed or periodic point of f^n in $(p,p']$. Moreover, by the above lemma, we have that p is restrictive since $p < f^n(c) < c < p'$.

(3) This last case is just the classic result of Guckenheimer, who proved in [32] that:

Theorem 4.2.16 (Guckenheimer [32]). *Suppose an S-unimodal map f has no stable periodic orbit. Then f has sensitivity to initial conditions if and only if there is an integer N such that $n \geq N$ implies f^n does not have a restrictive central point.*

We do not repeat the prove here, but just point out why his condition is qualified in our case, although the definition of restrictive central points is not exactly same.

We have shown that, it is only in the case above that f can have a stable periodic orbit, hence in this third case f has no stable periodic orbit. For the problem that Guckenheimer requires a restrictive central point being repelling, the arguments of above case also imply that all the restrictive central points in this case must be repelling too, hence there is no contradiction with his definition. Finally, taken the period of the restrictive central periodic point p closest to c as the required N in the theorem, we have that all the restrictive central points of f have period no more than N, because $[p',p]$ is inside all of the central intervals, so N is a multiple of period of any restrictive central point. That is, for any $n \geq N$, f^n does not have a restrictive central point, so the attractors in this case must be cycles of intervals.

□

Last, we present a description of the dynamical structures of attractors with their patterns, which can exhibit more characters of an attractor than only its type and period. Theorem 4.2.14 implies that, using the operation we defined for extension patterns, we can display the detailed block structure of an attractor. Namely, for the cases of periodic orbits and cycles of intervals, if the finite set of restrictive central periodic points of an S-unimodal map f is $\{p_0, p_1, \ldots, p_k\}$ with $p_{i+1} \in [p_i, p_i']$ for $0 \le i < k$, then

$$\mathcal{O}(p_k) = \mathcal{O}([p_0, p_0']) \ltimes \mathcal{O}_{[p_0,p_0']}([p_1, p_1']) \ltimes \ldots \ltimes \mathcal{O}_{[p_{k-1},p_{k-1}']}([p_k, p_k']) \ltimes \mathcal{O}_{[p_k,p_k']}(p_k)$$

gives the pattern of the stable periodic orbit of p_k. If the attractor is cycle of intervals inside these restrictive intervals, then its pattern is

$$\mathcal{O}([p_k]) = \mathcal{O}([p_0, p_0']) \ltimes \mathcal{O}_{[p_0,p_0']}([p_1, p_1']) \ltimes \ldots \ltimes \mathcal{O}_{[p_{k-1},p_{k-1}']}([p_k, p_k']).$$

For a solenoidal attractor, we allow to apply this operation infinitely times, thus its pattern can be written as

$$\mathcal{O}(A) = \mathcal{O}([p_0, p_0']) \ltimes \mathcal{O}_{[p_0,p_0']}([p_1, p_1']) \ltimes \ldots \ltimes \mathcal{O}_{[p_{k-1},p_{k-1}']}([p_k, p_k']) \ltimes \ldots.$$

Notice that, if an attracting periodic point p_k moves continuously cross c with a varying parameter of maps, the pattern denoted by $\mathcal{O}_{[p_{k-1},p_{k-1}']}([p_k, p_k'])$ actually changes from type n to type $2n$. However, it is usually apparent that for which case it represents when we use this notation later, so we will not distinguish these two cases unless necessary.

4.3 Bifurcations and transition of full family

In this section we discuss the bifurcations during the transition of an S-unimodal family. Theorem D is our first main result of this chapter, which explains the mechanism of bifurcations for attractors of cycles of intervals. A bifurcation of this type can be viewed as the reverse of a bifurcation of periodic orbit, their correspondence is shown by restrictive intervals. The bifurcations of periodic orbits product new orbits of periodic points together with new restrictive intervals linked to them, and the reverse bifurcations happen when the restrictive intervals become non-restrictive. We present these details in the first subsection.

The transition between each pair of such corresponding bifurcations is a kind of procedure of full family. This provides us an integrated perspective of the transition of S-unimodal families, particular the self-similarity during it. The second subsection is devoted to a short review of this transition process.

4.3.1 Reverse bifurcations as bands merging

As we have shown with particular examples in the first section of this chapter, the reverse bifurcations of attractors are in form of intervals merging, which yield cycles of bigger size and less period. In this subsection we exhibit precisely the mechanism of their changes. The key factor for the mechanism is just those restrictive intervals, actually it is the destruction of the innermost layer yields the merging of intervals. On the other hand, the birth of a new layer of restrictive intervals is due to the bifurcation of a new periodic orbit, hence there are natural relation between these two bifurcations which explains why a merging of intervals is called as "reverse" bifurcation.

First we review briefly the theories on the bifurcations of periodic orbits. These knowledge are commonly known in popular textbooks, so we do not go into much details. A special issue is on the intermittency phenomena linked with saddle-node bifurcations. The discussions on this issue are necessary for clarification of the mathematical nature with numerical appearance of the attractors. After that, we start to investigate the reserve bifurcations. In Theorem D we present their precise mechanism by extension patterns of restrictive intervals. At the last of this section, we give some examples to illustrate the situation much clear.

Bifurcations of periodic orbits

Generally, the qualitative changes of dynamics with respect to parameters are known as *bifurcations*. Here for us, this problem is given by how the state and periodicity of the attractor change with the parameter μ of an S-unimodal family f_μ varying. Roughly speaking, when there is a qualitative change at μ, one says that μ is a bifurcation or critical value of the parameter. Concerning those bifurcations which involve periodic orbits, assume $f_\mu^n(p) = p$ and that p is periodic with prime period n, the behaviours of orbits near p are usually decided by the number $\lambda(p) = \frac{d}{dx} f_\mu^n(p)$. If $|\lambda(p)| < 1$, then f^n is a contraction in some neighborhood of p. Hence, for x close enough to p, $f^{ni}(x) \to p$ as $i \to \infty$. On the other hand, if $|\lambda(p)| > 1$ there is some neighborhood of p such that p is the only point which stays always inside this neighborhood. Moreover, the implicit function theorem implies that when $\lambda(p) \neq 1$ there is a periodic point $p(\mu)$ of prime period n depending smoothly on μ.

Thus the bifurcations of periodic orbits for S-unimodal maps are of two sorts. The number of periodic orbits of a given prime period n can only change at a value of μ for which there is a periodic point p of period n with $\lambda(p) = 1$. The stability of a periodic orbit only changes when $|\lambda(p)| = 1$. Bifurcations take place "generically" in the two cases $\lambda(p) = \pm 1$ are the well-known saddle-node type and the period-doubling one correspondingly. The detailed analytic forms of their sufficient conditions and the proofs can be found in any common textbook. We refer to [15, 31] for discussions focused on S-unimodal families.

Briefly, a period-doubling bifurcation happens at the place that an originally existed periodic orbit of period n loses its stability with a new periodic orbit of period $2n$ occurring around it. When a periodic point $p(\mu)$ of prime period n depends smoothly on the parameter μ, the period-doubling bifurcation takes place at μ_0 where $\lambda(p(\mu_0)) = -1$. The orbit loses its stability if $\lambda(p(\mu)) < -1$ with parameter μ varying. From the restrictive intervals point of view, we know by Corollary 4.2.2 that, with the central point of an attracting periodic orbit moving to another side of c, the derivative λ of this orbit changes from positive to negative, and the original n type restrictive intervals of this orbit change to be $2n$ type. When finally this orbit loses its stability, a new attracting orbit occurs with each of these $2n$ type restrictive intervals containing one point of the new orbit inside. For typical S-unimodal families, for example, a quadratic one, the period-doubling scenario can be seen clearly in its transition diagram of attractors. Recall that the parameters of this scenario are inside each connected component of the dense set of parameter space, which consists of countably infinitely many nontrivial intervals.

A saddle-node bifurcation takes place when the graph of $f^n_{\mu_0}$ for some n touches the diagonal tangentially at some μ_0, so there occurs a periodic orbit of period n with $\lambda(p(\mu_0)) = 1$ if this derivative of point $p(\mu_0)$ exists. At one side of μ_0, this intersection does not happen, the orbit does not exist. As μ varies to the other side of μ_0, the graph of f^n_μ meets the diagonal at two points, the new orbit splits into two with one attracting and the other one repelling. In the transition diagram, this attracting one can be seen as the attractor provided its period n is rather small. Furthermore, unlike a period-doubling bifurcation at which the new orbit splits out of an existed attracting orbit, the new orbit of a saddle-node bifurcation is observed occurring from a chaotic attractor of cycle of intervals. The dynamical behaviour on such cycles of intervals is known as (type I) intermittency introduced by Pomeau and Manneville in [65].

Intermittency is regarded as a route to chaos in context of physics. The phenomena observed before a saddle-node bifurcation are as follows. There appears to be a chaotic orbit in the system, see Figure 4.6(a) for example. Examining the chaotic orbit for parameter close to the bifurcation value, the character of its transition is that: the orbit appears to be a period orbit for long stretches of time as the same period of the orbit born by the saddle-node bifurcation. But after that, there is a short burst (the "intermittent burst") of chaotic-like behaviour, followed by another long stretch of almost period behaviour, followed again by a chaotic burst, and so on. The average duration of the long stretches between the intermittent bursts becomes longer and longer, and approaches infinity with the pure periodic orbit appears at the bifurcation value.

Remark 4.3.1. Notice that, if one only looks at the picture of transition diagram, it seems that the parameters for intermittency occurring occupy quite a large nontrivial interval. But we know that, the parameter set of cycles of intervals is a completely disconnected set for a quadratic family, so cycles of intervals cannot appear in a whole connected components of the parameter space. In fact, a long stable cycle is indistinguishable from non-periodic motion, and is discernible due to both its high period and short occurrence. Hence, whether or not an attractor in this case is truly non-periodic is difficult to judge only by numerical simulation. One should be aware that, any nontrivial parameter interval does contain dense parameters for maps with stable periodic orbits. But close a parameter value with chaotic attractor, we are likely to find also interval attractors. Such values are almost all Lebesgue density points where measure accumulates. We will meet this similar situation when we deal with reverse bifurcations. $\diamond$

The reverse bifurcations

A reverse bifurcation happens when a set of restrictive intervals become non-restrictive, its performance is that some intervals of the attractor merge into a larger size one, as we show in Figure 4.1. Theorem D demonstrates the concrete mechanism of band merging, given by their extension patterns. The bifurcation value is the so-called Misiurewicz point. Before we exclusively investigate reverse bifurcations, we make a short exposition on the necessary known results on the dynamics at such critical values.

To be precise, we deal with a family of one-parameter S-unimodal maps $f_\mu(x)$ on the interval $I=[a,b]$, and assume that f_μ is also continuous with respect to the parameter μ. Denote by p_μ the restrictive central point of period n of some f_μ. For a value μ_0 such that $f_{\mu_0}^{K_p}(c)=p'_{\mu_0}$ with $K_p=n$ or $2n$ the period of restrictive intervals of orbit p_{μ_0}, we study what happens locally when the parameter μ passes through μ_0. Here at μ_0, we have $f_{\mu_0}^{K_p+1}(c)=f(p_{\mu_0})$, so μ_0 is a *Misiurewicz point* which means that the critical point c is preperiodic (i.e., it becomes periodic after finitely many iterates but is not periodic itself). The map f_{μ_0} at this point belongs to the set of uncountably many Misiurewicz maps in generic one-parameter families. In [59] Misiurewicz proved that Misiurewicz maps admit absolutely continuous invariant measures. Now it is known that the parameters of such maps are Lebesgue density points of parameters corresponding to absolutely continuous invariant measures.

Moreover, at the two sides of such μ_0, there are different types of behaviours. For values of μ at the side of $f_\mu^{K_p}(c)\in[p'_\mu,p_\mu]$, the intervals are kept to be restrictive, and with μ approaching to μ_0 the homoclinic bifurcation takes place. A good introduction on such situations can be found in the textbook [15] of Devaney. It shows that this critical value μ_0 is an accumulation point of infinitely many saddle-node and period-doubling bifurcations, and $f_{\mu_0}^{K_p}|_{[p'_{\mu_0},p_{\mu_0}]}$ is conjugate with the shift map of the sequence space on two symbols, hence is chaotic in the sense of Devaney. Particularly, the attractor at μ_0 is exactly the cycle of intervals of $[p'_{\mu_0},p_{\mu_0}]$. Notice also that, at this side until μ comes to μ_0, all the attractors have block structures over $\mathcal{O}([p'_\mu, p_\mu])$, which are the same with $\mathcal{O}([p'_{\mu_0}, p_{\mu_0}])$ in pure combinatorial sense.

At the side of $f_\mu^{K_p}(c) \notin [p'_\mu, p_\mu]$, the intervals $[p'_\mu, p_\mu]$ is no longer restrictive for the value of μ. A reverse bifurcation occurs at this side when μ is close to the critical value μ_0. In this case, we know that p_μ is unstable. By general theory, an unstable periodic orbit (in fact any hyperbolic set) persists and moves smoothly under small perturbations of the map. But since the intervals of periodic orbit of p_μ are no more restrictive, the attractor is not limited inside those intervals any longer. While we know that the attractor has to be contained in the innermost layer of restrictive intervals, hence it must be inside the originally second innermost restrictive intervals now. This yields the mechanism of the change of attractors at reverse bifurcations.

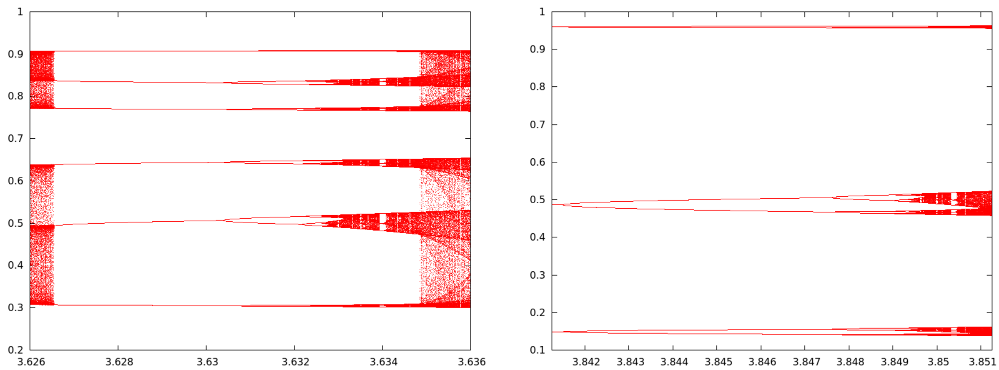
Theorem D. *Suppose that μ_0 is a critical value such that $f_{\mu_0}^{K_p}(c) = p'_{\mu_0}$ for a central periodic point p_{μ_0}. Denote by q_{μ_0} the restrictive central period point who is the second closest to c, that is, with p_{μ_0} the only restrictive central point in (q'_{μ_0}, q_{μ_0}). For value of μ (in Lebesgue measure sense) arbitrarily close to μ_0 with $f_\mu^{K_p}(c) \notin [p'_\mu, p_\mu]$, the attractor changes from a cycle of period $K_{p_{\mu_0}}$ intervals of $[p'_{\mu_0}, p_{\mu_0}]$, to a cycle of intervals of period $K_{q_{\mu_0}}$, contained in the restrictive intervals of orbit $[q'_\mu, q_\mu]$. Precisely for the patterns, as $\mu_0 \to \mu$, we have*

$$\mathcal{O}([p_{\mu_0}]) = \mathcal{O}([q'_{\mu_0}, q_{\mu_0}]) \ltimes \mathcal{O}_{[q'_{\mu_0}, q_{\mu_0}]}([p'_{\mu_0}, p_{\mu_0}]) \to \mathcal{O}([q_{\mu_0}]) = \mathcal{O}([q'_{\mu_0}, q_{\mu_0}]).$$

Proof. We show that, for those parameters μ (in Lebesgue measure sense) arbitrarily close to μ_0 with $f_\mu^{K_p}(c) \notin [p'_\mu, p_\mu]$, the cycle of intervals of orbit $[q'_\mu, q_\mu]$ is the innermost restrictive one of f_μ, and there exists non-restrictive central point inside $[q'_\mu, q_\mu]$. Therefore, our assertion follows by Theorem 4.2.14.

Consider $p_\mu(R)$ the most right endpoint of the set of restrictive intervals of p_μ. By Proposition 4.2.4, $f_{\mu_0}^{K_p}(c) = p'_{\mu_0}$ is equivalent to $f_{\mu_0}(c) = p_{\mu_0}(R)$. Meanwhile, we also have that, $f_{\mu_0}(c) = p_{\mu_0}(R) > s_{\mu_0}(R)$ for any non-restrictive central point s_{μ_0}, and $f_{\mu_0}(c) < q_{\mu_0}(R)$ since q_{μ_0} is restrictive by assumption. That is, $s_{\mu_0}(R) < p_{\mu_0}(R) = f_{\mu_0}(c) < q_{\mu_0}(R)$.

Now for μ arbitrarily close to μ_0 with $f_\mu^n(c) \notin [p'_\mu, p_\mu]$, due to the continuity of $f(\mu, x)$, the above inequality changes to be $s_\mu(R) \leq p_\mu(R) < f_\mu(c) \leq q_\mu(R)$. It means that q_μ is still a restrictive central period point, p_μ becomes non-restrictive, and all the other non-restrictive central points of f_{μ_0} remain to be non-restrictive. So q_μ is innermost restrictive central period point now, with all the other central period points inside (q_μ, q'_μ) being non-restrictive.



(a) Reverse bifurcation of saddle-node type with a period 6 cycle.

(b) Reverse bifurcation of period-doubling type with a period 6 cycle.

Figure 4.6: Examples of nested restrictive intervals.

By "in Lebesgue measure sense", we mean the following. It is similar with what we introduce for intermittency and also the homoclinic bifurcation at the other side of μ_0: in any S-unimodal family, parameters corresponding to cycles of intervals cannot appear in any whole connected component of the parameter space. In fact, in any connected interval of μ_0, there are infinitely many saddle-node and period-doubling bifurcations of periodic orbits, discernible with high periods and extremely short life. These infinitely many tiny windows open and then close very quickly, and all locate in restrictive intervals of orbit $[q'_\mu, q_\mu]$ which comes continuously from the orbit of $[q'_{\mu_0}, q_{\mu_0}]$ with the same pattern. The closed window gives attractor of cycle of intervals exactly with this pattern. The opening one cannot be distinguished hence also look like this cycle of intervals. μ_0 is a Lebesgue density point where measure accumulates, which means that one can observe chaotic interval attractors in positive possibility, thus those nearby attractors all look like chaotic bands in the transition diagram by numerical simulation (see Figure 4.6 for examples). $\square$

Maybe the mechanism given above by patterns of restrictive intervals is lack of intuition, because it is hard to know actually the related restrictive intervals at an abstractly given parameter. The situation becomes much clear if one dates back to the birth of these intervals in the transition of family. Since any set of restrictive intervals must be linked to a periodic orbit, hence its state is certainly decided by the bifurcation which gives birth of this orbit. The next examples that we display illustrate clearly the relation between the birth of restrictive intervals via bifurcation and their destruction as the reverse. Moreover, this is very helpful for the understanding of the overall transition structure of a family.

We give two examples of reverse bifurcations of the logistic family $f_\mu(x) = \mu x(1-x)$, each corresponds to a generic bifurcation of periodic orbits, the saddle-node and the period-doubling one respectively. Apparently, it looks like they exhibit two different behaviours, so they are thought as two sorts of phenomena in physical context. We will explain their detailed mechanism, which exhibits that their different appearances are natural from the point of view of the restrictive intervals.

Example 4.3.2. Our first example is shown in Figure 4.6(a), which is a reverse bifurcation of a cycle of intervals of period 6, occurring at the value about $\mu = 3.6348$ for $f_\mu(x) = \mu x(1-x)$.

In Figure 4.6(a), the attractor appears as a disjoint six-bands, with each three in two groups. It suddenly becomes a two-bands at $\mu = 3.6348$, where the three disjoint bands of each group merge directly into one. This is an example of a saddle-node type, because the restrictive intervals of those six bands originally occurs with a saddle-node bifurcation at about $\mu = 3.6265$. Notice that before this bifurcation of periodic orbit at $\mu = 3.6265$, the attractor is a two-bands. After this bifurcation, a periodic orbit of period six occurs, with each three points of the new orbit from an original band.

From the restrictive intervals point of view, the saddle-node bifurcation at $\mu = 3.6265$ takes place inside a set of restrictive intervals with two intervals, which contains the two-bands in Figure 4.6(a). After this bifurcation, new sets of restrictive intervals linked to the new periodic orbits appear. Figure 4.4(a) displays one case of this situation, where we plot in boxes the old two restrictive intervals (due to the repelling fixed point, refer to Example 4.2.6) and the new six restrictive intervals of the repelling periodic orbit from the saddle-node bifurcation.

This repelling orbit of period six cannot be seen in transition diagram Figure 4.6(a), but it persists since its appearance from the saddle-node bifurcation. Also it is the case for the restrictive intervals of its orbit, which last until the reverse bifurcation happens. During this process, a series of complicated dynamical transition presents in order, which are what we see in transition diagram Figure 4.6(a). The attractor bifurcates, starting from the period-doubling bifurcation of the attracting orbit by the same saddle-node bifurcation at $\mu = 3.6265$, ending with chaotic bands completely coincident with restrictive intervals of the repelling period six orbit at $\mu = 3.6348$. Notice that, all these bifurcations of transition happen inside these restrictive intervals of this repelling period six orbit, and for each one of them the transition diagram has the same picture of the general logistic family for $\mu \in (1, 4]$. Moreover, the reverse bifurcation makes the attractor go back to a two-bands just like that before $\mu = 3.6265$. This is because the innermost restrictive intervals come back to exact those before the saddle-node bifurcation.

This type of reverse bifurcation for a saddle-node bifurcation is common refer as (interior) "crisis" in physics, because it always presents a sudden change of jump in size of a chaotic attractor, with several bands merge into one piece. There is also evidence in the diagram that the density of points in the large attractor near the crisis concentrates in the original bands, and gradually spreads out, indicating another form of intermittency. These are its main character, which is for all the $K = n$ type of restrictive intervals as well. Another example of this type can be seen in Figure 4.1(b). $\diamond$

Example 4.3.3. Our next example in Figure 4.6(b) is also a case of period six. Differently, it is a reverse of period-doubling one, corresponds to the $K = 2n$ type.

The reverse bifurcation takes place at about $\mu = 3.851$. The periodic orbit to which these restrictive intervals linked occurs at about $\mu = 3.8415$, it is the attracting orbit of period three who comes from the saddle-node bifurcation which starts the big period three window.

The type of these intervals changes to be $2n$ (hence period six) since it moves to different side of c with the repelling orbit which comes from the same bifurcation of it. But the restrictive intervals of the attracting orbit are always contained in those restrictive intervals of the repelling orbit, whose period is kept to be three(see Figure 4.4(b)). Also after a series of complicated dynamical transition in order, the reverse bifurcation comes when the chaotic intervals attractor is completely coincident with its restrictive intervals. In this case, every pairs of the restrictive intervals bear at each two sides of the three points of original attracting orbit, the outer series of restrictive intervals is of the repelling period three orbit. Hence at the critical value, the pairs inside each outer intervals merge into one, the periodicity of the chaotic intervals attractor changes from six to three.

This reverse bifurcation of period-doubling type is usually called as band merging. Apparently, it is more "smooth" than the reverse of saddle-node one. This is only because that, the pairs of the inner restrictive intervals are not totally disjoint, but with a common endpoint instead. Therefore, as the attractor spreads inside and occupies the intervals finally, every pairs meet naturally and continuously. $\diamond$

Remark 4.3.4. In the literature, phenomena similar as these reverse bifurcations are considered to be typical and common in nonlinear dynamics, the popular opinion on their reason is that they are caused by apparent collisions between attractors and unstable periodic motions. In S-unimodal case, that is the chaotic intervals attractor touches the unstable periodic orbit. Notice that, for repelling periodic orbits, all the points nearby move away from them. The possible way for an attractor to touch a repelling orbit is that, it hits some preimage of a point of the orbit. The c-symmetric points are exactly the preimages of the central points in S-unimodal case. This mechanism for touching the repelling orbits can also be seen in the quasi-periodically forced systems later. $\diamond$

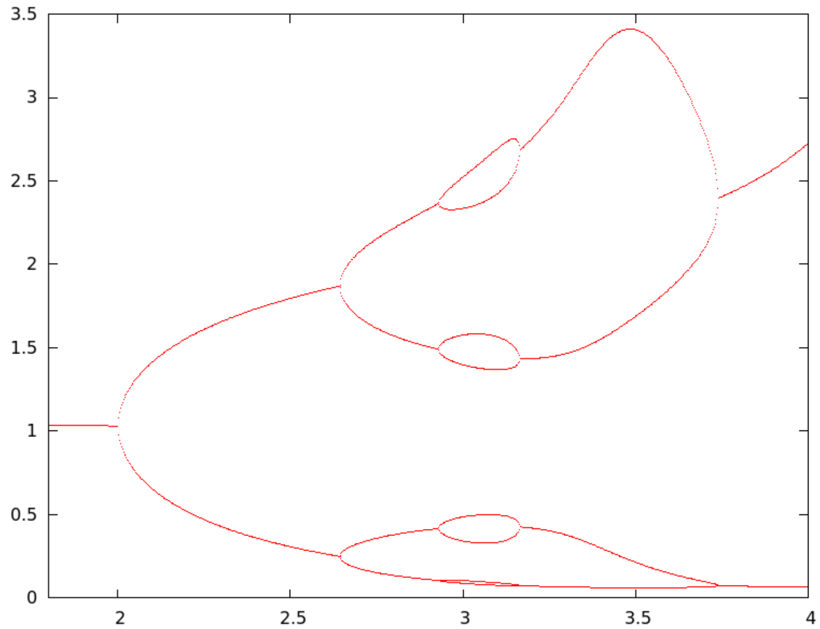


Figure 4.7: Example of not generic family with period-halving.

4.3.2 Self-similarity in transition of S-unimodal family

In this subsection we make a brief descriptive exposition of the transition of a generic S-unimodal family, and the self-similarity during it.

From the restrictive intervals point of view, the transition in a life cycle of each restrictive interval can form a full family. Namely, if $[p, p']$ is a restrictive periodic interval of period K, then from its creation with the occurrence of the periodic orbit of p, to its destruction at reverse bifurcation of $[p, p']$ becoming non-restrictive, $f^K|_{[p,p']}$ has the same transition with a general S-unimodal family. Hence each of such cycle is a unit of self-similarity, since the dynamics evolves with similar structures. This fact in turn provides us an integrated perspective of the transition of S-unimodal families.

For S-unimodal maps, the transition of a family is also an important issue besides the classification of their behaviours, that is, how the overall dynamics evolves with respect to a parameter. The works of Guckenheimer [31] and Devaney [14] are devoted to the systematic genealogy of periodic orbits in the transition, which display that there exists particular regularity on the order of the occurrence and also coexistence for periodic orbits of unimodal maps. People also notice the obvious self-similarity in the transition diagram. In [13], Derrida Gervois and Pomeau exhibited this internal similarity with a composition law of MSS (Metropolis-Stein-Stein) sequences (which is the same as the popular kneading sequences nowadays). Now we exhibit such structures with those restrictive intervals, which is more simple but integrated.

For simplicity, we only consider a generic family as Devaney in [14]. We assume that $f_\mu(c)$ goes from a to b with the increasing of μ. Furthermore, any periodic orbit must always lasts until $f_\mu(c) = b$ after its occurrence at a bifurcation, and must remain unstable since it changes to be so. Hence, the situation like period-halving in Figure 4.7 cannot occur.

Self-similarity of full families

We know that, a set of restrictive intervals occurs with its periodic endpoints at some bifurcation and destructs at the reverse bifurcation where it becomes non-restrictive. For each parameter unit between such two corresponding bifurcations, this is called as an "window" in the transition diagram. It turns out that every such period is a unit of similarity, because of a significant theorem of the full families.

More precise, given a family of one-parameter S-unimodal maps $f_\mu(x)$ on the interval $I = [a, b]$ such that f_μ is continuous with respect to the parameter μ too, a *full family* is the one that its extreme $f_\mu(c)$ goes continuously from one endpoint a to the other b with μ. A prototype is the logistic family $f_\mu(x) = \mu x(1-x)$ for $\mu \in [0, 4]$. For a full family of S-unimodal maps, a theorem (see Guckenheimer [32]) says that, for any unimodal map g, there exists a map f_{μ_0} of this family such that f_{μ_0} and g have the same kneading sequence. This means that all the possible combinatorial dynamics of unimodal maps will occur in a full family. Moreover, the results of Guckenheimer in [32] also show that, there must be some map of a full family which is topologically conjugate to any map of $f_\mu(x) = \mu x(1-x)$ for $\mu \in [0, 4]$.

For our purpose, we may relax a little the restriction of $f_\mu(c)$ moving from a to b on the definition of a full family, and ask $f_\mu(c)$ for going from any $r \in [a, c)$ to b instead. Doing so, we only miss at most one trivial case: a is the only attracting point for all $x \in [a, b]$. Corresponding to $f_\mu(x) = \mu x(1-x)$, it is the case of $\mu \in (1, 4]$. We regard any of such family as a full family also.

Notice that, the transition of a full family of S-unimodal maps can be viewed as inside a (the biggest) restrictive interval, their common domain of definition $I = [a, b]$.

Moreover, if $[p_\mu, p'_\mu]$ is a restrictive interval of period K of f_μ, the maps $f^K_\mu|_{[p_\mu, p'_\mu]}$ form certainly a full family from the creation of $[p_\mu, p'_\mu]$ to its destruction at reverse bifurcation. This is because that, any restrictive interval $[p_\mu, p'_\mu]$ can only occur together with the orbit of p_μ by some bifurcation at μ_0, by Lemma 4.2.15 we know that it must be $f^K_{\mu_0}(c) < c$ whatever p_{μ_0} itself is attracting or is the repelling one at a saddle-node bifurcation. On the other hand, the reverse bifurcation is at some μ_1 where $f^K_{\mu_1}(c) = p'_{\mu_1}$.

This fact demonstrates the self-similarity in the transition of a full S-unimodal family. Since on every life cycle of such a restrictive interval $[p_\mu, p'_\mu]$ above, $f_\mu^K|_{[p_\mu, p'_\mu]}$ is a full family, so all of them has the same transition structure with the full family of S-unimodal maps f on $I = [a, b]$.

In terms of extension pattern, we consider those windows in transition diagram, which display the evolution of attractors with the parameter. For any window, there is a pattern of it, which is given by the restrictive intervals that this window corresponds. In this window, all the pattens of attractors are then extension patterns who have some block structure over its pattern. More precise, if p_μ is a central restrictive periodic point of f_μ, we denote by $\{\mathcal{O}(p_\mu)\}$ the set of all the patterns of the attractors who have a block structure over $\mathcal{O}([p_\mu, p'_\mu])$. That is,

$$\{\mathcal{O}(p_\mu)\} := \{\text{patterns with form denoted by } \mathcal{O}([p_\mu, p'_\mu]) \ltimes *\},$$

here $*$ is any admissible patten for an S-unimodal map, that is, a pattern of some attractor of $f_\mu(x) = \mu x(1-x)$ for at least $\mu \in (1, 4]$.

Structures of transition

Below we make some short arguments on the overall structures of the transition diagram for full S-unimodal families, by considering the pattens inside a window.

The following two facts are keys for understanding the structure of a full family and transition of topological attractors:

- similarity: for every restrictive central periodic point p, $\{\mathcal{O}(p)\}$ is one-to-one with $\{\mathcal{O}(I)\}$.

- For sets of $\{\mathcal{O}(p)\} \neq \{\mathcal{O}(q)\}$ of two central restrictive points p and q, either there is an inclusion relation of $\{\mathcal{O}(p)\} \subset \{\mathcal{O}(q)\}$ or $\{\mathcal{O}(p)\} \supset \{\mathcal{O}(q)\}$; or they are disjoint ($\{\mathcal{O}(p)\} \bigcap \{\mathcal{O}(q)\} = \emptyset$) with an order relation of $\{\mathcal{O}(p)\} < \{\mathcal{O}(q)\}$ or $\{\mathcal{O}(p)\} > \{\mathcal{O}(q)\}$. Here $<$ and $>$ refer to the forcing relation when $\{\mathcal{O}(p)\}$ and $\{\mathcal{O}(q)\}$ are sets of patterns.

  That is, if a window does not embed into another one as a part of it, then they are completely independent.

  Notice that, if $p \in [q, q']$, then $[p, p'] \subset [q, q']$, and so $p(R) = f(c) < q(R)$ implies that q is still restrictive at the moment of $[p, p']$ becoming non-restrictive.

So we can try to arrange the parameter space of a full family according to the patterns of restrictive intervals.

Using the recursive method on the above two rules, we can know more exactly the general structures. In the general interval $I = [a, b]$, if it is $\mu \in$

$[0, 1]$ such that $f_0(c) = a$ and $f_1(c) = b$, there are two series of parameters μ_i^s and μ_i^e for $i \geq 0$ with

$$0 = \mu_0^s < \mu_1^s \ldots < \mu_i^s < \ldots < \mu_i^e \ldots < \mu_1^e < \mu_0^e = 1,$$

such that a new restrictive central interval $[p_i, p_i']$ occurs at μ_i^s and comes to the end at μ_i^e. That is, when $\mu \in (\mu_i^s, \mu_i^e]$, the attractors all have some block structure over $\mathcal{O}([p_i, p_i'])$. Correspondingly, we have the following results on the attractors:

- for $\mu \in (\mu_i^s, \mu_i^e]$, the period of a periodic attractor (orbit or cycle of intervals) is $s \cdot 2^i$ for some $s \in \mathbb{Z}^+$;
- for $\mu \in (\mu_i^s, \mu_{i+1}^s]$, the attractor is a period orbit of period 2^i;
- if the period of a periodic attractor (orbit or cycle of intervals) is $s \cdot 2^i$ for some prime number s, then it can only occur at $\mu \in (\mu_{i+1}^e, \mu_i^e]$.

Notice that, any periodic orbit with prime number period can only occur after the central interval of the repelling fixed point becomes non-restrictive, that is, after the reverse bifurcation terminates the period 2 window. For the logistic family $f_\mu(x) = \mu x(1-x)$, it is after $\mu_1 \approx 3.6785735$, just as we discussed in Example 4.2.6. For any two such orbits at this part, each of them corresponds an independent window. More generally, this is also the case for all periodic orbits who are not a period-doubling type, for example, the periodic orbit of period 4 with points $p_2 < p_3 < p_0 < p_1$.

The final thing is, this is the same structure in every window of restrictive intervals $\mu \in [\mu_0, \mu_1]$. We just need replace $\mu \in [0, 1]$ by $[\mu_0, \mu_1]$, and change all those period with 2^i to be with $2^i n$ for n the period of the pattern of the window (refer to Figure 4.6). Although, there are infinitely countably many windows inside every window, and there are also infinitely countably many windows between any two independent windows, we still get a clear perspective of the general structure.

4.4 Quasi-periodically forced S-unimodal maps

This final section is devoted to the investigation of periodicity of attractors of quasi-periodically forced S-unimodal maps. We propose the mechanism of the change of periodicity of the attractor with respect to the increasing of forcing terms by Claim E, which says that generally the pattern of attractor goes from inner layer to outer according to the extension pattern of the unforced S-unimodal map. Our analyses demonstrate the reason of this mechanism, which is similar with reverse bifurcations of S-unimodal maps and is clearly illustrated by numerical evidences.

More precise, a quasi-periodically forced S-unimodal system is given by map $F\colon \mathbb{S}^1 \times I \to \mathbb{S}^1 \times I$ of the form:

$$\begin{cases} \theta_{n+1} &= R(\theta_n) = \theta_n + \omega \pmod 1, \\ x_{n+1} &= \psi(\theta_n, x_n), \end{cases} \tag{4.2}$$

where $(\theta, x) \in \mathbb{S}^1 \times I$. Here $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is the unit circle and $\omega \in \mathbb{R} \setminus \mathbb{Q}$ is a fixed irrational number. The function $R\colon \mathbb{S}^1 \to \mathbb{S}^1$ denotes an irrational rotation of the circle $\mathbb{S}^1$ by the fixed angle ω as usual. Furthermore, $\psi(\theta_n, x_n)$ is a continuous function on both x and θ. For a fixed θ, the fibre map $\psi_\theta(x)$:

$$\psi(\theta, \cdot)\colon \{\theta\} \times I \to \{R(\theta)\} \times I$$

is a function of S-unimodal map $f(x)$ perturbed by some function of θ, which is in form of $\psi_\theta(x) = f(x) \cdot g_\epsilon(\theta)$ or $\psi_\theta(x) = f(x) + g_\epsilon(\theta)$. Here $\epsilon \geq 0$ is used as a parameter to control the perturbation given by the forcing function $g(\theta)$. Particularly, if $\psi_\theta(x) = f(x) \cdot g_\epsilon(\theta)$, we assume that $g_\epsilon(\theta) \geq 0$ so that the S-unimodal structure can be maintained in the fibre maps. In any case, we suppose that $\psi_\theta(x) = f(x)$ for all $\theta \in \mathbb{S}^1$ if $\epsilon = 0$.

For such systems, when f is fixed, the general mechanism of the change of periodicity with respect to the increasing of ϵ is as below.

Claim E. *Suppose the attractor of the unforced S-unimodal map $f(x)$ is contained in restrictive intervals of pattern*

$$\mathcal{O}([p_k]) = \mathcal{O}([p_0, p_0']) \ltimes \mathcal{O}_{[p_0, p_0']}([p_1, p_1']) \ltimes \ldots \ltimes \mathcal{O}_{[p_{k-1}, p_{k-1}']}([p_k, p_k']).$$

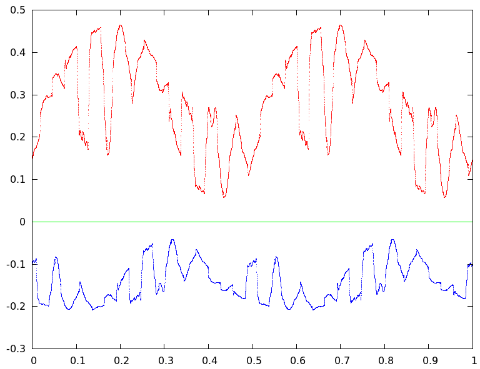
With f being fixed, let ϵ increase from 0, we can see a process that the attractor of quasi-periodically forced systems (4.2) *becomes stripes on the cylinder with patterns step by step (maybe not monotonically)*

$$\mathcal{O}([p_k]) \to \mathcal{O}([p_{k-1}]) \to \ldots \to \mathcal{O}([p_1]) \to \mathcal{O}([p_0]).$$

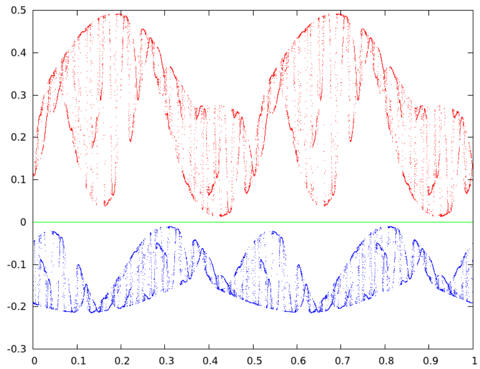
That is, the attractor merges into stripes with bigger size and less period, according to the block structure of $f(x)$ in general order.

Moreover, each of these merging bifurcations happens at the time that the attractor goes beyond the old block region containing them, similar with these reverse bifurcations of restrictive intervals in S-unimodal family.

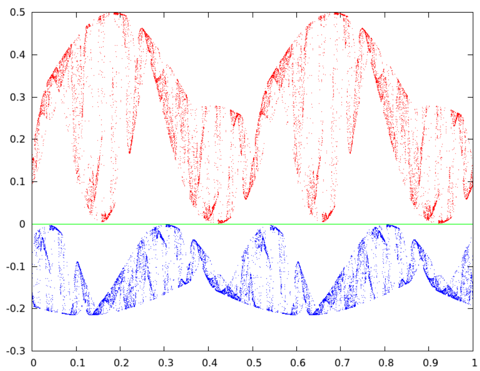
Such systems were studied extensively with numerical methods, and many phenomena were reported already, which can be seen in an exclusive summary book [24] and references therein. It was found that, similar phenomena like bands merging, interior crisis in S-unimodal families also occur for the attractors of such systems. Different with S-unimodal families, the successive period-doubling bifurcations of periodic orbit, which occur with the increasing of the maximum of S-unimodal maps, cannot always



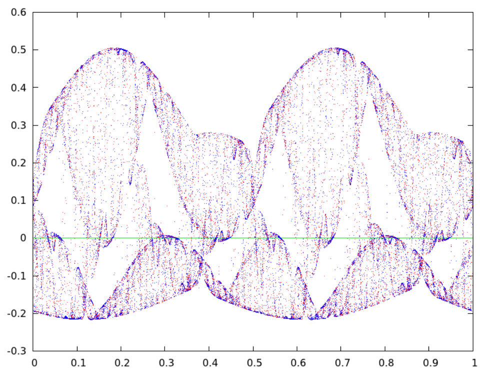
(a) Attractor for $\epsilon = 0.59$.



(b) Attractor for $\epsilon = 0.63$.



(c) Attractor for $\epsilon = 0.64$.



(d) Attractor for $\epsilon = 0.65$.

Figure 4.8: Attractors of system (4.3) for different parameter values.

continue in any forced systems. We will make simple and informal analysis of the reason of these bifurcations, it can be seen that this process according to the extension pattern is certainly reasonable.

We have known that, due to the irrational rotation of the base map, there cannot exist any fixed or periodic points in a quasi-periodically forced system. As a closed invariant subset, the projection of any attractor to the base space has to be the whole circle $\mathbb{S}^1$. The simplest invariant closed subset can only be the graph of a map from $\mathbb{S}^1$ to $\mathbb{R}$. Particularly, the attractors of quasi-periodically forced S-unimodal maps (4.2) are in forms of strips introduced in [3, 21]. That is, let $\pi : \mathbb{S}^1 \times I \to \mathbb{S}^1$ the projection from $\mathbb{S}^1 \times I$ to the base circle $\mathbb{S}^1$, a *strip* in $\mathbb{S}^1 \times I$ is a closed set $A \subset \mathbb{S}^1 \times I$ such that $\pi(A) = \mathbb{S}^1$ and $\pi^{-1}(\theta) \cap A$ is a closed interval (perhaps degenerate to a point) for a residual set of $\theta \in \mathbb{S}^1$. Thus an attractor of system (4.2) is in a form of union of n strips $A = A_1 \cup A_2 \cup \ldots \cup A_n$, such that $A_i \neq A_j$ if $i \neq j$ and $F(A_i) = A_{i+1 \pmod 1}$ for $1 \leq i \leq n$. Moreover, A is also transitive, which is the closure of a dense orbit. We call n the period of this attractor A and each A_i a periodic strip of the attractor. Notice that, $A_i \neq A_j$ for $i \neq j$ does not mean that $A_i \cap A_j = \emptyset$, this intersection may be a nowhere dense set.

To analyze the mechanism we claim, first let us look at a particular example, whose transition is shown briefly in Figure 4.8.

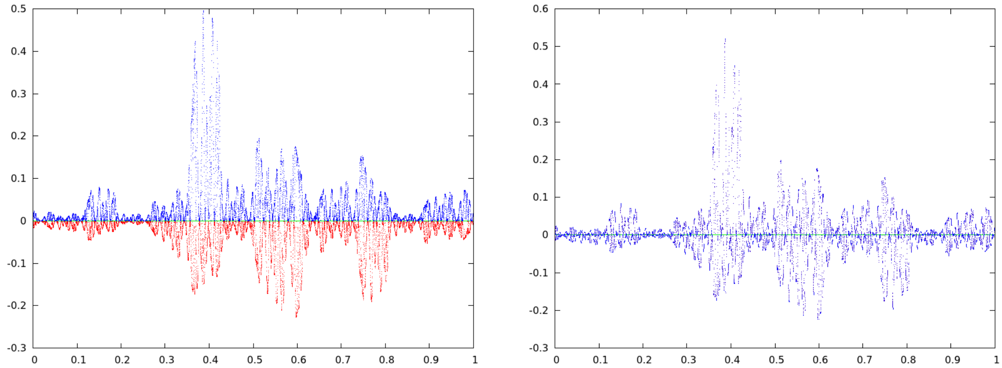
Example 4.4.1. Let the system given by the following map:

$$\begin{cases} \theta_{n+1} & = \theta_n + \omega \pmod 1, \\ x_{n+1} & = (1 + \epsilon|\cos(2\pi\theta_n)|)\, 2.1 x_n(x_n - 0.5). \end{cases} \tag{4.3}$$

In this example, the forcing term is $1 + \epsilon|\cos(2\pi\theta_n)|$ with parameter ϵ. We choose specially the unforced map to be $2.1x_n(x_n - 0.5)$, for the reason that $x = 0$ is always an invariant graph, with a preimage $x = 0.5$ in this system.

In fact, it is equivalent to a change of variables of the common logistic maps $\mu x(1 - x)$, so that its second fixed point $x = 1 - 1/\mu$ is moved and fixed to $x = 0$ in new system given by map $2.1x_n(x_n - 0.5)$. In forced systems (4.3), now the region of $\mathbb{S}^1 \times [0, 0.5]$ is equivalent to the central interval $[0, 0.5]$ of the fixed point $x = 0$ of the unforced map $2.1x_n(x_n - 0.5)$.

Notice that, we have $x_{n+1} < 0$ whenever $0 < x_n < 0.5$ and $x_{n+1} > 0$ if $x_n < 0$. Moreover, there must be a graph φ in the negative part $\mathbb{S}^1 \times (-\infty, 0)$ who is one of the preimage of the graph $x = 0.5$. When graph $x = 0$ is repelling and the region enclosed by φ and $x = 0.5$ is invariant, any attractor inside this region must be periodic strips whose period is a multiple of 2. This is displayed clearly in the pictures of the attractors for ϵ from 0.59 to 0.63, which we plot in Figure 4.8 using different colors for the even and odd iterates. Certainly, with the top of the attractor more close to



(a) Attractor for $\epsilon = 1.236$.

(b) Attractor for $\epsilon = 1.237$.

Figure 4.9: Attractors of $|1 + \epsilon \cos(2\pi\theta_n))|\, 3.3x(x - 0.5)$ for two parameters.

$x = 0.5$, the lower bound of its upper strip becomes more close to $x = 0$. During the process that the top of the attractor approaches to $x = 0.5$, the attractor looks more and more complicated, their pictures illustrate the fractalization process proved by Jorba and Tatjer in [41]. However, notice that the periodicity never changes as long as the top does not touch $x = 0.5$.

The change comes when the attractor finally goes beyond $x = 0.5$. This terminates the invariance of the region enclosed by φ and $x = 0.5$, then the period of the attractor reduces accordingly. We notice that, in the case that there is some iterates of point with $x_n > 0.5$, it must be $x_{n+1} > 0$ too, hence the period must change for such orbit. Furthermore, this implies that the attractor must intersect with the graph $x = 0$ and hence at a dense set on it. What follows is, the two bands separated by $x = 0$ before, now both cross over $x = 0$ and mix at the neighborhood of it and finally at other region everywhere, as in Figure 4.8(d).

Another example with $x_{n+1} = |1 + \epsilon \cos(2\pi\theta_n))|\, 3.3x_n(x_n - 0.5)$ show also this critical point of $x = 0.5$ clearly. This system is pinched since $\epsilon \geq 1$. We can see in Figure 4.9(a) that the attractor is a periodic SNA of period 2, whose top is very close to but still lower than $x = 0.5$. Notice that, in this situation, the two parts of the attractor, which are upper and lower semicontinuous graphs respectively, are not mixed, although they intersect at $x = 0$ densely. While the two parts do merge into one piece when the top of the attractor goes forward over $x = 0.5$ as in Figure 4.9(b). $\diamond$

Remark 4.4.2. The models of this example are modified version of the so-called "HH" model introduced by Heagy and Hammel in [35], where they tried to find an SNA by intersection of two periodic curves. They observed that the two curves mix into one piece immediately after their collision, and related this phenomenon with the bands merging introduced by Lorenz([51]). After them, this kind of bands merging were extensively found between couples of adjacent curves which are corresponding to period-doubling points of unforced logistic maps. A band merging is widely thought as the sign of terminal of period-doubling cascade of periodic invariant curves in quasi-periodically forced systems. $\diamond$

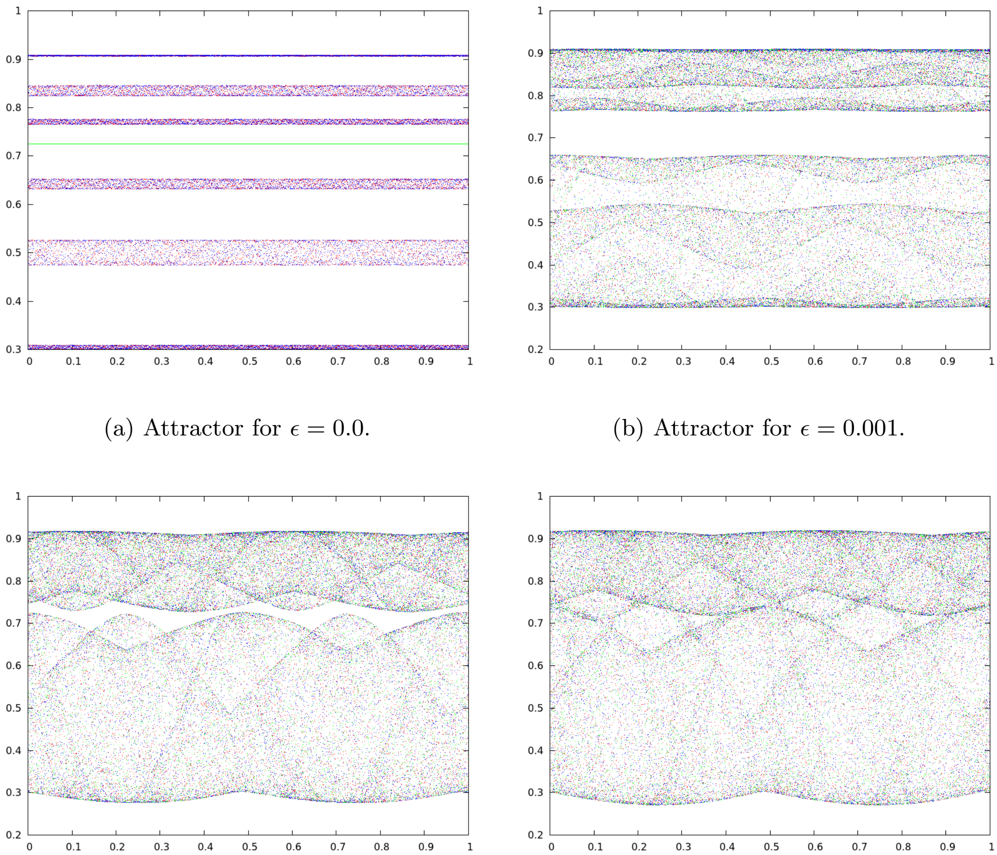
With the original "HH" model, all these analyses have to be carried out on the base of numerical simulations, since all the invariant and periodic curves and their preimages as boundary of invariant region cannot be expressed analytically. Unfortunately, this is the case for almost all the quasi-periodically forced systems that we meet. Even in this artificial example that we choose the models particularly, we cannot get the formula of any invariant periodic curve other than $x = 0$ either. Even though, this example still throws light on our investigation of the periodicity problem of the forced systems.

Analysis on mechanism With the S-unimodal map $f(x)$ of fibre maps $\psi_\theta(x)$ fixed, the sequent bifurcations of S-unimodal families can hardly take place with the varying of forcing term in forced system (4.2). It is because that, those sequent bifurcations are cause by the family parameter μ of function $f_\mu(x)$, but the change of the system (4.2) comes from the parameter ϵ of forcing term now. The effect of perturbation cause by ϵ is mostly shown on the block structure of the attractor of forced system (4.2), rather than to change this structure of $f(x) = f_{\mu_0}(x)$.

Precisely, let us consider the forced system (4.2) starting with the parameter of the forcing term being $\epsilon = 0$. Thus each fibre map is in fact unforced, which is just the original S-unimodal map f itself. In this case, each periodic point of f corresponds exactly a constant periodic curve in the forced system (4.2), with the same periodicity and stability. The pattern structure of all the restrictive intervals of f is also preserved by corresponding strips in system (4.2). In above example, they are strips enclosed by $x = 0$ and $x = 0.5$ and by $x = 0$ and φ.

If we increase ϵ gradually, by the implicit function theorem, all of those repelling periodic invariant curves in (4.2) persist and move smoothly as hyperbolic sets. Particularly those ones who correspond to the endpoints of restrictive intervals preserve the same block structure. Meanwhile, the attractor also changes gradually in its shape and size with respect to the perturbation, until it goes beyond the invariant region which corresponds a forward invariant set of restrictive intervals.

When the attractor breaks through the old invariant region, its period also changes, with its pattern being the new invariant block which is the innermost layer containing it. Due to the continuity of the system, such



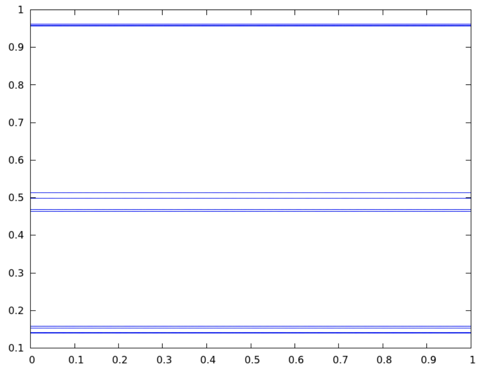
(a) Attractor for $\epsilon = 0.0$.

(b) Attractor for $\epsilon = 0.001$.

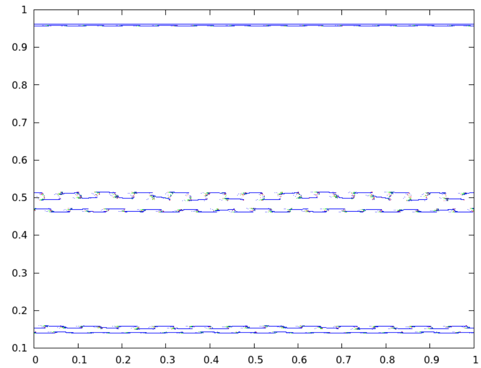
(c) Attractor for $\epsilon = 0.01$.

(d) Attractor for $\epsilon = 0.012$.

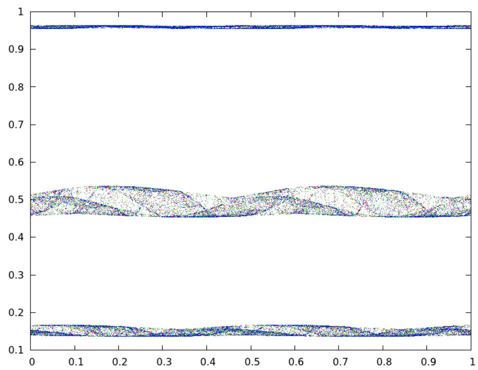
Figure 4.10: Attractors of system (4.4) for $\mu = 3.635$ with different ϵ values.



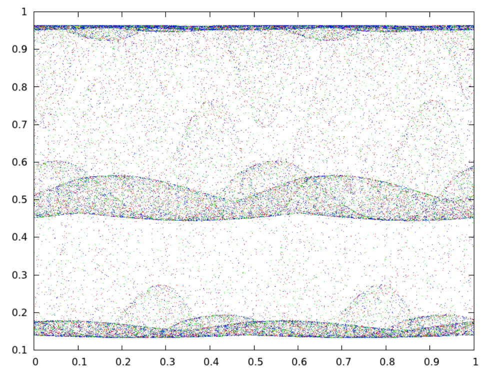
(a) Attractor for $\epsilon = 0.0$.



(b) Attractor for $\epsilon = 0.0001$.



(c) Attractor for $\epsilon = 0.001$.



(d) Attractor for $\epsilon = 0.002$.

Figure 4.11: Attractors of system (4.4) for $\mu = 3.85$ with different parameter values.

breakthrough leads only the attractor to the very outside layer. This means that what we can observe for its combinatorial dynamics is the same as the reverse bifurcation of unforced S-unimodal map f. That is, if the pattern of the attractor before bifurcation corresponds to some $\mathcal{O}([p_0])$ of the unforced map f, then it becomes $\mathcal{O}([p_1])$ after the bifurcation, where the central points $p_0 \in (p_1, p_1')$ with no other central restrictive point inside (p_1, p_1').[1]

More precisely, if the original blocks are of period-doubling type, then what happens is the bands merging of halving its period; for the case of saddle-node type, it is the interior crisis with several disjoint bands suddenly merging into one. This mechanism works for blocks of each layer, and hence the attractor goes all the way out to the final biggest invariant region, and comes into one piece. The example below illustrates intuitively this process of the mechanism we present.

Example 4.4.3. We show the transitions of attractors of two systems both in form of:

$$\begin{cases} \theta_{n+1} &= \theta_n + \omega \quad (\text{mod } 1), \\ x_{n+1} &= (1 + \epsilon \cos(2\pi\theta_n))\, \mu x_n (1 - x_n). \end{cases} \tag{4.4}$$

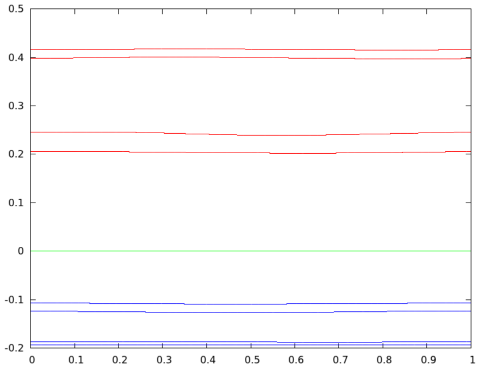
In Figure 4.10 and Figure 4.11 we plot the pictures of some attractors for $\mu = 3.635$ and $\mu = 3.85$ respectively. The corresponding unforced one-dimensional logistic maps have been discussed in Example 4.2.8, and their block structures of restrictive intervals can refer to Figure 4.4.

As we have known, for $\mu = 3.635$, the unforced map has an extension pattern with two layers, the period two blocks form the lower one, with a period three pattern over each of the two blocks. This is shown clearly in Figure 4.10(a) too with $\epsilon = 0$. This picture illustrates in addition that, the attractor of $\mu x_n(1 - x_n)$ at $\mu = 3.635$ is not an orbit of periodic points, but a cycle of intervals instead. So we start here with an attractor of strips of period six. With increasing of ϵ, quickly the three strips inside each block merge into one piece, the period of the attractor becomes two, and then these two bands also merge into one at last.

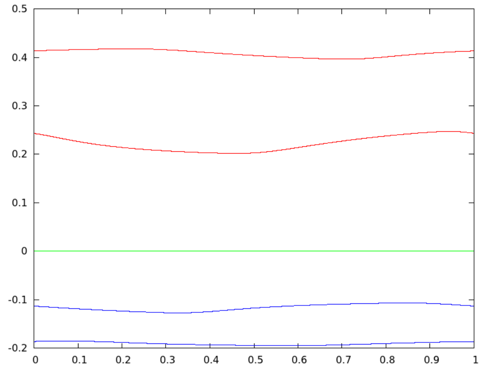
For $\mu = 3.85$, the outer two layers are a period-doubling pattern over each block of period three. In Figure 4.11(a) for $\epsilon = 0$, we see there are periodic curves of period twelve, this means that there are still two more layers by period-doubling bifurcations over the two layers that we have known above. The other pictures of Figure 4.11 exhibit each of these reverse bifurcations in order. That is, three times of the reverse period-doubling, followed by the final merging of the three strips of the outermost layer. $\diamond$

Remark 4.4.4. Finally, we mention some facts as necessary supplements to clarify the general situation more completely.

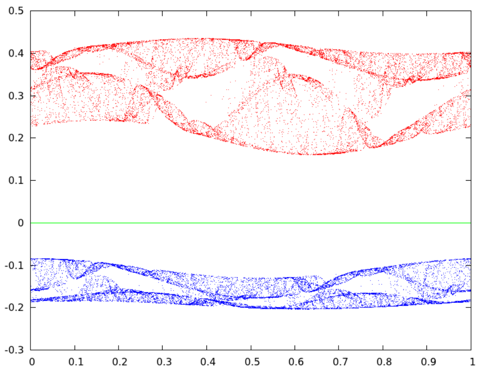
[1] In rare occasion, there may be bifurcation which yields an increase of its periodicity. We shortly explore this non-monotonicity problem in the final Remark 4.4.4.



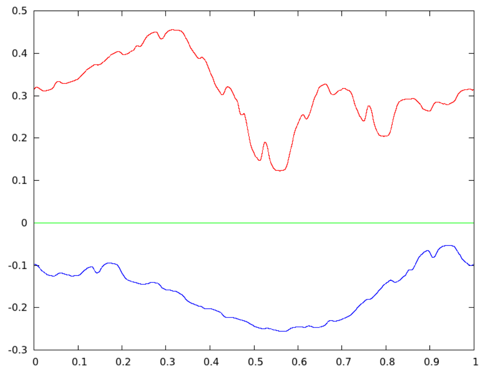
(a) Attractor for $\epsilon = 0.001$.



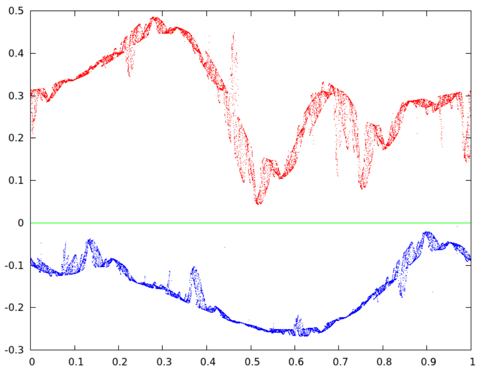
(b) Attractor for $\epsilon = 0.005$.



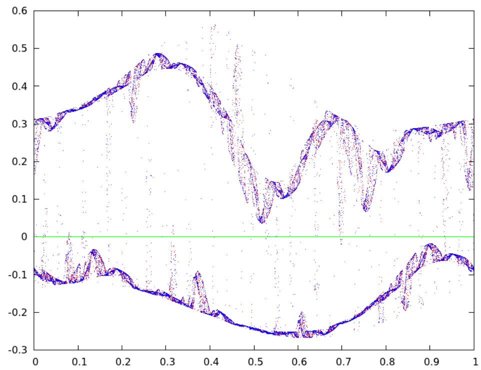
(c) Attractor for $\epsilon = 0.05$.



(d) Attractor for $\epsilon = 0.36$.



(e) Attractor for $\epsilon = 0.377$.



(f) Attractor for $\epsilon = 0.378$.

Figure 4.12: Attractors of system with fibre maps $x_{n+1} = (1 + \epsilon\cos(2\pi\theta_n))\,3.3x_n(x_n - 0.5)$.

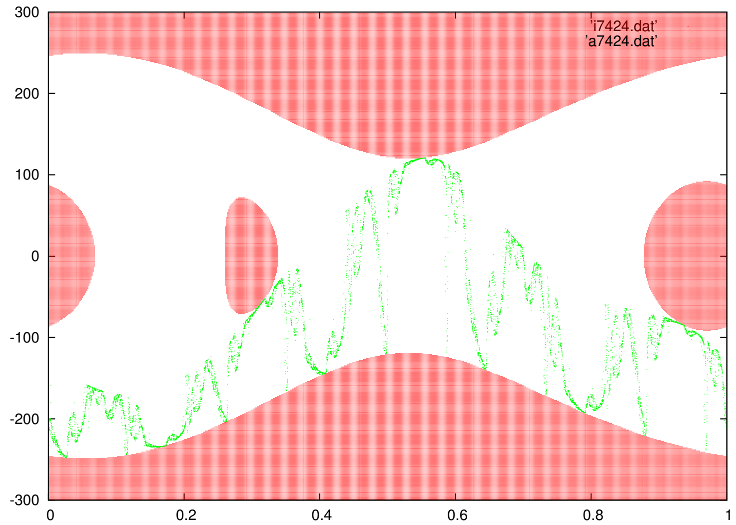


Figure 4.13: Attractor touch boundary of basin of attraction inside invariant region.

(1) Neither the size nor the chaoticity of the attractor increases monotonically with respect to the increasing of the parameter of forcing term completely. Instead, if the attractor lasts in a specific window for a rather long period, its size and chaoticity may change back and forth.

Furthermore, the periodicity does not decrease all the way in every system with a fixed forced map either. Definitively, it is the general tendency that, we can expect bigger size, more chaoticity and less period of the attractor with larger parameter of the forcing term. However, backward behaviours do exist occasionally at certain periods of time.

The precise dynamics of (4.2) depends on both the S-unimodal map fixed and the perturbation, particularly on their actions at the position of the attractor. The attractors in Figure 4.12 are given by fibre maps of $x_{n+1} = (1 + \epsilon \cos(2\pi\theta_n))\, 3.3x_n(x_n - 0.5)$. Starting from $\epsilon = 0$, the attractor consists of periodic curves of period eight, whose corresponding periodic orbit of $x_{n+1} = 3.3x_n(x_n - 0.5)$ comes from four times period-doubling bifurcations of the repelling fixed point $x = 0$. Notice the situation from Figure 4.12(c) to Figure 4.12(d). At $\epsilon = 0.05$ in Figure 4.12(c), the attractor consists of complicated strips just after a reverse bifurcation of bands merging. If the attractor is made of smooth periodic curves before the bands merging of period-doubling type, this bifurcation is normally accompanied with fractalization close to the critical point. However, after a long time of transition to $\epsilon = 0.36$, the two bands of the attractor become very simple and smooth curves.

We also point out that, sometimes the increasing of forcing parameter even can cause a period-doubling bifurcation on the curves of some attractors. However, usually at the next bands merging with large enough perturbations, it takes the attractor back to the same combinatorial type as before. So the mechanism that the pattern decreases according to the block structures is still valid in general.

(2) With any a fixed forcing term, whatever how small it is, the period-doubling cascade of system (4.2) terminates at some moment.

By this we mean that, we give a parameter of the forced S-unimodal map and increase it as what we do for S-unimodal families. Notice that, the period-doubling cascade requires smaller and smaller spaces for their occurrences, which tends to infinitely small finally. This certainly cannot be satisfied by a fixed forcing term, which in general brings a fixed amplitude in any forced system.

(3) A crisis of attractor may be caused by collision inside the block strips, not only by collision at the boundary.

We have shown that, whenever the attractor touches and then goes beyond the boundary of invariant blocks, it breaks the limit of original invariant region. However, it is not the only way for the attractor to break through. Another observed possibility exists, which is illustrated in the case of Figure 4.13. Inside the boundary of invariant region, there are some other points (plotted in red) mapped out of the invariant region, instead of being attracted by the attractor. If such points meet the attractor, then they take way all the points inside the original invariant region.

$\diamond$

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