Pub. Mat. UAB
Nº 13, Juliol 1979

CRITICAL CASE FOR PERIODIC SOLUTIONS OF A CLASS OF NEUTRAL EQUATIONS WITH A SMALL PARAMETER

Pedro Martínez Amores
Sección de Matemáticas - Universidad de Granada

Introduction

In this paper we study a class of neutral functional differential equations which arises from a coupled system of differential-difference and ordinary difference equations that occur in various applications, as electrical circuits with lossless transmision lines, [1]. In [10], [15] is studied the stability of such system and in [13] are given conditions for the existence of periodic solutions. In this paper, we consider a nonlinear system with a small parameter and we study the existence of periodic solutions when the corresponding linear system can have periodic solutions (critical case). This is done by applying the method of Hale [5,6] and its extension to delay diferential equations (see [14]) to the neutral equations obtained from the coupled system after taking into account the results of [10].

1. Notation and summary of known results

Let E^n be a complex n-dimensional linear vector space with norm $|\cdot|$ and let r be a fixed positive number. $C = C([-r, o], E^n)$ is the space of continuous functions $\phi: [-r, o] \longrightarrow E^n$ with norm $|\phi| = \sup\{|\phi(\theta)| : \theta \in [-r, o]\}$.

Suppose D, L are bounded linear operators from C to E^n,

$$D(\phi) = H\phi(0) - \int_{-r}^{o} [d\mu(\theta)]\phi(\theta)$$

$$L(\phi) = \int_{-r}^{o} [d\eta(\theta)]\phi(\theta)$$

where H is an $n \times n$ matrix, $\det H \neq 0$, μ, η are $n \times n$ matrix functions of bounded variation on $[-r, 0]$ with μ nonatomic at zero. We assume μ has no singular part.

If x is a continuous function mapping $[\sigma - r, \infty)$ into E^n, then for any $t \in [0, \infty)$ we define x_t in C by $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$. An autonomous linear homogeneous neutral funtional differential equation (NFDE) is defined to be

$$\frac{d}{dt} D(x_t) = L(x_t) \tag{1.1}$$

A solution $x = x(\phi)$ of (1.1) through a point $\phi \in C$ at $t = 0$ is a continuous function taking $[-r, A)$, $A > 0$, into E^n such that $x_0 = \phi$, $D(x_t)$ is continuously differentiable on $[0, A)$ and equation (1.1) is satisfied on this interval. It is proved in [2,4] that there is a unique solution $x(\phi)$ through ϕ and $x(\phi)(t)$ is continuous in (t, ϕ).

If the transformation $T(t) : C \to C$ is defined by $T(t)\phi = x_t(\phi)$, then it is shown in [11] that $\{T(t), t \geq 0\}$ is a strongly continuous semigroup of linear operators with the infinitesimal generator $A : \mathscr{D}(A) \to C$, $A\phi(\theta) = \dot{\phi}(\theta)$, $\theta \in [-r, 0]$, $\mathscr{D}(A) = \left\{ \phi \in C : \dot{\phi} \in C,\ D(\dot{\phi}) = L(\phi) \right\}$

and the spectrum $\sigma(A)$ of A consist of those λ which satisfy

$$\det \Delta(\lambda) = 0, \quad \Delta(\lambda) = \lambda H - \lambda \int_{-r}^{0} e^{\lambda\theta} d\mu(\theta) - \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta).$$

The fundamental matrix solution of (1.1) is defined to be the $n \times n$ matrix solution of the equation

$$\begin{aligned} D(X_t) &= I + \int_0^t L(X_s)\, ds, \quad t \geq 0 \\ X_0(\theta) &= \begin{cases} 0, & -r \leq \theta < 0 \\ H^{-1}, & \theta = 0 \end{cases} \end{aligned} \tag{1.2}$$

$$(1.6)\quad x_t - X_o G(t) = T(t-\sigma)[\phi - X_o G(\sigma)] + \int_\sigma^t T(t-s)X_o F(s)ds - \int_\sigma^t [d_s T(t-s)X_o] G(s)$$

for $t \geq \sigma$, $\phi \in C$, where it is always understood that the integrals in (1.6) are actually an integrals in E^n.

Formula (1.6) suggert the change of variables

$$x_t - X_o G(t) = z_t, \qquad \phi - X_o G(\sigma) = \psi$$

from $C \longrightarrow PC$. If this is done, equation (1.6) becomes

$$(1.7)\quad z_t = T(t-\sigma)\psi + \int_\sigma^t T(t-s)X_o F(s)ds - \int_\sigma^t [d_s T(t-s) X_o] G(s).$$

Definition 1.1. The operator D is said to be stable if there is a $\vartheta > 0$ such that all roots of the equations $\det D(e^{\lambda \cdot} I) = 0$ stisfy $\operatorname{Re}\lambda \leq -\vartheta$.

If $D(\phi) = H\phi(0) - M\phi(-r)$, then D is stable if the roots of the polynomial equation $\det(H - \rho M) = 0$ satisfy $|\rho| < 1$.

An important property of equation (1.1) when D is stable is the following (see [3]) : If D is stable, then there is a constant $a_D < 0$ such that for any $a > a_D$, there are only a finite nimber of roots of $\det \Delta(\lambda) = 0$ with $\operatorname{Re}\lambda > a$.

Let D be stable. If $\Lambda = \{\lambda : \det \Delta(\lambda) = 0,\ \operatorname{Re}\lambda \geq 0\}$, then Λ is a finite set and it is follows from [11] that the space C can be descomposed as $C = P \oplus Q$, where P, Q are subspaces of C invariant under T(t), the space P is finite dimensional and corresponds to the initial values of all those solutions of (1.1) which are of the form $p(t)e^{\lambda t}$, where p(t) is a polynomial in t and $\lambda \in \Lambda$. If Φ is a basis for P, then for every $\phi \in P$ there exists a vector $a \in E^n$ such that $\phi = \Phi a$. En this case, we can define $T(t)\phi = \Phi e^{Bt} a$, -

If $F, G : [0,\infty) \longrightarrow E^n$ are continuous, a nonhomogeneous linear NFDE is defined as

$$(1.3) \qquad \frac{d}{dt}\left\{D(x_t) - G(t)\right\} = L(x_t) + F(t).$$

A solution through ϕ at $t=\sigma$ of (1.3) is defined as before and is known to exist on $[\sigma - r, \infty)$.

The variation of constants formula for (1.3) (see [9]) states that the solution of (1.3) through (σ, ϕ) is given by

$$(1.4)\ x(t) = T(t-\sigma)\phi(0) + \int_{\sigma}^{t} X(t-s)F(s)ds - \int_{\sigma}^{t^+} [d_s X(t-s)]\, G(s) - G(\sigma),$$

for $t \geq \sigma$, where X is the fundamental matrix solution given by (1.2). Equation (1.4) can be written as

$$(1.5)\quad x(t) - X(0)\,G(t) = T(t-\sigma)\,\phi(0) - X(t-\sigma)G(\sigma) + \int_{\sigma}^{t} X(t-s)F(s)ds - \int_{\sigma}^{t}[d_s X(t-s)]\, G(s),$$

for $t \geq \sigma$.

Now, let PC be the space of functions taking $[-r, 0]$ into E^n which are uniformly continuous on $[-r, 0)$ and may be discontinuous at zero. With the matrix X_0 as defined before, it is clear that $PC = C + \langle X_0 \rangle$, where $\langle X_0 \rangle$ is the span of X_0; that is any $\psi \in PC$ is given as $\psi = \phi + X_0 b$, $\phi \in C$, $b \in E^n$. We make PC a normed vector space by defining the norm $|\psi| = \max\{|\phi|, b\}$.

Let us define $x_t(\psi) = T(t)\psi$, where $\psi \in PC$ and $x(\psi)$ is the solution of (1.1) through ψ. The operator $T(t) : PC \longrightarrow$ (functions on $[-r, 0]$) is linear, but $T(t)$ does not take $PC \longrightarrow PC$. It is an extension of the original semigroup $T(t)$ on C. If we use this notation, then the variation of constants formula (1.5) can be written as where B is an $n \times n$ matrix defined by $A\Phi = \Phi B$. The spectrum of B is Λ.

If C is decomposed by Λ as $C = P \oplus Q$ then equation (1.6) is equivalent to

(1.8)
$$x_t^P - X_o^P G(t) = T(t-\sigma)\left[\phi^P - X_o^P G(\sigma)\right] + \int_\sigma^t T(t-s)X_o^P F(s)ds - \int_\sigma^t [d_s T(t-s)X_o^P] G(s)$$

$$x_t^Q - X_o^Q G(t) = T(t-\sigma)\left[\phi^Q - X_o^Q G(\sigma)\right] + \int_\sigma^t T(t-s) X_o^Q F(s)ds - \int_\sigma^t [d_s T(t-s)X_o^Q] G(s)$$

where the superscrits P and Q designate the projections of the corresponding functions onto the subspaces P and Q, respectively, and they can be determined by means of adjoint differential equation to (1.1), see [11].

2. The linear problem

In this section, we consider the system

(2.1)
$$\text{a)}\quad \dot{x}(t) = A_1 x(t) + A_2\, y(t-r)$$
$$\text{b)}\quad y(t) - A_3 x(t) + A_4 y(t-r) = 0$$

where x, y are n-vector and all matrices are constants. For any $a \in E^n, \psi \in C$, one can define a solution of (2.1) with initial value $x(o) = a$, $y_o = \psi$. If we define, $C = C([-r, 0], E^n)$

$$D, L : E^n \times C \longrightarrow E^{n \times n}, \quad D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \quad L = \begin{bmatrix} L_1 \\ 0 \end{bmatrix},$$

(2.2)
$$D_1(a, \psi) = a$$
$$D_2(a, \psi) = \psi(0) - A_3 a - A_4 \psi(-r)$$
$$L_1(a, \psi) = A_1 a + A_2 \psi(-r)$$

then equation (2.1) is a special case of the NFDE

$$\frac{d}{dt} D\,(x(t),\ y_t) = L\,(x(t),\ y_t) \tag{2.3}$$

and one obtain the system (2.1) by requiring that

$$D_2(a,\psi) = 0$$

Equation (2.3) defines a semigroup $T(t)$ on $E^n \times C$. If we define $(E^n \times C)_o = \{(a,\psi)\in E^n \times C : D_2(a,\psi) = 0\}$ then $(E^n \times C)_o$ can be considered as a Banach space. Furthermose, for any $(a,\psi)\in(E^n \times C)_o$, the solution of (2.3) through (a,ψ) will be in $(E^n \times C)_o$ since it corresponds to the solution of (2.1) through (a,ψ). Consequently,

$$T_o(t) \overset{\text{def}}{=} T(t)\Big|_{(E^n\times C)_o} : (E^n\times C)_o \longrightarrow (E^n\times C)_o$$

is a strongly continuous semigroup. The infinitesimal generator A_o of $T_o(t)$ is $A_o = A\Big|_{(E^n\times C)_o}$ where A is the infinitesimal generator of $T(t)$. One shows that

$$\sigma(A_o) = \{\lambda\in \mathbb{C} : \det \Delta(\lambda) = 0\}, \qquad \Delta(\lambda) = \begin{bmatrix} \lambda I - A_1 & -A_2 e^{-\lambda r} \\ -A_3 & I - A_4 e^{-\lambda r} \end{bmatrix}$$

Observe that

$$D(a,\psi) = \begin{bmatrix} a \\ \psi(0) - A_3 a - A_4 \psi(-r) \end{bmatrix} = \begin{bmatrix} I & 0 \\ -A_3 & I \end{bmatrix} \begin{bmatrix} a \\ \psi(0) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & A_4 \end{bmatrix} \begin{bmatrix} a \\ \psi(-r) \end{bmatrix}$$

$$\overset{\text{def}}{=} H\,\phi(0) - M\,\phi(-r)$$

$$L(a,\psi) = \begin{bmatrix} a \\ \psi(0) - A_3 a - A_4 \psi(-r) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ \psi(0) \end{bmatrix} + \begin{bmatrix} 0 & A_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ \psi(-r) \end{bmatrix}$$

$$\overset{\text{def}}{=} N\,\phi(0) + P\,\phi(-r).$$

Thus, if the eigenvalues of the matrix A_4 have moduli less than 1, then D is stable.

Also, from (1.2), we can define the fundamental matrix solution X(t) of (2.3) as

$$X_o(\theta) = H^{-1} = \begin{bmatrix} I & 0 \\ A_3 & I \end{bmatrix}, \quad \theta = 0, \qquad X_o(\theta) = 0, \quad -r \leq \theta < 0.$$

If $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$, where X_{ij}, i, j = 1 , 2 , are n x n matrices, then X must be a solution of (2.3) with the initial - data specified above. Therefore, the matrices $X_{i,j}$ must sa-- tisfy

(2.4)

$$\text{a)}\ D_2 (X_{11}(t), X_{21,t}) = 0, \quad t \geq 0$$

$$\text{b)}\ D_2 (X_{12}(t), X_{22,t}) = \quad , \quad t \geq 0.$$

Notice that (2.4 a) implies X_{11}, X_{21} are solutions of (2.1). The functions X_{12}, X_{22} do not satisfy (2.1 b). This implies that the variation of X(t) satisfies the system (2.1).

Using Laplace transform or the same type of argu-- ments as in Hale [7] pag. 303, one can prove the following

<u>Lemma 2.1</u>. If the eigenvalues of A_4 have moduli less than 1 and all roots of det $\Delta(\lambda) = 0$ satisfy $Re\lambda \leq -\delta < 0$, then - there are positive constants K, α such that

$$|X_{11}(t)|, \quad |X_{21}(t)|, \quad |\dot{X}_{ij}(t)| \leq K e^{-\alpha t}, \quad \text{a.e} \quad t \geq 0, \quad i,j = 1,2.$$

Now we consider the nonhomogeneous system

(2.5)

$$\dot{x}(t) = A_1 x(t) + A_2 y(t-r) + f(t)$$

$$y(t) - A_3 x(t) - A_4 y(t-r) - g(t) = 0$$

where f, g are continuous functions from $[0,\infty)$ to E^n. With D, L defined as in (2.2), system (2.5) is a special case of the N F D E

$$(2.6) \qquad \frac{d}{dt}\left\{D(w_t) - G(t)\right\} = L(w_t) + F(t)$$

where $w_t = \mathrm{col}(x(t), y_t)$, $w_o = \phi = \mathrm{col}(a, \psi) \in E^n \times C$, $G = \mathrm{col}(0, g) \in E^{n \times n}$, $F = \mathrm{col}(f, 0) \in E^{n \times n}$ and one obtains the system (2.5) from (2.6) by requiring that $D_2(a, \psi) = g(0)$.

As in Section 1, if we extend the definition of $T(t)$ to $E^n \times (C + \langle X_o \rangle) \overset{\text{def}}{=\!=} Y$, then the general solution of (2.6) is given by the variation of constants formula

$$(2.7) \qquad w_t - X_o G(t) = T(t)\left[\phi - X_o G(0)\right] + \int_o^t T(t-s) X_o F(s)ds - \int_o^t \left[d_s\, T(t-s)\, X_o\right] G(s).$$

This last formula suggests the change of variables

$$W_t - X_o\, G(t) = z_t, \qquad \phi - X_o G(0) = \xi$$

from $E^n \times C$ to Y. If this is done, formula (2.7) becomes

$$(2.8) \qquad z_t = T(t) \quad + \int_o^t T(t-s)\, X_o F(s)ds - \int_o^t \left[d_s\, T(t-s) X_o\right] G(s), \quad t \geq 0.$$

One can give an explicit decomposition of (2.8) by using the adjoint equation to (2.3). For this, we write (2.3) in the form

$$\frac{d}{dt}\left\{H\, w(t) - M\, w(t-r)\right\} = N\, w(t) + P\, w(t-r), \quad w_o = \phi \in E^n \times C$$

which is equivalent to

$$(2.9) \qquad \frac{d}{dt}\left\{w(t) - M\, w\, (t-r)\right\} = \bar{N}\, w(t) + \bar{P}\, w(t-r)$$

since $H^{-1}M = M$ and where $H^{-1}N = \bar{N}$, $H^{-1}P = \bar{P}$.

We define the adjoint equation to (2.9) as

$$(2.10) \frac{d}{dt}\left\{v(t) - v(t+r)\, M\right\} = -\, v(t)\, \bar{N} - v(t+r)\, \bar{P}, \quad v_o = \alpha \in E^{n^*} \times C^*$$

In the same way, we may write (2.6) in the form

$$(2.11)\quad \frac{d}{dt}\left\{w(t) - M\,w(t-r) - H^{-1}G(t)\right\} = \bar{N}\,w(t) + \bar{P}\,w(t-r) + H^{-1}F(t)$$

Using the same arguments as in [7, 13], it is easily shown that if the eigenvalues of A_4 have moduli less than 1 and $E^n \times C$ is decomposed by $\Lambda = \{\lambda : \det \Delta(\lambda) = 0\ ,\ \mathrm{Re}\lambda \geq 0\}$ as $P \oplus Q$ then equation (2.8) is equivalent to

$$(2.12)\quad \begin{aligned} &a)\ z_t^P = T(t)\xi^P + \int_0^t T(t-s)\,X_0^P H^{-1} F(s)\,ds - \\ &\qquad - \int_0^t [d_s\, T(t-s)\,X_0^P] H^{-1}\, G(s) \\ &b)\ z_t^Q = T(t)\xi^Q + \int_0^t T(t-s)\,X_0^Q H^{-1} F(s)\,ds - \\ &\qquad - \int_0^t [d_s\, T(t-s)\,X_0^Q] H^{-1}\, G(s) \end{aligned}$$

where $\xi^P = \phi^P - X_0^P\, G(0)$, $\xi^Q = \phi^Q - X_0^Q\, G(0)$, $z_t = z_t^P + z_t^Q$. Also, if $z_t^P = \Phi\, u(t)$, where Φ is a basis for P and $T(t)\Phi = \Phi e^{Bt}$, the spectrum of B is Λ, then u satisfies the equation

$$(2.13)\quad \dot{u}(t) = B\,u\,(t) + \Psi(0)\,H^{-1}F(t) + B\Psi(0)\,H^{-1}G(t),\quad -\infty < t < \infty$$

where Ψ is a basis for the initial values of those solutions of (2.10) of the form $p(t)e^{-\lambda t}$, p a polynomial, $\lambda \in \Lambda$.

If $\mathscr{P}_T$ is the Banach space of continuous and T-periodic functions with norm $|f| = \sup\{|f(t)|\ ,\ t \in [0, T]\}$, then one can state the theorem on the Fredholm alternative for periodic solutions as:

<u>Theorem 2.1. [13]</u> If the eigenvalues of A_4 have moduli less than 1 and f, $g \in \mathscr{P}_T$, then system (2.5) has a solution in $\mathscr{P}_T$ if and only if

$$(2.14)\quad \int_0^T v(t)H^{-1}F(s)ds - \int_0^T [d\,v\,(t)]\,H^{-1}G(s) = 0.$$

for all T-periodic solutions v of the adjoint equation (2.10).

Since $\Psi(s) = e^{-Bs}\Psi(0)$ and Ψ is a basis for the T-periodic solutions of the adjoint equation, formula (2.14) can be written as

$$\int_0^T e^{-Bs}\Psi(0)H^{-1}F(s)\,ds - \int_0^T [d_s e^{-Bs}\Psi(0)]H^{-1}G(s) =$$

$$= \int_0^T e^{-Bs}[\Psi(0)H^{-1}F(s) + B\Psi(0)H^{-1}G(s)]\,ds = 0$$

or $$\frac{1}{T}\int_0^T e^{B(t-s)}[\Psi(0)\;H^{-1}F(s) + B\Psi(0)H^{-1}G(s)]\,ds = 0.$$

Thus, equation (2.14) defines a continuous projection operator $J : \mathscr{P}_T \to \mathscr{P}_T$, given by

$$J(\Psi(0)H^{-1}F + B\Psi(0)H^{-1}G)(t) \overset{def}{=} J(F, G) =$$

$$= \frac{1}{T}\int_0^T e^{B(t-s)}[\Psi(0)H^{-1}F(s) + B\Psi(0)H^{-1}G(s)]\,ds,$$

and system (2.5) has a solution in $\mathscr{P}_T$ if and only if $J(F, G) = 0$.

Furthermore, there is a continuous linear operator

$$K(I - J) : \mathscr{P}_T \to \mathscr{P}_T$$

such that $K(I-J)(F, G)$ is the unique solution of (2.5) which satisfies $JK(I-J)(F, G) = 0$. The unique solution z_t^* of (2.8) in $\mathscr{P}_T$ is

$$z_t^* = \Phi e^{Bt} a + K(I-J)(F,G), \quad a \in E^n.$$

The operator $K(I-J)$ is given by

$$K(I-J)(F,J) = z_t^* - \Phi e^{Bt}a = z_t^{*P} - \Phi e^{Bt} a + z_t^{*Q} =$$

$$=\Phi[u^*(t) - e^{Bt}\, a] + z_t^{*Q}$$

where $u^*(t)$ and z_t^{*Q} are the unique solutions of (2.13) and (2.12 a) in $\mathcal{P}_T$.

3. Critical case for the periodic system with a smamall parameter

Consider the nonlinear systems

$$(3.1)\quad \begin{aligned} \dot{x}(t) &= A_1 x(t) + A_2 y(t-r) + f(t, x(t), y_t, \varepsilon) \\ y(t) &- A_3 x(t) - A_4 y(t-r) - y(t, x(t), y_t, \varepsilon) = 0 \end{aligned}$$

where f, g are continuous in all their arguments and T-periodic in t and ε is a small real parameter. With D, L defined as before, system (3.1) is a special case of the nonlinear NFDE

$$(3.2)\quad \frac{d}{dt}\left\{D(w_t) - G(t, w_t, \varepsilon)\right\} = L(w_t) + F(t, w_t, \varepsilon)$$

where $w_t = \text{col}(x(t), y_t)$, $w_o = \phi = \text{col}(a, \psi) \in E^n \times C$, $G = \text{col}(0, g) \in E^{n \times n}$, $F = \text{col}(f, 0) \in E^{n \times n}$ and one obtains system (3.1) from (3.2) by requiring that $D_2(a, \psi) = g(0, a, \psi, \varepsilon)$ The solution of (3.2) is given by the variation of constants formula

$$(3.3)\quad w_t - X_o G(t, w_t, \varepsilon) = T(t)[\phi - X_o G(0, \phi, \varepsilon)] + \int_o^t T(t-s) X_o F(s, w_s, \varepsilon)\, ds - \int_o^t [d_s T(t-s) X_o] G(s, w_s, \varepsilon).$$

If we let

$$z_t = w_t - X_o G(t, w_t, \varepsilon), \qquad \xi = -X_o G(0, \phi, \varepsilon),$$

then we have defined the transformation $h: E^n \times C \longrightarrow Y$, $h(\phi) = \xi$, which is a homeomorphim (see [8, 10]). If this is done, formula (3.3) becomes

$$(3.4) \qquad z_t = T(t)\xi + \int_o^t T(t-s) X_o F(s, h^{-1}(z_s), \varepsilon)\, ds - \\ - \int_o^t [d_s T(t-s) X_o] G(s, h^{-1}(z_s), \varepsilon).$$

Suppose D is stable. If $\Lambda = \{\lambda : \det \Delta(\lambda) = 0, \ \mathrm{Re}\,\lambda = 0\}$, then Λ is a finite set and the space $E^n \times C$ can be decomposed by Λ as $P \oplus Q$. As in Section 2, equation (3.4) can be split as (2.8). Then (3.4) is equivalent to

$$z_t = \Phi u(t) + z_t^Q$$

$$(3.5) \quad \dot{u}(t) = B u(t) + \Psi(0) H^{-1} F(t, h^{-1}(z_t), \varepsilon) + B \Psi(0) H^{-1} G(t, h^{-1}(z_t), \varepsilon)$$

$$z_t^Q = T(t)\xi^Q + \int_o^t T(t-s) X_o H^{-1} F(s, h^{-1}(z_s), \varepsilon)\, ds - \\ - \int_o^t [d_s T(t-s) X_o] H^{-1} G(s, h^{-1}(Z_s), \varepsilon),$$

where $z_t = z_t^P + z_t^Q$, $z_t^P = \Phi u(t)$ and B is an $n \times n$ matrix whose spectrum is Λ. We assume that the eigenvalues of B are integral multiples of $2\pi i/T$ and that B is diagonalizable.

We are going to give conditions under which (3.1) has T-periodic solutions which are closed to T-periodic solutions of its linear part. We will do this for equation (3.2) and as $D_2(\xi) = D_2(\phi) - g(0, \phi, \varepsilon) = 0$, since equation (3.1) is satisfied, then we will have done it for system (3.1).

We will assume that f, g in system (3.1) fulfill the following conditions:

i) $f(t, 0, 0) = 0$

ii) $|f(t,\phi_1, \varepsilon) - f(t, \phi_2, \varepsilon)| \leq \eta(|\varepsilon|, \sigma)\, |\phi_1 - \phi_2|$

for t in $[0, \infty)$, $|\phi_1|$, $|\phi_2| < \sigma$ and $\eta(\rho, \sigma)$ is a continuous non-decreasing function for $\rho \geq 0$, $\sigma \geq 0$, $\eta(0,0) = 0$, and the same conditions for g. For equation (3.4) we obtain the following

Lemma 3.1. For any $\alpha > 0$, there is an $\varepsilon_o > 0$ such that for any $a \in E^n$, $|a| \leq \alpha$, $|\varepsilon| \leq \varepsilon_o$, there is a unique function $z^*_t = z^*_t(a, \varepsilon)$ which satisfies

$$z^*_t = \Phi e^{Bt} a + K(I - J)\Big[\Psi(0) H^{-1} F(t, h^{-1}(z^*_t), \varepsilon) + + B\Psi(0) H^{-1} G(t, h^{-1}(z^*_t), \varepsilon)\Big] \tag{3.6}$$

where K, J are defined as the and of Section 2. Furthermore $z^*_t(a, \varepsilon)$ is continuous in (a, ε).

Proof. Let β be a positive number such that $|\Phi e^{Bt} a| \leq \beta$ for $|a| \leq \alpha$. For any $\gamma > 0$, define

$$\mathcal{S}(\gamma) = \left\{ z_t \in \mathcal{P}_T : J\, z_t = \Phi\, e^{Bt},\ |z_t| \leq \gamma \right\}$$

and the operator $\mathcal{F}: \mathcal{S}(\gamma) \rightarrow \mathcal{P}_T$ by

$$\mathcal{F} z_t = \Phi\, e^{Bt} a + K(I-J)(\Psi(0) H^{-1} F(t, h^{-1}(z_t), \varepsilon) + + B\Psi(0) H^{-1} G(t, h^{-1}(z_t), \varepsilon))$$

for $z_t \in \mathcal{S}(\gamma)$. If $z^*_t(a, \varepsilon)$ is a fixed point of $\mathcal{F}$ in $\mathcal{S}(\gamma)$, then $z^*_t(a, \varepsilon)$ is a solution of (3.6). Since h is a homeomorphism there is a constant k_1 such that $|h^{-1}(z_t)| \leq k_1 |z_t|$

and since F satisfies conditions i), ii),

$$|F(t, h^{-1}(z_t),\varepsilon)| \leq |F(t, h^{-1}(z_t),\varepsilon) - F(t,0,\varepsilon)| + |F(t,0,\varepsilon)|$$

$$\leq \eta(|\varepsilon|,\sigma)\, k_1 |z_t| + k_2(|\varepsilon|) \leq \eta(|\varepsilon|,\sigma)\, k_1 \gamma + k_2(|\varepsilon|)$$

where $k_2(0) = 0$ and the same for G. Since K(I-J) is a continuous linear operator, we get

$$|\mathscr{F}z_t| \leq \beta + k_3 \Big\{ |\Psi(0)H^{-1}| (\eta(|\varepsilon|,\sigma)k_1\gamma + k_2(|\varepsilon|)) + |B\Psi(0)H^{-1}|(\eta(|\varepsilon|,\sigma)\, k_1\gamma +$$

$$+ k_2(|\varepsilon|))\Big\} = \beta + k_4\Big\{\eta(|\varepsilon|, \sigma)\, k_1\gamma + k_2(|\varepsilon|)\Big\}.$$

Now choose ε_o, σ_o positive and so small that

$$k_4[\eta(\varepsilon_o, \sigma_o)\, k_1\gamma + k_2(\varepsilon_o)] \leq \gamma - \beta$$

For this choice of ε_o, σ_o and since $J(\mathscr{F}z_t) = \Phi\, e^{Bt} a$, one easily shows that $\mathscr{F}: \mathscr{S}(\gamma) \to \mathscr{S}(\gamma)$ and is a uniform contraction with respect to a, ε for $|a| \leq \alpha$, $0 \leq |\varepsilon| \leq \varepsilon_o$. The continuity property of $z^*_t(a, \varepsilon)$ is a consequence of - the uniform contraction principle.

<u>Theorem 3.1</u>. If there are a, ε with $|a| \leq \alpha$, $0 \leq |\varepsilon| \leq \varepsilon_o$, and the solution $z^*_t(a, \varepsilon)$ of (3.6) satisfies

$$(3.7) \quad J\Big\{\Psi(0)\, H^{-1} F(\cdot\, , h^{-1}(z^*_\cdot(a,\varepsilon)), \varepsilon) +$$

$$+ B\Psi(0)H^{-1}G(\cdot\, , h^{-1}(z^*_\cdot(a, \varepsilon)), \varepsilon)\Big\} = 0,$$

then $z^*_t(a, \varepsilon)$ is a T-periodic solution of (3.2). Conversely, any T-periodic solution of (3.2) in $\mathscr{S}(\gamma)$ is a solution of (3.6) and (3.7).

<u>Proof</u> . The first part of the theorem is obvious, and the second follows grom the fact that any solution of (3.2) must

satisfy (3.6) and (3.7). The uniqueness of the solution in $\mathcal{S}(\gamma)$ implies the result.

Equation (3.7) is equivalent to

$$(3.8)\ \int_0^T e^{-Bt}\left\{\Psi(0)H^{-1}F(t,h^{-1}(z_t^*(a,\varepsilon)),\varepsilon)+\right.$$

$$\left.+\ B\Psi(0)H^{-1}G(t,h^{-1}(z_t^*(a,\varepsilon)),\varepsilon)\right\}dt = 0.$$

Equations (3.7) or (3.8) are called the bifurcation equa-tions of (3.2) and can be determined approximately by successive approximations since $\mathcal{F}$ is a contraction operator.

As $\Phi e^{-Bt}\Psi(0) = T(-t)\Phi\Psi(0) = T(-t)X_0^P = X^P(-t)$, where X is the fundamental matrix solution of (3.2) (this notation is justified in Henry[12]), equation (3.8) is equivalent to

$$\int_0^T X^P(-t)\,H^{-1}\,F(t,w_t^*(a,\varepsilon),\varepsilon)\,dt\ -$$

$$-\int_0^T [d_t\,X^P(-t)]\,H^{-1}G(t,w_t^*(a,\varepsilon),\varepsilon)\,dt = 0$$

which, taking into account that F = col(f, 0), G = col(0,g), is equivalent to

$$\int_0^T X_{11}^P(-t)\,H^{-1}\,f(t,w_t^*(a,\varepsilon),\varepsilon)\,dt\ +$$

$$+\int_0^T \dot{X}_{12}^P(-t)\,H^{-1}\,g(t,w_t^*(a,\varepsilon),\varepsilon)\,dt = 0$$

(3.9)

$$\int_0^T X_{21}^P(-t)\,H^{-1}f(t,w_t^*(a,\varepsilon)\,dt\ +$$

$$+\int_0^T \dot{X}_{22}^P(-t)H^{-1}g(t,w_t^*(a,\varepsilon),\varepsilon)\,dt = 0$$

Equations (3.9) are the bifurcation equations of (3.1).

If the function $z^*_t(a,\varepsilon)$ in Lemma 3.1 is differentiable with respect to a, we can apply the implicit function theorem to (3.9) to have $a = a(\varepsilon)$. In particular, if $f = \varepsilon \bar{f}$, $g = \varepsilon \bar{g}$, where $\bar{f}(t,\phi)$, $\bar{g}(t,\phi)$ are continuosly differentiable in ϕ and we define

$$(3.10) \quad F_1(a,\varepsilon) = \int_0^T X^P_{11}(-t)\, H^{-1}\, \bar{f}(t, w^*_t(a,\varepsilon))\, dt + \int_0^T \dot{X}^P_{12}\, H^{-1}\, \bar{g}(t, w^*_t(a,\varepsilon))\, dt = 0$$

$$F_2(a,\varepsilon) = \int_0^T X^P_{21}(-t)\, H^{-1}\, \bar{f}(t, w^*_t(a,\varepsilon))\, dt + \int_0^T \dot{X}^P_{22}\, H^{-1}\, \bar{g}(t, w^*_t(a,\varepsilon))\, dt = 0$$

then, we have the following theorem for the first approximation

<u>Theorem 3.2.</u> Let f, g satisfy the above conditions. If there is an a_o, $|\Phi e^{Bt} a_o| < \sigma$, such that

$$(3.11) \quad F_1(a_o, 0) = 0, \quad F_2(a_o, 0) = 0, \quad \det\left[\frac{\partial(F_1, F_2)}{\partial a}(a_o, 0)\right] \neq 0$$

then there is an $\varepsilon_o > 0$ such that system (3.1) has a T-periodic solution $w^*_t(a_o, \varepsilon)$, $0 \le |\varepsilon| \le \varepsilon_o$, continuous in ε and $w^*_t(a_o, 0) = \Phi e^{Bt} a_o$.

<u>Proof.</u> The hypothesis (3.11) and the implicit function theorem imply there is an $\varepsilon_o > 0$, such that equations (3.10) have a solution $a(\varepsilon)$, $|a(\varepsilon)| \le \alpha$, $0 \le |\varepsilon| \le \varepsilon_o$. Theorem 3.1. implies the assertions of the theorem.

REFERENCES

[1] Brayton, R. Small - signal stability criterion for elec-- trical natworks containing lossles transmision lines. I. B.M. J. Res. Develop. 12(1968), 431-440.

[2] Cruz, M. and J. Hale. Existence, uniqueness and conti- - nuous dependence for hereditary systems. Annaly Math. Pura Appl. 85 (1970), 63-82.

[3] Cruz, M. and J. Hale. Exponential estimates and the - - saddle point for neutral functional differential equa- - tions. J. Math. Anal. Appl. 34 (1971), 267-288.

[4] Hale, J. Forward and backward continuation for neutral - differential equations. J. Differential Equs. 9(1971) - 168-181.

[5] Hale, J. Ordinary differential equations. Interscience, 1969.

[6] Hale, J. Oscillations in nonlinear systems. McGraw-Hill, 1963.

[7] Hale, J. Theory of functional differential equations. - Appl. Math. Sci. Vol. Springer-Verlag, 1977.

[8] Hale, J. Critical cases for neutral functional differen- tial equations. J. Differential Equs. 10(1971), 59-82.

[9] Hale, J. and M. Cruz. Asymptotic behavior of neutral - functional differential equations. Arch. Rat. Mech. Anal. 34(1969), 331-353.

[10] Hale, J. and P. Martinez-Amores. Stability in neutral - equations. J. Nonl. Anal.- Theory Math. Appl. 1(1977), - 161-173.

[11] Hale, J. and K. Meyer. A class of functional differen-- tial equations. Memoirs Am. Math. Soc. No,76, 1967.

[12] Henry, D. Linear autonomous neutral functional diffe- rential equations. J. Differential Equs. 15(1974), 106 -128.

[13] Martinez-Amores, P. Periodic solutions for coupled - systems of differential-difference and difference - equations. To appear in Annly Math. Pura Appl.

[14] Perelló, C. Periodic solutions of differential equa-- tions with time lag containing a small parameter. J. - Differential Equs. 4(1968), 160-175.

[15] Rasvan, V. Absolute stability of a class of control - system described by functional differential equations of neutral type. Equations Differentielles et Fontione lles Nonlinaires. Herman, 1973.