Cohomological Characterisations of Classes of Finite Groups

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These notes are a more or less complete account of a series of talks
given at the Universitat Autònoma de Barcelona and at the Universidad
de Zaragoza in spring of 1979. In these talks recent work on cohomolo-
gical characterisations of classes of finite groups was presented.

The main results are as follows: We start with a brief treatment of
the Huppert-Thompson-Tate theorem (Theorem 2) which can be regarded
as a (co)homological characterisation of p-nilpotent groups. We
then continue with a characterisation of p-solvable groups in terms
of the cohomology of certain quotient groups with simple coefficient
modules (Theorem 5). This then leads us to characterisations in
terms of the cohomology with simple coefficient modules of the
classes of p-supersolvable (Proposition 12) and of p-nilpotent
groups (Proposition 14). In a similar way, we are able to characterise
certain locally defined formations (Theorem 13). Finally these same
classes are characterised in a way which generalises the well-known
result of Hoechsmann-Roquette-Zassenhaus on p-nilpotent groups
(Propositions 15,16, Theorem 19).

In our talks, as in these notes, we have not always chosen the
shortest possible proofs. For example, we have not made use of
modular representation theory except where it was absolutely necessary.
We have done so for reasons of clarity and also to facilitate access
to the results presented here.

1. In this first section we briefly recall the definition of certain classes of groups that play an important role in what follows.

Let G be a finite group. A <u>chief</u> <u>series</u> of G is a properly descending series of normal subgroups

$$G = N_o \supset N_1 \supset \ldots \supset N_{\ell-1} \supset N_\ell = e \quad , \qquad N_i \lhd G$$

with no proper refinement. The factors N_i/N_{i+1} are simple G-groups, the G-action being induced by conjugation. By Jordan-Hölder the isomorphism classes and the multiplicity of these factors are uniquely determined by G . They are called <u>chief</u> <u>factors</u> of G .

The classes of groups we shall deal with are defined by putting restriction on the structure of the chief factors. Let p be a prime. A group G is called p-<u>solvable</u> if the chief factors are either p-groups or p'-groups.

We note that p-chief factors (i.e. chief factors whose order is a power of p) are automatically elementary abelian p-groups. For, if P is a p-chief factor which is not elementary abelian, then the insertion of a normal subroup of G corresponding to $P' \cdot P^p$ would yield a proper refinement of the given chief series. We may thus regard the p-chief factors of a p-solvable group as modules over kG where k denotes the field of p-elements.

A group G is called p-<u>supersolvable</u> if G is p-solvable and the p-chief factors are of k-dimension one. A group G is called p-<u>nilpotent</u> if G is p-solvable and the p-chief factors are isomorphic to the trivial kG-module k .

It is clear that a group G is solvable, subpersolvable, nilpotent if and only if G is p-solvable, p-supersolvable, p-nilpotent for every prime p dividing the group-order.

The following proposition is well-known. For completeness we include a proof.

Proposition 1. A group G is p-nilpotent if and only if there
exists a normal subgroup N of p'-order and p-index. (N is
called a normal p-complement.)

Proof: If N is a normal subgroup of G of p-index, then G/N
is a p-group and hence has a chief series with factors
isomorphic to the trivial module k . Since N is a
p'-group, the group G is p-nilpotent.

For the converse we proceed by induction on the length of a
chief series. If there is only one chief factor it must be
a p-group or a p'-group. In both cases, the existence of
a normal p-complement is trivial. Thus suppose that M
is the last non-trivial term of a chief series of G .
By induction G/M has a normal p-complement, i.e. we have
a normal subgroup Q of G containing M , with G/Q a
p-group and G/M a p'-group. If the chief factor M is
a p'-group then clearly Q is a normal p-complement of G .
If M is isomorphic to k ,   then by Schur-Zassenhaus
$Q = M \times \overline{Q}$ where $\overline{Q} \cong Q/M$. It is then clear that $\overline{Q}$ is a
normal p-complement of G .

2. We now turn to the Huppert-Thompson-Tate-theorem which gives a
characterisation of p-nilpotent groups in terms of the map in
homology, induced by the embedding of a p-Sylow subgroup.

Theorem 2. [12]: Let P be a p-Sylow subgroup of G . Then the
following statements are equivalent:

> (i)   G is p-nilpotent;
> (ii) $\iota_*: H_1(P,\mathbb{Z}) \rightarrow H_1(G,\mathbb{Z})$ is injective;
> (iii) ι_* $H_1(P,k) \rightarrow H_1(G,k)$ is isomorphic.

Proof. (i) => (ii) Let G be p-nilpotent. Then the embedding
$\iota: P \rightarrow G$ has a left inverse $\pi: G \rightarrow P$; hence
$\pi_* \iota_*: H_1(P,\mathbb{Z}) \rightarrow H_1(G,\mathbb{Z}) \rightarrow H_1(P,\mathbb{Z})$ is the identity. This
implies that $\iota_*: H_1(P,\mathbb{Z}) \rightarrow H_1(G,\mathbb{Z})$ is injective.

(ii) => (iii) Since for any G-module A the map
$\iota_*: H_i(P,A) \to H_i(G,A)$ is surjective on the p-primary part
of $H_i(G,A)$, we conclude that $\iota_*: H_1(P,k) \to H_1(G,k)$
is surjective. It is also injective since
$H_1(-,k) = H_1(-,\mathbb{Z}) \otimes k$ and $\iota_*: H_1(P,\mathbb{Z}) \to H_1(G,\mathbb{Z})$ is
injective and surjective on the p-primary part.

(iii) => (i) We first note that by the above
$\iota_*: H_2(P,k) \to H_2(G,k)$ is surjective. We then apply a
result of Stallings and Stammbach (see for example [8],
p. 93) to show that ι induces isomorphisms
$P/P_j \xrightarrow{\sim} G/G_j$, $j \geq 1$, where P_j, G_j are the terms of the
lower central p-series of P,G respectively. Since this
series terminates with e for the p-group P , we may
conclude that ι induces an isomorphism $P \xrightarrow{\sim} G/\cap G_j$.
Hence $N = \cap G_j$ is a normal p-complement of G .

Proposition 3. [10]: Let P be a p-Sylow subgroup of G . Then
G is p-nilpotent if and only if $\iota_*: H_2(P,k) \to H_2(G,k)$ is
isomorphic.

Before we attempt to prove this proposition, we note the following
example. Let $G = \mathcal{Y}_3$, $k = \mathbb{Z}/(3)$, and let P be a 3-Sylow
subgroup of G . Then $\iota_*: \hat{H}_i(P,k) \to \hat{H}_i(G,k)$ is an isomorphism
for $i \equiv 0$ or $i \equiv 3$ (mod 4); but of course, G is not
3-nilpotent.

Proof: Consider the universal coefficient exact suequences

$H_2P \otimes k \rightarrowtail H_2(P,k) \twoheadrightarrow \text{Tor}(H_1P,k)$.

$\alpha\downarrow \qquad\qquad \beta\downarrow \qquad\qquad \gamma\downarrow$

$H_2G \otimes k \rightarrowtail H_2(G,k) \twoheadrightarrow \text{Tor}(H_1G,k)$

where H_iP , H_iG is the i-th integral homology group of P,G
respectively, and the maps α, β, γ are induced by ι .
Since $H_2P \to H_2G$ is surjective on the p-primary part,
α is surjective too. Thus if β is injective, γ is

injective. Now let $C = \text{im}(H_1P \to H_1G)$, i.e. let C be
the p-primary part of H_1G. Then $\text{Tor}(C,k) = \text{Tor}(H_1G,k)$,
and the exact sequence $D \rightarrowtail H_1P \twoheadrightarrow C$ gives rise to the
exact sequence

$$0 \to \text{Tor}(D,k) \to \text{Tor}(H_1P,k) \overset{\gamma}{\to} \text{Tor}(H_1G,k) \ .$$

Since γ is injective, it follows that $\text{Tor}(D,k) = 0$;
but D is a p-group, so that $\text{Tor}(D,k) = 0$ implies $D = 0$.
Thus $\iota_*: H_1P \to H_1G$ is injective, and G is p-nilpotent
by Theorem 2.

Corollary 4. [10]: The group G is p-nilpotent if and only if G
has a p-Sylow subgroup P such that every central extension
$k \rightarrowtail \overline{P} \twoheadrightarrow P$ of P by k can be embedded into a central
extension $k \rightarrowtail \overline{G} \twoheadrightarrow G$ of G by k .

Proof: Extensions are classified by the second cohomology group;
hence the second statement is equivalent to the statement
that $\iota_*: H^2(G,k) \to H^2(P,k)$ is surjective. But since
$H^2(G,k) = \text{Hom}(H_2(G,k) , k)$ and similarly for $H^2(P,k)$,
this is equivalent to the injectivity of
$\iota_*: H_2(G,k) \to H_2(P,k)$. By Proposition 3 this in turn is
equivalent to G being p-nilpotent.

3. In this section we obtain a characterisation of p-solvable groups.
We define the centralizer C_GM of a kG-module M by

$$C_GM = \{x \in G \mid xm = m \quad \text{for all} \quad m \in M\} \ .$$

We shall suppress the index G whenever there is no danger of
confusion, writing CM instead of C_GM . Clearly CM is a
normal subgroup in G . If $CM = G$, the module M is called
trivial; it is called faithful if $CM = e$. Our main result is
the following

Theorem 5. [9]: The group G is p-solvable if and only if
$H^1(G/CM,M) = 0$ for all simple kG-modules M .

For the proof of this theorem we need some lemmas, most of them well-known; again we include proofs for completeness.

Lemma 6. Let P be a p-group and let M be a simple kP-module. Then $M \cong k$.

Proof: Let M be a simple kP-module. Consider the semi-direct product $Q = M \rtimes P$. Clearly Q is a p-group; hence its chief factors are all isomorphic to k . But M is a chief factor of Q ; hence $M \cong k$.

Lemma 7. Let M be a faithful simple kP-module. Then G does not contain a non-trivial normal p-subgroup.

Proof: Let N be a normal p-subgroup of G . Consider M as kN-module. Then the last non-trivial term in a composition series of M is a simple kN-submodule and hence isomorphic to k . It follows that $M^N \neq 0$, so that M^N is a non-trivial kG-submodule of M . Since M is simple, we must have $M^N = M$, i.e. $N \subseteq CM = e$. This shows that $N = e$.

Lemma 8. (Baer-Gaschütz) Let $G \neq e$ be p-solvable and let M be a faithful simple kG-module. Then $H^i(G,M) = 0$ for all $i \geq 0$.

We note that Lemma 8 implies that if G is p-solvable, then $H^i(G/CM,M) = 0$ for all simple kG-modules M and all $i \geq 1$. In particular it implies one half of our Theorem 5. We also note the following group theoretic consequence: If G is p-solvable and if M is a faithful simple kG-module, then every extension of G by M splits (for $H^2(G,M) = 0$) and the complements of M in the extension group are all conjugate (for $H^1(G,M) = 0$) .

Proof of Lemma 8: By Lemma 7 the group G does not contain a non-trivial normal p-subgroup. Since G is p-solvable, there is a non-trivial normal p'-subgroup, say N .

Consider the extension $N \rightarrowtail G \twoheadrightarrow G/N$ and the associated Lyndon-Hochschild-Serre spectral sequence

$$H^r(G/N, H^s(N,M)) \implies H^{r+s}(G,M) .$$

Since $(|N|, |M|) = 1$ we have $H^s(N,M) = 0$ for $s \geq 1$, so that we obtain isomorphisms

$$\begin{aligned} H^i(G,M) &= H^i(G/N, H^0(N,M)) \\ &= H^i(G/N, M^N) \end{aligned}$$

Since M is faithful $M^N \neq M$; since M is simple, we must thus have $M^N = 0$. It follows that $H^i(G,M) = 0$ for all $i \geq 1$.

We may add that this result could also be obtained using modular representation theory, for it is a well-known result of Brauer [3] that $O_{p'p}G$ centralizes every simple kG-module in the first block. Hence if G is p-solvable and M is faithful, then M cannot belong to the first block. Hence its cohomology is trivial (see [5], p. 178).

<u>Lemma 9.</u> Let $N \triangleleft G$ and let B be a simple kN-module. Then the induced kG-module $A = \operatorname{Hom}_{kN}(kG, B)$, regarded as kN-module, is a direct sum of simple kN-module conjugate to B.

<u>Proof:</u> Let $G = \cup N x_i$ be a partition of G into cosets. Then $kG = \oplus (kN) x_i$ as kN-module, so that $A = \oplus \operatorname{Hom}_{kN}((kN) x_i, B) \cong \oplus B_i$, where $B_i \cong B$ as k-vectorspace, but, in general, with a new kN-module structure. Let $f: (kN) x_i \to B$, and let $f(x_i) = b_i \in B_i$. Then we obtain for $y \in N$

$$y \underset{B_i}{\circ} b_i = (y \circ f)(x_i) = f(x_i y) = f(x_i y x_i^{-1} \cdot x_i) = x_i y x_i^{-1} \underset{B}{\cdot} f(x_i) .$$

In other words: "y operates in B_i in the same way as $x_i y x_i^{-1}$ operates in B". It is clear that every B_i is simple.

We record for later use the following consequence for the
centralizer of B_i in N :

$$C_N(B_i) = \{y \in N \mid yb_i = b_i \quad \text{for all} \quad b_i \in B_i\}$$
$$= \{y \in N \mid x_i y x_i^{-1} b = b \quad \text{for all} \quad b \in B\}$$
$$= x_i^{-1} C_N B x_i \ .$$

Lemma 10. Let A be a kG-module with $H^n(G,A) \neq 0$. Then there
exists a simple kG-module M , a composition factor of A , such
that $H^n(G,M) \neq 0$.

Proof: We proceed by induction on the composition length of A .
Let B be a minimal submodule of A ; then the long exact
cohomology sequence reads

$$\cdots \to H^n(G,B) \to H^n(G,A) \to H^n(G,A/B) \to \cdots \ .$$

Thus $H^n(G,B) \neq 0$, in which case we set $M = B$, or
$H^n(G,A/B) \neq 0$, in which case we infer by induction that
there is a composition factor M of A/B and hence of A
with $H^n(G,M) \neq 0$.

Lemma 11. If $H^1(G/CM , M) = 0$ for all simple kG-modules M ,
then G is p-solvable.

Clearly this proves the other half of Theorem 5.

Proof. We first note that if G has the property that
$H^1(G/CM , M) = 0$ for all simple kG-modules M , then
every quotient group of G has this property too. So
in order to prove Lemma 11, we look at a group G of
smallest order which has this property but which is
not p-solvable, and deduce a contradiction.

We first claim that G cannot be simple. Indead, G
is not of prime order, since every such group is p-solvable.

Suppose that G is non-abelian and simple. Then every
non-trivial kG-module M is faithful, so that for every
simple kG-module $M \neq k$ we have $H^1(G,M) = 0$. Moreover
$H^1(G,k) = G/G' \otimes k = 0$. Hence by Lemma 10 the cohomolgy of
G in dimension 1 vanishes for all kG-modules. By
dimension shifting, the cohomology of G with kG-modules
is trivial in all dimension. It follows by a theorem
of Swan [11] that G is a p'-group. But then G would
be p-solvable, which is a contradiction.

Thus let $e \neq N \lhd G$, $N \neq G$ be a minimal normal subgroup.
Then G/N , being of smaller order than G , is p-solvable.
We may conclude that N is <u>not</u> p-solvable, otherwise G
would be p-solvable. Since N is of smaller order than
G , it follows that there is a simple kN-module B with
$H^1(N/CB , B) \neq 0$. In particular $C_N B \neq N$, i.e. $B \neq k$.
The beginning of the 5-term sequence associated with
$N \rightarrowtail G \twoheadrightarrow G/N$ reads

$$0 \to H^1(N/CB , B) \to H^1(N/B) \to \cdots ,$$

hence $H^1(N,B) \neq 0$.   Consider $A = \text{Hom}_{kN}(kG , B)$, then

$$H^1(G,A) = H^1(G,\text{Hom}_{kN}(kG , B)) = H^1(N,B) \neq 0 .$$

By Lemma 10 there exists a composition factor M of A
with $H^1(G,M) \neq 0$. Since, by hypothesis, $H^1(G/CM , M) = 0$,
we conclude that $C_G M \neq e$. The module A and hence M ,
regarded as kN-module, is a direct sum of kN-modules B_i ,
conjugate to B (Lemma 9). The centralizers of B_i in
N are certain G-conjugates of $C_N B$, which, by the above,
is a proper subgroup of N . Hence, we may infer that
$C_N M = N \cap C_G M$ is properly contained in N . But $C_N M$ is
a normal subgroup of G , so that by minimality of N ,
we have $N \cap C_G M = e$. In other words, $G/C_G M$ contains a
copy of N , so that it cannot be p-solvable. Since
$C_G M \neq e$ this is a contradiction to the minimality of G .

We recall that a group is solvable if and only if it is p-solvable for every prime. Since a finite simple $\mathbb{Z}G$-module is automatically a simple kG-module for some finite prime field k, our Theorem 5 easily yields the following

Corollary: A group G is solvable if and only if $H^1(G/CM, M) = 0$ for all finite simple $\mathbb{Z}G$-modules M.

4. In this Section, we shall give cohomological characterisations of p-supersolvable groups and, more generally, of groups in certain local formations. These results are obtained as consequences of our Theorem 5.

Proposition 12. [1][9]: For a finite group G the following statements are equivalent:

    (i) G is p-supersolvable;

    (ii) $H^1(G,M) = 0$ for all simple kG-modules M with $\dim_k M \geq 2$;

    (iii) $H^i(G,M) = 0$ for all simple kG-modules M with $\dim_k M \geq 2$ and all $i \geq 1$.

Proof: (i) $\Rightarrow$ (iii) Let G be p-supersolvable and let M be a simple kG-module with $\dim_k M \geq 2$. By Lemma 8 we have $H^i(G/CM, M) = 0$ for all $i \geq 0$. We may thus assume that $CM \neq e$. We proceed by induction on the group order. Let $e \neq N \subseteq CM$ be a minimal normal subgroup of G; in particular N is a chief factor of G . Consider $N \rightarrowtail G \twoheadrightarrow G/N$ and the associated Lyndon-Hochschild-Serre spectral sequence

$$G^r(G/N, H^s(N,M)) \Rightarrow H^{r+s}(G,M) \ .$$

If N is a p'-group, then $H^s(N,M) = 0$ for $s \geq 1$. Since $N \subseteq CM$, we may conclude by induction that $H^1(G,M) = H^1(G/N, H^0(N,M)) = H^1(G/N, M) = 0$. If N is a p-group, it must be cyclic of order p . We then have $H^s(N,M) = \mathrm{Hom}(H_s(N,k), M) = \mathrm{Hom}(k,M) = \overline{M}$, where $\bar{}$ indicates

that the G-action may have changed. Since $\overline{M}$ is again a simple kG-module, it follows by induction that
$H^r(G/N, H^s(N,M)) = H^r(G/N, \overline{M}) = 0$ for $r \geq 1$. Hence $H^i(G,M) = 0$ for $i \geq 1$.

Since (iii) $\Rightarrow$ (ii) is trivial, it remains to prove (ii) $\Rightarrow$ (i). Let G be a group satisfying (ii), and let M be a kG-module with $\dim_k M \geq 2$. Since $0 \to H^1(G/CM, M) \to H^1(G,M)$ is exact, we have that for any such module $H^1(G/CM, M) = 0$. If $M = \overline{k}$, then G acts via $\mathrm{Aut}\, k = C_{p-1}$, so that in this case $|G/CM| / p-1$.
It follows that $H^1(G/CM, M) = 0$. By Theorem 5 we may thus conclude that G is p-solvable. It remains to prove that the p-chief-factors are one-dimensional. Note that property (ii) is inherited by quotient groups, so that we may proceed by induction on the group order. Let N be a minimal normal subgroup of G. Then we may assume by induction that G/N is p-supersolvable. If N is a p'-group then G is p-supersolvable too. If N is a p-group, we consider N as a simple kG-module. The 5-term sequence associated with $N \rightarrowtail G \twoheadrightarrow G/N$ then is

$0 \to H^1(G/N, N) \to H^1(G,N) \to \mathrm{Hom}_G(N,N) \to H^2(G/N,N) \to H^2(G,N)$.

If $\dim_k N \geq 2$, then $H^i(G/N,N) = 0$ by induction and (i) $\Rightarrow$ (iii). Hence $H^1(G,N) \cong \mathrm{Hom}_G(N,N)$ is non-trivial. This is a contradiciton. Hence $\dim_k N = 1$ and G is p-supersolvable.

We note that the implication (i) $\Rightarrow$ (iii) could also be obtained using modular representation theory, for it is well-known that the simple modules in principal p-block of a p-supersolvable group are of k-dimension one (see also Lemma 17). Hence if M is a simple kG-module with $\dim_k M \geq 2$ it does not belong to the principal p-block and hence its cohomology is trivial (see [5], p. 178).

Next we give a generalization to certain local formations.

Let $\mathcal{C}$ be an arbitrary formation (see [7], VI. 7). Define F
to be the local formation defined by $F(p) = \mathcal{C}$ and $F(q) = \mathcal{G}r$,
the formation of all finite groups for $q \neq p$. In other words,
the group G is in F if and only if for every chief factor
M of G with $p/|M|$ we have $G/C_G M \in \mathcal{C}$, or equivalently, if
$G/O_{p'p}G \in \mathcal{C}$ (see [7]).

Theorem 13. [2]: For a finite group G the following statements
are equivalent:

(i) $G \in F$;

(ii) $H^1(G,M) = 0$ for all simple kG-modules M with
       $G/CM \notin \mathcal{C}$;

(iii) $H^i(G,M) = 0$ for all simple kG-modules M with
       $G/CM \notin \mathcal{C}$ and all $i \geq 1$.

Proof: (i) => (iii) Let $G \in F$ and let M be a simple kG-module
with $G/CM \notin \mathcal{C}$. We claim that M does not belong to
the principal p-block of G . By [5], p. 178 this then
implies (iii). Were M contained in the principal p-block
of G , we would have $O_{p'p}G \subseteq C_G M$ (see [3]). Since $G \in F$
we have $G/O_{p'p}G \in \mathcal{C}$, so that we would have $G/CM \in \mathcal{C}$.
This is a contradicition.

Since (iii) => (ii) is trivial it remains to prove
(ii) => (i) . We first note that property (ii) is inherited
by quotient groups. Hence we may consider a group G
satisfying (ii) with $G \notin F$ of smallest order. Let N be
a minimal normal subgroup of G . Since G/N is in F
and F is a formation, N is the unique minimal normal
subgroup of G . Clearly $p/|N|$. If N is non-abelian,
then $H^1(N,k) = N/N' \otimes k = 0$. Hence, there exists a
non-trivial simple kN-module B with $H^1(N,B) \neq 0$. Set
$A = Hom_{kN}(kG,B)$. Then $H^1(G,A) = H^1(N,B) \neq 0$. By Lemma 10
there exists a composition factor M of A with
$H^1(G,M) \neq 0$. Moreover, M regarded as kN-module is a

direct sum of kN-modules conjugate to B. Since $H^1(G,M) \neq 0$, we conclude from (ii) that $G/CM \in \mathcal{C}$. Since $C_N M$ is contained in a certain conjugate of $C_N B$ (see Lemma 9) it is properly contained in N and hence trivial. We conclude $e = C_N M = C_G M \cap N$. Since N is the unique minimal normal subgroup, it follows that

$C_G M = e$. But then $G = G/CM \in F$. This is a contradiction. Thus N is abelian and hence a simple kG-module. Since F is saturated and $G/N \in F$, G splits over N. Since N is the unique minimal subgroup, we must thus have $C_G N = N$. Hence $G/C_G N = G/N \notin \mathcal{C}$, otherweise G would be in F. But now the 5-term sequence associated with the split extension $N \rightarrowtail G \twoheadrightarrow G/N$

$$0 \to H^1(G/N,N) \to H^1(G,N) \to \mathrm{Hom}_{kG}(N,N) \overset{\delta}{\to} H^2(G/N,N) \to H^2(G,N)$$

has the property that $\delta(1_N) = 0$, so that $H^1(G,N) \neq 0$. This is a contradiction.

It is clear that Proposition 12 is a special case of Theorem 13. Also, we note that if we take $\mathcal{C} = \{e\}$, then F is the formation of p-nilpotent groups. Hence we obtain the following

Proposition 14. [9]: For a finite group G the following statements are equivalent:

    (i) G is p-nilpotent;

    (ii) $H^1(G,M) = 0$ for all non-trivial simple
                       kG-modules M ;

    (iii) $H^i(G,M) = 0$ for all non-trivial simple kG-modules
                     M and all $i \geq 1$.

We note that this result can be regarded as a variant of Brauer's theorem that a group is p-nilpotent if and only if k is the only simple module in the principal p-block.

5. There is a well-known cohomological characterisation of
   p-nilpotent groups due to Hoechsmann-Roquette-Zassenhaus. For
   our purposes it is convenient to state it in the following form.

   Proposition 15. [6]: For a finite group G the following statements
   are equivalent:

          (i)   G is p-nilpotent;

          (ii)   if A is a kG-module with $\hat{H}^i(G,A) \neq 0$ for
                     some i , then $\hat{H}^1(G,A) \neq 0$;

          (iii)   if A is a kG-module with $\hat{H}^i(G,A) \neq 0$ for
                     some i , then $\hat{H}^{\ell}(G,A) \neq 0$ for all ℓ .

   Here $\hat{H}^i(G,A)$ denotes, as usual, the i-th Tate cohomology group.

   A proof of this proposition may be obtained by proceeding in a manner
   analogous to the proof of Proposition 16 , which gives a similar
   characterisation for p-supersolvable groups.

   Proposition 16. For a finite group G the follwing statements are
   equivalent:

          (i)   G is p-supersolvable;

          (ii)   if A is a kG-module with $\hat{H}^i(G,A) \neq 0$ for some
                   i , then there is a one-dimensional kG-module $\overline{k}$
                   such that $\hat{H}^1(G , \text{Hom}(\overline{k},A)) \neq 0$.

          (iii)   if A is a kG-module with $\hat{H}^i(G,A) \neq 0$ for some
                   i, then for every ℓ there is a one-dimensional
                   kG-module $\overline{k}$ depending on ℓ such that
                   $\hat{H}^{\ell}(G , \text{Hom}(\overline{k},A)) \neq 0$.

   Proof: (iii) => (i) We apply Proposition 12. Let M be a simple
        kG-module with $H^1(G,M) \neq 0$. By (iii) there exists a
        one-dimensional kG-module $\overline{k}$ such that $\hat{H}^0(G , \text{Hom}(\overline{k},M)) \neq 0$
        Hence $H^0(G , \text{Hom}(\overline{k},M)) = \text{Hom}_{kG}(\overline{k},M) \neq 0$, so that $M \cong \overline{k}$.
        By Proposition 12 the group G is p-supersolvable.

        (ii) => (iii) This implication is easily obtained by

dimension shifting and by using that for a projective
(and injective) kG-module P , the module $\text{Hom}(\overline{k},P)$ is
again projective (and injective).

For the implication (i) => (ii) we need some preparations.
We first recall the definition of a (p-)block of a finite
group G .   Let $M = \{M_1, \ldots, M_\ell\}$ be the set of isomorphism
classes of simple kG-modules. Define a graph Γ with vertices
M_i and with an edge joining M_r and M_s if and only if
$\text{Ext}^1_{kG}(M_r,M_s) \neq 0$ or $\text{Ext}^1_{kG}(M_s,M_r) \neq 0$.   We say that M_i and
M_j belong to the same block if they belong to the same connected
component of the graph Γ .   Also, we say that M_i belongs to the
principal block if M_i and   k   belong to the same connected
component of the graph Γ .

We note as an example that the only simple module in the principal
block of a p-nilpotent group G is k .   This is easily proved
using Proposition 14.   For p-supersolvable groups we have the
following well-known result.

Lemma 17. The simple modules in the principal block of a p-super-
solvable group are one-dimensional.

Proof: Let the simple module M be in the principal block of G .
Then there is a path in Γ joining M and k . We may
thus suppose by induction that there is a one-dimensional
kG-module $\overline{k}$ such that $\text{Ext}^1_{kG}(\overline{k},M) \neq 0$ or $\text{Ext}^1_{kG}(M,\overline{k}) \neq 0$.
Suppose first that $\text{Ext}^1_{kG}(\overline{k},M) \neq 0$.   then

$$0 \neq \text{Ext}^1_{kG}(\overline{k},M) = \text{Ext}^1_{kG}(k , \text{Hom}(\overline{k},M)) = H^1(G , \text{Hom}(\overline{k},M))$$

where $\overline{M} = \text{Hom}(\overline{k},M)$ is again a simple kG-module. Using
Proposition 12, we conclude that $\overline{M}$ and hence M is
one-dimensional. The proof in the other case is similar.
This completes the proof of Lemma 17.

Lemma 18. Let A be a kG-module. Then $A = A' \oplus A''$, where the composition factors of A' belong to the principal block of G and the composition factors of A" do not belong to the principal block.

Proof. We proceed by induction on the composition length of A. Thus let $B \subsetneq A$ be a minimal submodule of A. By induction we know that $A/B = C' \oplus C''$, where C' and C" have the obvious meaning. Let $\pi: A \to A/B$ be the canonical projection. If B belongs to the principal block, we set $A' = \pi^{-1}(C')$ and get an exact sequence $A' \rightarrowtail A \twoheadrightarrow C''$. If B does not belong to the principal block, we set $A'' = \pi^{-1}(C'')$ and get an exact sequence $A'' \rightarrowtail A \twoheadrightarrow C'$. We now claim that these sequences split. We show this in the first case, the proof of the second case is similar. Thus suppose that $A' \rightarrowtail A \twoheadrightarrow C''$ does not split; then $\text{Ext}^1_{kG}(C'',A') \neq 0$. By the argument used in the proof of Lemma 10 we may conclude that there is a composition factor M" of C" and a composition factor M' of A' such that $\text{Ext}^1_{kG}(M'',M') \neq 0$. But this is a contradiction since M' and M" do not belong to the same block.

We are now ready to complete the proof of Proposition 16 by showing the implication (i) $\Rightarrow$ (ii). Thus suppose $\hat{H}^i(G,A) \neq 0$. We write $A = A' \oplus A''$ using Lemma 18. Since $\hat{H}^i(G,A'') = 0$ (see [5], p.178), we conclude that $\hat{H}^i(G,A') \neq 0$. Thus A' is not injective. Hence there exists a non-trivial extension $A' \rightarrowtail E \twoheadrightarrow C$, i.e. $\text{Ext}^1_{kG}(C,A') \neq 0$. By the argument used in the proof of Lemma 10 we infer that there is a composition factor M" of C such that $\text{Ext}^1_{kG}(M'',A') \neq 0$ and a composition factor M' of A' such that $\text{Ext}^1_{kG}(M'',M') \neq 0$. Since M' belongs to the principal block, M" belongs to the principal block too. Since G is p-supersolvable, Lemma 17 implies that M" is one-dimensional, $M'' = \bar{k}$. Thus $\text{Ext}^1_{kG}(\bar{k},A) \neq 0$, and we obtain

$$0 \neq \text{Ext}^1_{kG}(\bar{k},A) = \text{Ext}^1_{kG}(k, \text{Hom}(\bar{k},A)) = \hat{H}^1(G, \text{Hom}(\bar{k},A)).$$

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This completes the proof of the proposition.

We note that there is a generalisation of Propositions 15,16 to formations. Let, as in section 4 , $\mathcal{C}$ be an arbitrary formation and let F be the local formation defined by $F(p) = \mathcal{C}$ and $F(q) = \mathcal{G}l$ for $q \neq p$. Recall that a kG-module M is called F-central if $G/C_G M \in \mathcal{C}$.

Theorem 19. For a finite group G the following statements are equivalent:

(i) $G \in F$;

(ii) if A is a kG-module with $\hat{H}^i(G,A) \neq 0$ for some i, then there exists a simple F-central kG-module M such that $\hat{H}^1(G, \operatorname{Hom}(M,A)) \neq 0$;

(iii) if A is a kG-module with $\hat{H}^i(G,A) \neq 0$ for some i , then for every ℓ there exists a simple F-central kG-module M, depending on ℓ, such that $\hat{H}^\ell(G, \operatorname{Hom}(M,A)) \neq 0$.

The proof is similar to the proof of Proposition 16 .

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