

IMMERSING HOMOGENEOUS SPACES IN EUCLIDEAN SPACE

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In [4], Lam proves, among other things, an immersion result for the real flag manifold

$$G_{\mathbb{R}}(n_1, \dots, n_s) = O(n_1 + \dots + n_s) / O(n_1) \times \dots \times O(n_s)$$

where $O(n)$ is the real orthogonal group. His result is a special case of the following more general observation.

Proposition 1. Let G be a compact, connected semisimple Lie group and H a closed subgroup. Then either G/H is a π -manifold (and so immerses in codimension one) or G/H immerses in $\mathbb{R}^{\dim(\mathfrak{g})}$, where \mathfrak{g} is the Lie algebra of G .

Remark 2. The dimension of the ambient Euclidean space is independent of the subgroup H , so one expects the strongest results for small H . But if H is a maximal torus (all $n_i = 1$ in above example) then G/H is a π -manifold [1]. See Remark 4 below.

Proof. A real vector bundle over G/H is determined by an action of H on a real vector space. The tangent bundle $T(G/H)$ comes from the adjoint action of H on $\mathfrak{g}/\mathfrak{h}$, $\mathfrak{h} = \text{Lie } H$. Let η denote the bundle over G/H coming from the adjoint action of H on \mathfrak{g} . Clearly η is trivial since the action extends to all of G . There is a bundle epimorphism $\eta \rightarrow T(G/H)$

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which necessarily splits (see for example [5,p.46]). Since $\dim(\eta) = \dim(\theta)$, the theorem of Hirsch [3] yields the desired immersion.

Corollary 3. (Lam) The manifold $G_{\mathbb{R}}(n_1, \dots, n_s)$ immerses in $\mathbb{R}^{\binom{n}{2}}$; where $n = n_1 + \dots + n_s$.

Proof. Observe $\dim O(n) = \binom{n}{2}$.

Remark 4. That these immersions are really interesting follows from [2] where it was shown that if $s = 2$, $n_1 = n_2$, one obtains a best possible immersion for the real Grassmannian.

Similarly, one obtains in the complex and quaternionic cases.

Corollary 5. (a) $G_{\mathbb{C}}(n_1, \dots, n_s)$ immerses in $\mathbb{R}^{\binom{n}{2}}$
 (b) $G_{\mathbb{H}}(n_1, \dots, n_s)$ immerses in $\mathbb{R}^{2n^2 + n}$

These results are not as strong as those of Lam [4].

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