Pub. Mat. UAB
Vol. 28 Nº 2-3 Set. 1984

ON THE LOGARITHMIC CONVERGENCE EXPONENT AND GEOMETRIC MEANS OF AN INTEGRAL FUNCTION OF ORDER ZERO

M.I.Rizvi*

1. For a non constant integral function of order zero, the logarithmic order ρ^* and the lower logarithmic order λ^* are given as [1],

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \log M(r,f)}{\log \log r} = \begin{matrix}\rho^* \\ \lambda^*\end{matrix} \tag{1.1}$$

where $M(r,f) = \max_{z=r} |f(z)|$.
The geometric means of $|f(z)|$ for $0 < K < \infty$, are defined as

$$G(r) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta\right\} \tag{1.2}$$

and

$$g_K(r) = \exp\left\{\frac{(K+1)}{r^{K+1}}\int_0^r x^K \log G(x)dx\right\}. \tag{1.3}$$

Another geometric mean of $|f(z)|$ is defined as [2],

---

* This work has been supported by JRF of C.S.I.R., New Delhi (INDIA).

$$(1.4)\quad g_K^*(r) = \exp\left\{\frac{(K+1)}{(\log r)^{K+1}}\int_1^r (\log x)^K \log G(x)\frac{dx}{x}\right\}.$$

The logarithmic convergence exponent ρ_1^* and lower logarithmic convergence exponent λ_1^* are given as ([3] , p.58)

$$(1.5)\quad \limsup_{r\to\infty} \frac{\log n(r)}{\log\log r} = \begin{matrix}\rho_1^* \\ \lambda_1^*\end{matrix},$$

where $(0 \leqslant \lambda_1^* \leqslant \rho_1^* \leqslant \infty)$.

Jain, P.K. and Chungh, V.D. ([2] ,[3] ,[4]) have discussed some properties of these geometric means. In this paper we have also studied few properties of $g_K^*(r)$ which are given in the form of the theorems.

2. Theorem 1: - Let $f(z)$ be an integral function of order zero. Then, for $0 < r_1 < r_2$, we have

$$(2.1)\quad \left\{(\log r_2)^{K+1} - (\log r_1)^{K+1}\right\} \log G(r_1) \leqslant$$

$$\leqslant (\log r_2)^{K+1} \log g_K^*(r_2) - (\log r_1)^{K+1} \log g_1^*(r_1) \leqslant$$

$$\left\{(\log r_2)^{K+1} - (\log r_1)^{K+1}\right\} \log G(r_2),$$

where K is any positive number.

Proof. From (1.4), we have

$$(2.2)\quad (\log r_1)^{K+1} \log g_K^*(r_1) = (K+1)\int_1^{r_1} \log G(x)\,(\log x)^K \frac{dx}{x}.$$

Similarly,

$$(2.3)\quad (\log r_2)^{K+1} \log g_K^*(r_2) = (K+1)\int_1^{r_2} \log G(x)(\log x)^K \frac{dx}{x}.$$

From (2.2) and (2.3) we get,

$$(2.4)\quad (\log r_2)^{K+1} \log g_K^*(r_2) - (\log r_1)^{K+1} \log g_K^*(r_1) = (K+1)\int_{r_1}^{r_2} \log G(x)(\log x)^K \frac{dx}{x}.$$

From (2.4), (2.1) follows since $G(x)$ is an increasing function of x.

3. Theorem 2: - Let $f(z)$ be an integral function of order zero and logarithmic convergence exponent ρ_1^* and lower logarithmic convergence exponent λ_1^*, then,

$$(3.1)\quad \lim_{r\to\infty} \sup_{\inf} \frac{\log \log \left\{\left\{\frac{G(r)}{g_K^*(r)}\right\}^{1/(\log r)}\right\}}{\log \log r} = \frac{\rho_1^*}{\lambda_1^*}.$$

In order to prove the above theorem we first prove the following lemma.

Lemma 1. $$\left\{\frac{g_K^*(r)}{G(r)}\right\}^{1/\log r} = \exp\left\{\frac{-1}{(\log r)^{K+2}}\int_1^r (\log x)^{K+1} n(x)\frac{dx}{x}\right\}.$$

Proof of Lemma 1 : -

$$\frac{-1}{(\log r)^{K+1}}\int_1^r (\log x)^{K+1}\,\frac{d}{dx}\,(\log G(x))dx$$

$$= \frac{-1}{(\log r)^{K+1}}\left\{(\log r)^{K+1}\log G(r) - \int_1^r (K+1)(\log x)^K \log G(x)\frac{dx}{x}\right\}$$

$$= -\log G(r) + \frac{(K+1)}{(\log r)^{K+1}}\int_1^r (\log x)^K \log G(x)\,\frac{dx}{x}$$

$$= -\log G(r) + \log g_K^*(r)$$

$$= \log\left\{\frac{g_K^*(r)}{G(r)}\right\}.$$

Hence,

$$(3.2)\quad \exp\left\{\frac{-1}{(\log r)^{K+2}}\int_1^r (\log x)^{K+1}\frac{d}{dx}(\log G(x))dx\right\} = \left\{\frac{g_K^*(r)}{G(r)}\right\}^{1/\log r}$$

From (1.2) and using Jensen's formula we get $\frac{d}{dx}(\log G(x))=\frac{n(x)}{x}$. Hence, form (3.2), we get

$$\left\{\frac{g_K^*(r)}{G(r)}\right\}^{1/\log r} = \exp\left\{\frac{-1}{(\log r)^{K+2}}\int_1^r (\log x)^{K+1}\,n(x)\,\frac{dx}{x}\right\}.$$

Proof of theorem 2 : - From Lemma 1, we have

$$\left\{\frac{G(r)}{g_K^*(r)}\right\}^{1/\log r} = \exp\left\{\frac{1}{(\log r)^{K+2}}\int_1^r (\log x)^{K+1}\,n(x)\,\frac{dx}{x}\right\}$$

$$\leqslant \exp\left\{\frac{1}{(\log r)^{K+2}}\,n(r)\int_1^r (\log x)^{K+1}\,\frac{dx}{x}\right\}$$

$$= \exp\left\{\frac{n(r)}{(K+2)}\right\}.$$

Hence, using (1.5), we get

$$\limsup_{r\to\infty\,\inf} \frac{\log\log\left\{\frac{G(r)}{g_K^*(r)}\right\}^{1/\log r}}{\log\log r} \leq \limsup_{r\to\infty\,\inf} \frac{\log n(r)}{\log\log r} = \frac{\rho_1^*}{\lambda_1^*} \tag{3.3}$$

Further,

$$\left\{\frac{G(r^2)}{g_K^*(r^2)}\right\}^{1/\log(r^2)} = \exp\left\{\frac{1}{(\log r^2)^{K+2}}\int_1^{r^2} (\log x)^{K+1}\, n(x)\, \frac{dx}{x}\right\}$$

$$> \exp\left\{\frac{1}{(\log r^2)^{K+2}}\int_r^{r^2} (\log x)^{K+1}\, n(x)\, \frac{dx}{x}\right\}$$

$$\geq \exp\left\{\frac{1}{(\log r^2)^{K+2}}\, n(r) \int_r^{r^2} (\log x)^{K+1}\, \frac{dx}{x}\right\}$$

$$= \exp\left\{\frac{n(r)}{(\log r^2)^{K+2}}\left\{\frac{(\log r^2)^{K+2} - (\log r)^{K+2}}{(K+2)}\right\}\right\}$$

$$= \exp\left\{\frac{n(r)}{K+2}\,\left(1 - \left(\tfrac{1}{2}\right)^{K+2}\right)\right\}.$$

Hence,

$$\limsup_{r\to\infty\,\inf} \frac{\log\log\left\{\frac{G(r^2)}{g_K^*(r^2)}\right\}^{1/\log r^2}}{\log\log(r^2)} \geq \tag{3.4}$$

$$\geq \limsup_{r\to\infty\,\inf} \frac{\log n(r)}{\log\log r} = \frac{\rho_1^*}{\lambda_1^*}.$$

From (3.3) and (3.4), (3.1) follows.

4. Theorem 3:- Let us set

$$\lim_{r\to\infty}\ \sup_{\inf}\ \frac{\log\left\{\frac{G(r)}{g_K^*(r)}\right\}^{1/\log r}}{(\log r)^{\rho_1^*}} = \frac{p}{q}, \tag{4.1}$$

and

$$\lim_{r\to\infty}\ \sup_{\inf}\ \frac{n(r)}{(\log r)^{\rho_1^*}} = \frac{c}{d}. \tag{4.2}$$

Then we have,

$$\text{(i)}\quad \frac{d}{(K+\rho_1^*+2)} \leqslant q \leqslant p \leqslant \frac{c}{(K+\rho_1^*+2)},$$

$$\text{(ii)}\quad \left(\frac{c}{d}\right)^{\frac{K+2}{\rho_1^*}} q \leqslant \frac{d}{(K+\rho_1^*+2)} + d\left\{\frac{\left(\frac{c}{d}\right)^{\frac{K+2}{\rho_1^*}}-1}{K+2}\right\},$$

$$\text{(iii)}\quad \left\{\frac{(K+2)(c-d)+c}{c\rho_1^*}^{\rho_1^*}\right\} \geqslant \frac{c}{(K+\rho_1^*+2)}.$$

Proof. From Lemma 1, for $h > 0$, we have

$$\log\left\{\frac{G(r^{1+h})}{g_K^*(r^{1+h})}\right\} = \frac{1}{\{\log(r^{1+h})^{K+1}\}}\int_1^{r^{1+h}} (\log x)^{K+1}\, n(x)\frac{dx}{x}$$

$$= \frac{1}{\{\log(r^{1+h})\}^{K+1}}\left\{\left\{\int_1^{r_o} + \int_{r_o}^{r} + \int_{r}^{r^{1+h}}\right\}(\log x)^{K+1} n(x)\frac{dx}{x}\right\}.$$

$$< O\{(\log r)^{-K-1}\} + \frac{(c+\epsilon)}{\{\log(r^{1+h})\}^{K+1}}\int_{r_o}^{r} (\log x)^{K+\rho_1^*+1}\frac{dx}{x}$$

$$+ \frac{n(r^{l+h})}{\{\log(r^{l+h})\}^{K+1}} \int_{r}^{r^{l+h}} (\log x)^{K+1} \frac{dx}{x}$$

$$= 0\left\{(\log r)^{-K-1}\right\} + \frac{(c+\epsilon)(\log r)^{K+\rho_1^*+2} - (\log r_o)^{K+\rho_1^*+2}}{(l+h)^{K+1}(K+\rho_1^*+2)(\log r)^{K+1}}$$

$$+ \frac{n(r^{l+h})[(l+h)^{K+2} - 1]\log r}{(K+2)(l+h)^{K+1}} .$$

Hence,

$$\frac{\log\left\{\frac{G(r^{l+h})}{g_K^*(r^{l+h})}\right\}^{1/\log r^{l+h}}}{\{\log (r^{l+h})\}^{\rho_1^*}} <$$

$$< \frac{(c+\epsilon)}{(l+h)^{\rho_1^*+K+2}(K+\rho_1^*+2)} + \frac{n(r^{l+h})[(l+h)^{K+2}-1]}{(K+2)(l+h)^{\rho_1^*+K+2}(\log r)^{\rho_1^*}} .$$

Taking limits of both sides and using (4.1) and (4.2) we get

$$(4.3)\quad (l+h)^{K+\rho_1^*+2}\, p \leqslant \frac{c}{(K+\rho_1^*+2)} + c(l+h)^{\rho_1^*}\frac{(l+h)^{K+2}-1}{K+2}$$

and

$$(4.4)\quad (l+h)^{K+\rho_1^*+2}\, q \leqslant \frac{c}{(K+\rho_1^*+2)} + d(l+h)^{\rho_1^*}\left\{\frac{(l+h)^{K+2}-1}{K+2}\right\}$$

Similarly, we obtain

$$(4.5)\qquad (l+h)^{K+\rho_1^*+2}\, p \geqslant \frac{d}{(K+\rho_1^*+2)} + c\left\{\frac{(l+h)^{K+2}-1}{K+2}\right\}$$

and

$$(4.6)\qquad (1+h)^{K+\rho_1^*+2}\, q \geqslant \frac{d}{(K+\rho_1^*+2)} + d\left\{\frac{(1+h)^{K+2}-1}{K+2}\right\}.$$

It can be seen that minima of right hand expansion of (4.3) and (4.4) occurs at $h = 0$ and $(1+h)^{\rho_1^*} = c/d$. Substituting $h = 0$ in (4.3) and $(1+h)^{\rho_1^*} = c/d$ in (4.4), we get second parts of (i) and (ii) respectively. Taking

$$(1+h)^{K+2} = \frac{(K+2)(c-d) + c^{\rho_1^*}}{c^{\rho_1^*}}$$

in (4.5) and $h = 0$ in (4.6) we get (iii) and first part of (1) respectively.

5. Theorem 4 : - If $f(z)$ is an integral function of order zero and logarithmic convergent exponent ρ_1^* and lower logarithmic convergent exponent λ_1^*, then

$$(5.1)\qquad \lim_{r\to\infty} \frac{\sup}{\inf} \frac{\log\log\{\frac{G(r)}{g_K(r)}\}}{\log\log r} = \frac{\rho_1^*}{\lambda_1^*}.$$

Proof. From (1.2) and (1.3) and as $\frac{d}{dx}(\log G(x)) = \frac{n(x)}{x}$, we get

$$\frac{G(r)}{g_K(r)} = \exp\left[\frac{1}{r^{K+1}}\int_0^r x^{K+1}\frac{d}{dx}\left\{\log G(x)\right\}dx\right]$$

$$= \exp\left[\frac{1}{r^{K+1}}\int_0^r x^{K+1}\, n(x)dx\right]$$

$$\leqslant \exp_{r\to\infty}\left\{\frac{1}{r^{K+1}}\, n(r)\int_0^r x^{K+1}\,dx\right\}$$

$$= \exp\left\{\frac{n(r)}{(K+1)}\right\}.$$

Hence,

$$\lim_{r\to\infty}\sup_{\inf}\frac{\log\log\{\frac{G(r)}{g_K(r)}\}}{\log\log r} \leqslant \tag{5.2}$$

$$\lim_{r\to\infty}\sup_{\inf}\frac{\log n(r)}{\log\log r} = \frac{\rho_1^*}{\lambda_1^*}$$

On the other hand

$$\frac{G(2r)}{g_K(2r)} = \exp\left\{\frac{1}{(2r)^{K+1}}\int_0^{2r} x^K n(x)\,dx\right\}$$

$$> \exp\left\{\frac{1}{(2r)^{K+1}}\int_r^{2r} x^K n(x)\,dx\right\}$$

$$\geqslant \exp\left\{\frac{1}{(2r)^{K+1}}\, n(r)\int_r^{2r} x^K dx\right\}$$

$$= \exp\left\{\frac{n(r)}{2^{K+1}}\,\frac{2^{K+1}-1}{K+1}\right\}$$

Hence,

$$\lim_{r\to\infty}\sup_{\inf}\frac{\log\log\{\frac{G(2r)}{g_K(2r)}\}}{\log\log(2r)} \geqslant \tag{5.3}$$

$$\geqslant \lim_{r\to\infty} \frac{\sup}{\inf} \frac{\log n(r)}{\log \log r} = \frac{\rho_1^*}{\lambda_1^*} .$$

Therefore from (5.2) and (5.3), (5.1) follows.

ACKNOWLEDGEMENT

I take this opportunity to express my sincere thanks to Dr. S.N.Srivastava for his kind help and valuable suggestions in the preparation of this paper.

REFERENCES

1. S.M. Shah & M. Ishaq — On the maximun modulus and the coefficient of an entire series. J. Indian Math. Soc. 16(1952), 177-182.

2. Jain, P.K. & Chungh, V.D. — The geometric mean of an entire function of order zero. Collect. Math. 24(1973).

3. Jain, P.K. & Chugh, V.D. — Logarithmic order and geometric mean of an entire functionof order zero.

4. Jain, P.K. & Chugh, V.D. — The geometric mean of an entire function of order zero. Collect. Math. 28(1977) nº. 2, 115-120.

Rebut el 8 de març del 1984

Department of Mathematics
and Astronomy,
Lucknow University
Lucknow (INDIA)