Publ. Mat. **66** (2022), 457–540
DOI: 10.5565/PUBLMAT6622202

TOPOLOGICALLY SEMISIMPLE AND TOPOLOGICALLY PERFECT TOPOLOGICAL RINGS

Leonid Positselski and Jan Šťovíček

Abstract: Extending the Wedderburn–Artin theory of (classically) semisimple associative rings to the realm of topological rings with right linear topology, we show that the abelian category of left contramodules over such a ring is split (equivalently, semisimple) if and only if the abelian category of discrete right modules over the same ring is split (equivalently, semisimple). Our results in this direction complement those of Iovanov–Mesyan–Reyes. An extension of Bass' theory of left perfect rings to the topological realm is formulated as a list of conjecturally equivalent conditions, many equivalences and implications between which we prove. In particular, all conditions are equivalent for topological rings with a countable base of neighborhoods of zero and for topologically right coherent topological rings. Considering the rings of endomorphisms of modules as topological rings with the finite topology, we establish a close connection between the concept of a topologically perfect topological ring and the theory of modules with perfect decomposition. Our results also apply to endomorphism rings and direct sum decompositions of objects in certain additive categories more general than the categories of modules; we call them *topologically agreeable* categories. We show that any topologically agreeable split abelian category is Grothendieck and semisimple. We also prove that a module Σ-coperfect over its endomorphism ring has a perfect decomposition provided that either the endomorphism ring is commutative or the module is countably generated, partially answering a question of Angeleri Hügel and Saorín.

2020 Mathematics Subject Classification: Primary: 16W80; Secondary: 16K40, 16L30, 16N40, 18E10.

Key words: topological rings, discrete modules, contramodules, semisimple abelian categories, perfect decompositions, projective covers, topological perfectness.