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POWERS OF THE AUGMENTATION IDEAL

Summary of talk given at Universitat

Autònoma de Barcelona, 19 April, 1978, by B. Hartley.

Let G be a group, R a commutative ring with 1 (we will in fact only be concerned with the cases when R is a field or $\mathbb{Z}$).

The group ring RG is the set of normal finite linear combinations $\sum_{g\in G} \lambda_g g$ ($\lambda_g \in R$), with the definitions $(\Sigma\lambda_g g) + (\Sigma\mu_g g) = \Sigma(\lambda_g + \mu_g)g$ and $(\Sigma\lambda_g g)(\Sigma\mu_h h) = \sum_k (\sum_{gh=k} \lambda_g\mu_h)k$. The augmentation $\varepsilon: RG \to R$ given by $\varepsilon(\Sigma\lambda_g g) = \Sigma\lambda_g$ is a ring homomorphism and its kernel $\Delta(R, G)$ is the augmentation ideal of RG. We are interested in the powers $\Delta^n(R, G)$ and the associated dimension subgroups

$$D_n(R, G) = G \cap (1 + \Delta^n(R, G)).$$

Note that $D_n(R, G)$ is the kernel of the natural map of G into the group of units $U(RG/\Delta^n(R, G))$ of $RG/\Delta^n(R, G)$.

This is not intended to be a detailed survey of this area, but simply a discussion of some sample results.

I. BACKGROUND

First we mention some connections between this subject and other problems.

1. Connection with matrix representations

It is often useful to know that certain groups can be represented faithfully by matrices over some well behaved ring.

Theorem. If G is a finitely generated nilpotent group, then G can be embedded in some $GL(n, \mathbb{Z})$ (much more general results are true).

Indication of proof. We easily reduce to the case when G is torsion-free. Let G have nilpotency class c. Then $D_{c+1}(\mathbb{Z}, G) = 1$ (quite a difficult result). Hence G can be embedded in the group of units of $\mathbb{Z}G/\Delta^{c+1}(\mathbb{Z}, G) = A$. The additive group of A is finitely generated, and G operates faithfully on this additive group by right multiplication. The additive torsion subgroup T of A is finite, and it is easy to see that G operates faithfully on A/T.

See P. Hall "The Edmonton Notes on Nilpotent Groups" (Queen Mary College, London) for the details.

2. Connection with residual nilpotency of groups

Theorem (Baumslag, Passi, see J. Pure App. Algebra 6 (1975)). Let F be a non-abelian free group, $R \lhd F$. Then F/R' is residually nilpotent if and only if $\bigcap_{n=1}^{\infty} \Delta^n(\mathbb{Z}, G) = 0$, where $G = F/R$.

II. DIMENSION SUBGROUPS

For any G, let $G_1 = G$ and $G_{n+1} = [G_n, G]$, so that $G_1 \geq G_2 \geq \dots$ is the lower central series of G. Then

Lemma 2.1. $G_n \leq D_n(R, G)$

This follows by induction from the identity

$$1-x^{-1}y^{-1}xy = x^{-1}y^{-1}((y-1)(x-1)-(x-1)(y-1)).$$

Let $\sqrt{G_n}/G_n$ be the torsion subgroup of G/G_n.

Theorem 2.2. If K is a field of characteristic zero, then $D_n(K, G) = \sqrt{G_n}$.

This is due to Jennings; a proof can be found in P. Hall's Edmonton Notes, or D.S. Passman "The Algebraic Structure of Group Rings" (Interscience).

Theorem 2.3. (Lazard) If K is a field of characteristic $p > 0$, then

$$D_n(K, G) = \prod_{ip^j \geq n} G_i^{p^j}$$

where if H is a group, $H^m = \langle h^m : h \in H \rangle$. Another description of $D_n(K, G)$ was given earlier by Jennings. Theorem 2.3 can also be found in Passman's book. Note that in Theorem 2.3, if G has exponent p, then $D_n(K, G) = G_n$ for all n.

Theorem 2.2 and 2.3 show that over fields, the dimension subgroups can be completely described in group theoretic terms.

For the case $R = \mathbb{Z}$ the situation is more complicated.

It was at one time conjectured that $D_n(\mathbb{Z}, G) = G_n$ for all G (Dimension subgroup conjecture) and several false proofs have been given.

Theorem 2.4. (i) If F is free, then $D_n(F) = F_n$ for all n. (Magnus, 1937)

(ii) If G is any group, then $D_n(G) = G_n$ for n = 1,2,3 (I don't know who first proved these results; quite a number of proofs are now available, for example A.H.M. Hoare, J. London Math. Soc.)

iii) If G is a finite p-group, then $D_n(G) = G_n$ for $n \leq p$ (Moran, Proc. Cambridge Philos. Soc.)

I have written $D_n(G)$ for $D_n(\mathbb{Z}, G)$.

Eventually the Dimension Subgroup Conjecture was refuted by

Theorem 2.5 (Rips, 1972) There exists a finite 2-group G of class 3 such that $|D_4(G)| = 2$. (Israel J. Math.)

The best result to date is contained in a recent and difficult paper of Sjogren (J. Pure Appd. Algebra (1979)).

Theorem 2.6 (Sjogren) There exists an (explicitly given) function c_n such that if G is any group, then $D_n(G)/G_n$ has exponent dividing c_n.

The proof uses some elementary spectral sequence ideas and some complicated Lie-theoretic methods. I have a simplified version of the proof.

We have $c_1 = c_2 = c_3 = 1$, $c_4 = 2$, so 2.4(ii) follows from Sjogren's work, and Rips' example **shows** that c_4 is best possible. Also by examining the function c_n one can replace the restriction $n \leq p$ in 2.4(iii) by $n \leq p+1$.

Some open problems

Problem 1. Does there exist an integer valued function $f(c)$ such that if G is a nilpotent group of class c, then $D_{f(c)}(G) = 1$? Rips' example shows that $f(c) = c+1$ will not do, but there is no counterexample known to rule out the possibility $f(c) = c+2$.

Problem 2. (a weaker version of Problem 1). If G is nilpotent, is it true that $\bigcap_{n=1}^{\infty} D_n(G) = 1$?

Problem 3. Does there exist a finite p-group ($p \neq 2$) such that $D_n(G) \neq$ $\neq G_n$ for some n? The smallest possibility is p = 3, n = 5.

III. THE LIE RING AND THE GRADED RING

Let G be a group, with lower central series

$$G_1 \geq G_2 \geq \cdots$$

We write G_n/G_{n+1} additively and form the abelian group

$$\mathcal{L}(G) = \bigoplus_{n=1}^{\infty} G_n/G_{n+1}$$

(restricted direct sum), and define an operation (,) on $\mathcal{L}(G)$ by

$$(xG_{n+1}, yG_{m+1}) = [x, y]G_{n+m+1}$$

($x \in G_n$, $y \in G_m$) and extending by additivity. This turns $\mathcal{L}(G)$ into a Lie ring; the Jacobi identity follows from the Witt identity.

Now writing $\Delta = \Delta(\mathbb{Z}, G)$, we form the abelian group

$$\mathcal{G}(G) = \bigoplus_{n=0}^{\infty} \Delta^n/\Delta^{n+1},$$

and define

$$(\alpha+\Delta^{n+1})(\beta+\Delta^{m+1}) = \alpha\beta+\Delta^{n+m+1}.$$

We extend this operation by additivity again, and obtain this time an associative ring, called the associated graded ring of $\mathbb{Z}G$. We can think of $\mathcal{G}(G)$ as a Lie ring under usual uv-vu operation, and then the map

$$\phi : xG_{n+1} \to (x-1) + \Delta^{n+1} \qquad (x \in G_n)$$

extends to a Lie homomorphism of $L(G)$ into $G(G)$. The kernel of the restriction of ϕ to G_n/G_{n+1} is $(D_{n+1}(G) \cap G_n)/G_{n+1}$, and so in this way we obtain a connection between dimension subgroups and Lie algebras.

The map ϕ also determines a homomorphism from $L(G) \otimes_{\mathbb{Z}} K \to G(G) \otimes_{\mathbb{Z}} K$, if K is any field of characteristic zero, and we have the following beautiful result of Quillen (J. Algebra 10 (1968)).

Theorem 3.1. With the above notation, ϕ is injective and $G(G) \otimes K$ is the universal enveloping algebra of $L(G) \otimes K$.

It is in fact more natural for these purposes to construct the Lie ring from the characteristic zero dimension series

$$\sqrt{G_1} \geq \sqrt{G_2} \quad \ldots,$$

though the two Lie rings become isomorphic on tensoring with a field of characteristic zero. However using the dimension series gives the right analogy with characteristic $p > 0$; here we get a result like Theorem 3.1 involving the restricted universal enveloping algebra. For a more detailed survey of these matters, see I.B.S. Passi (J. London Math. Soc. 1979).

IV. POWERS OF THE AUGMENTATION IDEAL

Here we confine ourselves to the question: When is it true that $\bigcap_{n=1}^{\infty} \Delta^n = 0$, where $\Delta = \Delta(\mathbb{Z}, G)$? In dealing with this question, it seems unavoidable to consider also the ring $\mathbb{Z}/p^m\mathbb{Z}$.

Theorem 4.1. $\bigcap_{n=1}^{\infty} \Delta^n(G, \mathbb{Z}/p^m\mathbb{Z}) = 0$ if and only if G is residually a nilpotent p-group of finite exponent.

This was proved by B. Hartley (Proc. London Math. Soc. 1969) and also by K.W. Gruenberg (unpublished). The case m = 1 was done much earlier by Mal'cev.

Theorem 4.2. If G is residually torsion-free nilpotent, then $\bigcap_{n=1}^{\infty} \Delta^n(\mathbb{Z}, G) = 0$.

This is also due to Hartley in the above paper. The main part of the proof consists of adapting a construction of P. Hall to obtain, for a finitely generated torsion-free nilpotent group G, a $\mathbb{Z}$-basis of $\mathbb{Z}G$ which is closely related to the powers of Δ. This can now be done rather better. Let $\sqrt{\Delta^n}(\mathbb{Z}, G)/\Delta^n(\mathbb{Z}, G)$ denote the additive torsion subgroup of $\mathbb{Z}G/\Delta^n(\mathbb{Z}, G)$.

Theorem 4.3. Let G be a finitely generated torsion-free nilpotent group. Then $\mathbb{Z}G$ has a $\mathbb{Z}$-basis $\mathcal{B}$ such that, for each $n \geq 1$, $\sqrt{\Delta^n}(\mathbb{Z}, G)$ is spanned by a subset of $\mathcal{B}$. (See Symposia Math., 1976)

Other proofs of 4.2 have since been given by A.L. Smel'kin using techniques from Lie Algebras (Trans. Moscow Math. Soc. 1973, and also in a recent issue of Uspekhi Mat. Nauk.)

Finally, A.I. Lichtman showed that sufficient conditions given by 4.1 and 4.2 are very nearly necessary. If G is a group, then we say that G is discriminated by nilpotent p-groups of finite exponent if and only if for each finite set of elements $x_1, \ldots, x_m \in G$, there exists a prime p and a homomorphism θ of G into a nilpotent p-group of finite exponent, such that

$$\theta(x_i) \neq 1 \qquad (1 \leq i \leq m).$$

Theorem (Israel J. Math. 1977)

$\bigcap_{n=1}^{\infty} \Delta^n(\mathbb{Z}, G) = 0$ if and only if G is either residually torsion-free nilpotent or discriminated by nilpotent p-groups of finite exponent.