Journal of Dynamical and Control Systems, Vol. 14, No. 4, October 2008, 505–535 (©2008)
DOI: 10.1007/s10883-008-9049-5

THE POINCARÉ CENTER PROBLEM

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Abstract. We present two examples of real planar polynomial vector fields with an orbitally linearizable saddle point such that they are neither rationally reversible nor Liouvillian integrable. We show that vector fields from one of these examples form an isolated component of the so-called integrable saddle variety. Next, we discuss the problem of partial duality between real centers and real integrable saddles and the problem of continuous moduli for the center variety.

1. Introduction and statement of the main results

A singular point 0 of a real planar analytic vector field V is called a *center* if there exists a neighborhood U of 0 such that $U \setminus 0$ is filled with periodic integral curves of the field. The center is said to be *elementary* if the linearization $dV(0)$ of the vector field is a rotation, otherwise the center is said to be *nonelementary*. A nonelementary center is said to be *nilpotent* if $dV(0)$ is nonzero.

The *center problem* relies on finding the conditions on the coefficients of the Taylor expansion of V at 0 which imply that 0 is a center. Here, we must distinguish between local and global problems.

The *local center problem* (or the *center-focus problem*) was formulated by V. Arnold [1] as follows. The space J^k of k-jets j^kV at 0 of the germs $V : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is divided into three subspaces J^k_s, J^k_u, and J^k_n. The jets from J^k_s (respectively, from J^k_u) are such that any germ W with $j^kW = j^kV$ has 0 as an asymptotically stable (respectively, unstable) point. The subspace J^k_n consists of "neutral" jets. Arnold asked whether the sets $J^k_\#$ are semi-algebraic, or semi-analytic, and conjectured that the codimension of J^k_n grows to infinity together with k. In the case $dV(0) \neq 0$, the problem is algebraically solvable (see [16,18,21,22]). In general, the sets $J^k_{s,u,n}$ may not

2000 *Mathematics Subject Classification.* 35B37.

Key words and phrases. Polynomial vector field, center, integrable saddle.

The first author is supported by the Polish MNiSzW Grant No. 1 P03A 015 29, and the second by a MCYT/FEDER grant number MTM2005–06098–C02–01 and by a CIRIT grant number 2005SGR 00550.