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SOME EXTENSIONS OF THE MODULAR METHOD AND FERMAT EQUATIONS OF SIGNATURE $(13, 13, n)$

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Abstract: We provide several extensions of the modular method which were motivated by the problem of completing previous work to prove that, for any integer $n \geq 2$, the equation

$$x^{13} + y^{13} = 3z^n$$

has no non-trivial primitive solutions. In particular, we present four elimination techniques which are based on: (1) establishing reducibility of certain residual Galois representations over a totally real field; (2) generalizing image of inertia arguments to the setting of abelian surfaces; (3) establishing congruences of Hilbert modular forms without the use of often impractical Sturm bounds; and (4) a unit sieve argument which combines information from classical descent and the modular method.

The extensions are of broader applicability and provide further evidence that it is possible to obtain a complete resolution of a family of generalized Fermat equations by remaining within the framework of the modular method. As a further illustration of this, we complete a theorem of Anni–Siksek to show that, for $\ell, m \geq 5$, the only primitive solutions to the equation $x^{2\ell} + y^{2m} = z^{13}$ are trivial.

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