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AUTOMORPHISMS OF THE POLYNOMIAL RING IN TWO VARIABLES*

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Let k be a field, k[x,y] the polynomial ring in two variables, and Aut k[x,y] the group of all its k-algebra automorphisms. Such an automorphism will be denoted by the ordered pair (p,q) where $p,q \in k[x,y]$ are the respective images of x,y.

THEOREM. The group Aut k[x,y] is generated by (y,x), $(x,y-\mu x^n)$ $\mu \in k$, $n \geq 0$. Moreover Aut $k[x,y] = A *_C B$ where

$$A = \{(\lambda_{11}x+\lambda_{12}y+\lambda_1, \lambda_{21}x+\lambda_{22}y+\lambda_2) \mid \lambda_{11}\lambda_{22} \neq \lambda_{21}\lambda_{12}\},$$

$$B = \{(\lambda_{11}x+\lambda_1, \lambda_{22}y+f(x)) \mid \lambda_{11}\lambda_{22} \neq 0, f(x) \in k[x]\},$$

$$C = A \cap B = \{(\lambda_{11}x+\lambda_1, \lambda_{21}x+\lambda_{22}y+\lambda_2) \mid \lambda_{11}\lambda_{22} \neq 0\}.$$

The elements of A are called affine automorphisms, the elements of B de Jonquières automorphisms, and the elements of the subgroup generated by $A \cup B$ are called tame automorphisms. The fact that all k-algebra automorphisms of k[x,y] are tame was proved by Jung [2] for char k = 0, and then by Van der Kulk [8] in the general case. From their work the coproduct decomposition follows fairly easily, but it is not clear who first made the observation. (Kambayashi [3] gives the credit to Shafarevitch [7].)

Rentschler [5] gave a very simple proof of tameness for char k = 0, and then along slightly different lines Makar-Limanov [4] gave a fairly simple proof for arbitrary characteristic. (News of Van der Kulk's result seems not to have reached Moscow at that time, for Makar-Limanov refers to the result as

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unpublished work of Shafarevitch.) In the spirit of Serre [6], Roger Alperin [1] gave an explicit example of a tree acted on by Aut $k[x,y]$ from which the coproduct decomposition can be read off.

In §1 below we give a modified version of Makar-Limanov's proof, and in §2 recall Alperin's example.

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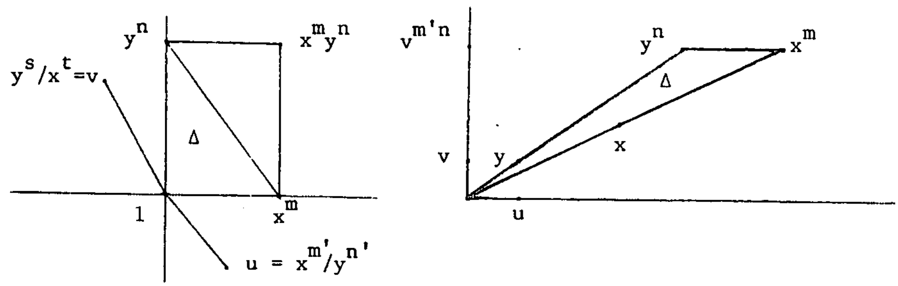
§1 The support of a primitive element

Let (f,g) be an automorphism of $k[x,y]$. We can write $f = \sum \lambda_{ij} x^i y^j$, $\lambda_{ij} \in k$ and define $\text{supp}(f) = \{x^i y^j \mid \lambda_{ij} \neq 0\} \subseteq \langle x,y \rangle$, where $\langle x,y \rangle$ is the free abelian group generated by x,y. Let $m = x\text{-deg}(f)$, $n = y\text{-deg}(f)$, that is, m is the highest exponent of x occurring in $\text{supp}(f)$, and similarly for n. Set $\Delta = \{x^i y^j \mid ni+mj \leqslant mn,\ i \geqslant 0,\ j \geqslant 0\} \subseteq \langle x,y \rangle$. Geometrically, $\text{supp}(f)$ lies in the rectangle determined by $1, x^m, x^m y^n, y^n$ and Δ occupies the triangle determined by $1, x^m, y^n$.

The objective of this section is to show $x^m, y^n \in \text{supp}(f) \subseteq \Delta$ and $m|n$ or $n|m$.

If $mn = 0$ this is clear.

Thus we may assume $mn > 0$. Let $m' = m/(m,n)$, $n' = n/(m,n)$. These are coprime natural numbers, so we can choose natural numbers s,t such that $sm'-tn' = 1$. Let $u = x^{m'}/y^{n'}$, $v = y^s/x^t$ in $\langle x,y \rangle$ so $x = u^s v^{n'}$, $y = u^t v^{m'}$.



Thus $k[x,y] \subseteq k[u,v]$ and we can write $f = \sum \mu_{ij} u^i v^j$ so $\mathrm{supp}(f) = \{u^i v^j \mid \mu_{ij} \neq 0\}$. We define the leading v-component of f to be $|f| = (\sum_i \mu_{ij} u^i) v^j \in k[u]^\times \times \langle v \rangle$ where $j = \text{v-deg}(f)$. If then $\text{u-deg}(|f|) = i$ we define $\|f\| = u^i v^j \in \langle u,v \rangle$ called the leading term of f. This extends to a group homomorphism $\| \ \| : k(u,v)^\times \to \langle u,v \rangle$. (Notice the superscript $^\times$ is being used to denote the set of nonzero elements.) The following statement indicates the steps in Makar-Limanov's argument.

THEOREM 1. (i) There exist $\alpha, \beta \in k(u)^\times \times \langle v \rangle \subseteq k(u,v)^\times$ such that $|f| = \lambda\alpha^a$ ($\lambda \in k^\times$, $a \in \mathbb{N}^+$) and $x,y \in k[\alpha^{\pm 1}, \beta]$.

(ii) There then exist $w,z \in \langle u,v \rangle$ such that $\langle w \rangle = \langle \|\alpha\| \rangle$ or $\langle \|\alpha\|, \|\beta\| \rangle$ and $x,y \in \mathrm{semigp}\langle w^{\pm 1}, z \rangle$.

(iii) Then $x^m, y^n \in \mathrm{supp}(f) \subseteq \Delta$ and $\|f\| = x^m$ and $\langle w \rangle = \langle x \rangle$.

(iv) If $\langle w \rangle = \langle \|\alpha\| \rangle$ then $m \mid n$.

(v) If $\langle w \rangle = \langle \|\alpha\|, \|\beta\| \rangle$ then $n \mid m$.

PROOF. (i) Let $K = k(u)$ and consider the Laurent series field $K((v^{-1}))$. In a natural way $k(u,v) \subseteq K((v^{-1}))$ and there are maps v-deg: $K((v^{-1}))^\times \to \mathbb{Z}$, $| \ | : K((v^{-1})) \to K^\times \times \langle v \rangle$ extending the corresponding maps on $k[u,v]$. We view $k^\times$ as a subgroup of $K^\times \times \langle v \rangle \subseteq K((v^{-1}))^\times$. Since $\text{v-deg}(f) > 0$ there exists $\alpha \in K^\times \times \langle v \rangle$ such that the image of α in $(K^\times \times \langle v \rangle)/k^\times$ generates a maximal cyclic subgroup containing the image of $|f|$, say $|f| = \lambda\alpha^a$ $\lambda \in k^\times$, $a \in \mathbb{N}^+$. By induction on a we shall show that for any $f,g \in K((v^{-1}))$ with $|f| = \lambda\alpha^a$ $\lambda \in k^\times$, $a \in \mathbb{N}^+$ there exists $\beta \in K^\times \times \langle v \rangle$ such that $|k[f^{\pm 1}, g]| \subseteq k[\alpha^{\pm 1}, \beta]$.

The case $a = 0$ is vacuous.

Let us now define a (possibly finite) sequence inductively. Let $g_1 = g$. Suppose we have g_i for some $i \geq 1$. If $|g_i| = \lambda_i(\lambda\alpha)^{an_i}$ for some $\lambda_i \in k^\times$, $n_i \in \mathbb{Z}$ we set $g_{i+1} = g_i - \lambda_i f^{n_i}$; if $g_i = 0$ or $g_i \neq 0$ and $|g_i|$ is not of this form we let the sequence end at g_i. Since v-deg(g_1) > v-deg(g_2) > ... the sequence $g_1, g_2, \ldots$ has a limit g_* in $K((v^{-1}))$, $g_* = g - \lambda_1 f^{n_1} - \lambda_2 f^{n_2} - \ldots$.

If $g_* = 0$ then $k[f^{\pm 1}, g] \subseteq k((f^{-1}))$ so $|k[f^{\pm 1}, g]^{\times}| \subseteq |k((f^{-1}))^{\times}| \subseteq k[|f|^{\pm 1}] \subseteq k[\alpha^{\pm 1}]$ and we can take β arbitrary.

Thus we may assume $g_* \neq 0$ so the sequence is finite and $k[f^{\pm 1}, g] \subseteq k[f^{\pm 1}, g_*]$.

If $|f|, |g_*|$ are algebraically independent over k then it is easy to see $|k[f^{\pm 1}, g_*]^{\times}| \subseteq k[|f|^{\pm 1}, |g_*|]$ and we can take $\beta = |g_*|$.

This leaves the case where $|f|, |g_*|$ are algebraically dependent over k. If c = v-deg(f), d = v-deg(g_*) then $|f|^d, |g_*|^c$ are algebraically dependent over k and are v-homogeneous with the same v-degree. It follows that $|f|^d/|g_*|^c$ lies in K and is algebraic over k so lies in k. Thus $|g_*|^c \equiv |f|^d \equiv \alpha^{ad} (\text{mod } k^{\times})$. But $(K^{\times} \times \langle v \rangle)/k^{\times}$ is a torsion-free abelian group, and the image of α generates a maximal cyclic subgroup, so $c \mid ad$ and $|g_*| \equiv \alpha^b (\text{mod } k^{\times})$ where $b = ad/c$. Say $|g_*| = \mu\alpha^b$, $\mu \in k^{\times}$. By the definition of g_* we know $a \nmid b$, say $b = aq+r$ $\quad 0<r<a$. $\quad$ Let $h = g_*/f^q$. Then $|h| \equiv \alpha^r (\text{mod } k^{\times})$ and the induction hypothesis applies to the pair (h,f). Hence there exists $\beta \in K^{\times} \times \langle v \rangle$ such that $|k[h^{\pm 1}, f]^{\times}| \subseteq k[\alpha^{\pm 1}, \beta]$. Now $|k[f^{\pm 1}, g]^{\times}| \subseteq |k[f^{\pm 1}, h]^{\times}| \subseteq |k[f,h]^{\times}| \langle |f| \rangle \subseteq k[\alpha^{\pm 1}, \beta]$. By induction $|k[f^{\pm 1}, g]^{\times}| \subseteq k[\alpha^{\pm 1}, \beta]$ for some $\beta \in k(u)^{\times} \times \langle v \rangle$, and (i) is proved since $x, y \in |k[f,g]^{\times}|$.

(ii) Recall that two elements of $\langle u,v \rangle$ are said to be dependent if they generate a cyclic subgroup, and otherwise they are independent, that is, freely generate a free abelian subgroup. If $\|\alpha\|$, $\|\beta\|$ are independent then it is clear that $x, y \in \|k[\alpha^{\pm 1}, \beta]^{\times}\| \subseteq \text{semigp}\langle \|\alpha\|^{\pm 1}, \|\beta\| \rangle$ and we can take $w = \|\alpha\|$, $z = \|\beta\|$. This leaves the case where $\|\alpha\|$, $\|\beta\|$ are dependent. Let w be a generator of $\langle \|\alpha\|, \|\beta\| \rangle$, say $\|\alpha\| = w^i$, $\|\beta\| = w^j$, $w = \|\alpha\|^c \|\beta\|^d$. Here

$\|\alpha^j\| = \|\beta^i\| = w^{ij}$ so there is a unique $\mu \in k^\times$ such that $z = \|\alpha^j - \mu\beta^i\| \neq w^{ij}$. But z and w^{ij} have the same v-degree so w,z are independent. Let $\alpha' = \alpha^c\beta^d$, $\beta' = \alpha^j/\beta^i - \mu$. Then $\|k[\alpha^{\pm 1},\beta^{\pm 1}]^\times\| = \|k[\alpha'^{\pm 1},(\beta'+\mu)^{\pm 1}]^\times\| \subseteq \|k[\alpha',\beta']^\times\|\langle w\rangle$ $\subseteq \text{semigp}\langle w^{\pm 1},z\rangle$. Thus $x,y \in \text{semigp}\langle w^{\pm 1},z\rangle$ and $\langle w\rangle = \langle\|\alpha\|, \|\beta\|\rangle$.

(iii) Geometrically $x,y \in \text{semigp}\langle w^{\pm 1},z\rangle$ means that one of the two half-planes determined by w contains both x and y. Now by (ii) $\|\alpha\| = w^i$ for some integer i and on replacing w with w^{-1} if necessary we may assume $i \geq 0$. By (i), $\|f\| = \|\alpha\|^a = w^{ia}$ and $\|f\| \in \text{semigp}\langle x,y\rangle$ so $w \in \text{semigp}\langle x,y\rangle$. The only way this can happen is for w to lie along the x or y axis, that is, w is a power of x or y. But $\langle w,z\rangle \supseteq \langle x,y\rangle$ so w is x or y. Thus $\|f\|$ is a power of x or y. But the only place supp(f) meets the x or y axes is in Δ so $\|f\| \in \Delta$ and this forces $\text{supp}(f) \subseteq \Delta$. The only way x-deg(f) can be m is for x^m to be in supp(f), and similarly $y^n \in \text{supp}(f)$. Thus $\|f\| = x^m$ or y^n. But $\text{u-deg}(x^m) = \text{u-deg}(u^{ms}v^{mn'}) = ms$, $\text{u-deg}(y^n) = \text{u-deg}(u^{nt}v^{m'n}) = nt = ms-(m,n)$ $< ms$ so $\|f\| = x^m$. Hence $\langle w\rangle = \langle x\rangle$.

(iv) If $\langle\|\alpha\|\rangle = \langle w\rangle = \langle x\rangle$ then $\|\alpha\| = x$. But by (i) $\|f\| = \|\alpha\|^a = x^a$ and by (iii) $\|f\| = x^m$ so $a = m$. Thus $|f| = \lambda\alpha^m$ in $k(x,y)$ so $\text{y-deg}(|f|) = m(\text{y-deg}(\alpha))$. And $\text{y-deg}|f| = n$ since $y^n \in \text{supp}|f|$, so $m|n$.

(v) If $\langle\|\alpha\|, \|\beta\|\rangle = \langle w\rangle = \langle x\rangle$ then $n'\mathbb{Z} = \text{v-deg}(\langle x\rangle) = \text{v-deg}(\langle\|\alpha\|, \|\beta\|\rangle)$ $= \text{v-deg}(\langle\alpha,\beta\rangle)$. By (i) $y \in k[\alpha^{\pm 1},\beta]$ and this is v-homogeneous so $\text{v-deg}(y) \in \text{v-deg}(\langle\alpha,\beta\rangle)$, that is, m' is a multiple of n' so $n|m$.

§2 The Automorphism Group

For any $p = \sum\mu_{ij}x^iy^j \in k[x,y]^\times$ we define $\deg(p) = \max\{i+j \mid \mu_{ij} \neq 0\}$; if $\deg p = d$ we define $p_0 = \sum\mu_{i\,d-i}x^iy^{d-i}$ called the leading component of p.

THEOREM 2 ([2],[8]). Let (p,q) be a k-algebra automorphism of $k[x,y]$ with $\deg p \leq \deg q$. Then either (p,q) is affine or there is a unique $\mu \in k^\times$ and positive integer r such that $\deg(q-\mu p^r) < \deg(q)$.

PROOF. Let (f,g) be the inverse of (p,q) and let f be as in §1. If $\deg(p^m) \neq \deg(q^n)$ then $\deg(f(p,q)) = \max\{\deg(p^m), \deg(q^n)\}$. But $f(p,q) = x$ so p or q is a polynomial in x of degree 1 and the desired conclusion follows easily. This leaves the case where $\deg(p^m) = \deg(q^n)$. Here $m \geq n$ so $n|m$ and $\deg(p^r) = \deg(q)$ for $r = \frac{m}{n}$. We may assume (p,q) is not affine so $\deg q > 1$. Since $f(p,q) = x$ it follows that p_0, q_0 are algebraically dependent over k. Hence q_0/p_0^r is algebraic over k so lies in k, say $q_0/p_0^r = \mu$. Then $\deg(q-\mu p^r) < \deg(q)$ as desired.

By induction on $\deg(q)$ it follows easily from Theorem 2 that all k-algebra automorphisms of $k[x,y]$ are tame. It is even a simple matter to obtain the decomposition.

THEOREM 3. $\operatorname{Aut} k[x,y] = A *_C B$.

PROOF. Let Γ be the oriented graph whose vertices are the k-subspaces of $k[x,y]$ and whose edges are the inclusion maps. Then $\operatorname{Aut} k[x,y]$ acts in a natural way on the graph Γ. Let T be the orbit of $k+kx \to k+kx+ky$. We claim that T is a tree.

Any vertex of T is of the form $k+kp$ or $k+kp+kq$ where (p,q) is some automorphism. We define $\deg(k+kp) = \deg(p)$ and $\deg(k+kp+kq) = \max\{\deg(p), \deg(q)\} - \frac{1}{2}$. It is easy to see these are well-defined.

Consider a vertex of the form $k+kp$. We can find an automorphism (p,q) with $\deg(q)$ minimal, so $\deg(q) < \deg(p)$ or (p,q) is affine. All the neighbours of $k+kp$ are of the form $k+kp+k(q+h)$ where $h \in k[p]$. The only neighbour of $k+kp$ with smaller degree is $k+kp+kq$; all the others have greater degree.

Consider a vertex of the form $k+kp+kq$ where $\deg(q) < \deg(p)$. The neighbours are of the form $k+k(\alpha p+\beta q)$ where $\alpha, \beta \in k^{\times}$ are not both zero; only $k+kq$ has smaller degree, all the others have greater degree.

Finally, the vertex k+kx+ky has smaller degree than all its neighbours. Thus every path from k+kx+ky is strictly increasing (so T has no circuits) and from each vertex there is a strictly decreasing path which must necessarily arrive at k+kx+ky (so T is connected). Hence T is a tree.

Now k+kx → k+kx+ky is a transversal in T for the action of Aut k[x,y] and the stabilizer of k+kx is B while the stabilizer of k+kx+ky is A. This implies $G \simeq A *_C B$. cf [6].

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