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PLANAR VECTOR FIELD VERSIONS OF CARATHÉODORY'S AND LOEWNER'S CONJECTURES

Carlos Gutierrez and Federico Sánchez-Bringas

Abstract

Let $r = 3, 4, \dots, \infty, \omega$. The C^r-Carathéodory's Conjecture states that every C^r convex embedding of a 2-sphere into $\mathbb{R}^3$ must have at least two umbilics. The C^r-Loewner's conjecture (stronger than the one of Carathéodory) states that there are no umbilics of index bigger than one. We show that these two conjectures are equivalent to others about planar vector fields. For instance, if $r \neq \omega$, C^r-Carathéodory's Conjecture is equivalent to the following one:

Let $\rho > 0$ and $\beta : U \subset \mathbb{R}^2 \to \mathbb{R}$, be of class C^r, where U is a neighborhood of the compact disc $D(0, \rho) \subset \mathbb{R}^2$ of radius ρ centered at 0. If β restricted to a neighborhood of the circle $\partial D(0, \rho)$ has the form $\beta(x, y) = (ax^2 + by^2)/(x^2 + y^2)$, where $a < b < 0$, then the vector field (defined in U) that takes (x, y) to $(\beta_{xx}(x, y) - \beta_{yy}(x, y), 2\beta_{xy}(x, y))$ has at least two singularities in $D(0, \rho)$.

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