Publicacions Matemàtiques, Vol 32 (1988), 91–110.

APPROXIMATION OF Z_2-COCYCLES AND SHIFT DYNAMICAL SYSTEMS

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Abstract

Let $\overline{G} = G\{n_t, n_t \mid n_{t+1}, t \geq 0\}$ be a subgroup of all roots of unity generated by $\exp(2\pi i/n_t), t \geq 0$, and let $\tau: (X, \mathcal{B}, \mu) \circlearrowleft$ be an ergodic transformation with pure point spectrum $\overline{G}$. Given a cocycle φ, $\varphi: X \longrightarrow \mathbf{Z}_2$, admitting an approximation with speed $0(1/n^{1+\epsilon}, \epsilon > 0)$ there exists a Morse cocycle ψ such that the corresponding transformations τ_φ and τ_ψ are relatively isomorphic. An effective way of a construction of the Morse cocycle ψ is given. There is a cocycle φ oddly approximated with an arbitrarily high speed and without roots.

This note delivers examples of φ's admitting an arbitrarily high speed of approximation and such that the power multiplicity function of τ_φ is equal to one and the power rank function is oscillatory. Finally, we also prove that if φ is a Morse cocycle then each proper factor of τ_φ is rigid. In particular continuous substitutions on two symbols cannot be factors of Morse dynamical systems.

Introduction and statement of results

Let $T: (X, \mathcal{B}, \mu) \circlearrowleft$ be an ergodic transformation of a Lebesgue space. We will denote by $Sp(T)$ the group of all the eigenvalues of the unitary operator $U_T: L^2(X, \mu) \circlearrowleft$, $U_T f = f \circ T$. T is said to have rational pure point spectrum (r.p.p.s) if $L^2(X, \mu)$ is generated by the eigenfunctions of U_T and, besides,

$$Sp(T) = G\{n_t: t \geq 0\}$$

where $G\{n_t: t \geq 0\}$, $n_t \mid n_{t+1}$, $t \geq 0$, denotes the subgroup of all roots of unity generated by $\exp(2\pi i/n_t)$, $t \geq 0$.

Let $D^{n_t} = (D_0^{n_t}, D_1^{n_t}, \ldots, D_{n_t 1}^{n_t})$ be a partition, i.e. $D_i^{n_t} \cap D_j^{n_t} = \emptyset$, $i \neq j$, $D_i^{n_t} \epsilon \mathcal{B}$. Then D^{n_t} is called a partition of X if, besides, ${}^{n_t}{}_{i=0}^{1}$ $D_i^{n_t} = X$. We call D^{n_t} a T–tower (with height n_t) if $T^i D_o^{n_t} = D_i^{n_t}$, $i = 0, \ldots, n_t - 1$ and a T–tower of X if, besides, D^{n_t} is a partition of X. The ergodicity of T says that if D^{n_t} is a T–tower of X with height n_t then this is the only T–tower of X with that height (reordering elements of D^{n_t}, if necessary). It turns out (cf. [15]) that:

(1) T has r.p.p.s. with $S_p(T) = G\{n_t \colon t \geq 0\}$ iff there is a sequence $(D^{n_t})_{t\geq 0}$ of T–towers of X such that $D^{n_t} \nearrow \mathcal{B}$.

Let K denote the class of all the $\mathbf{Z}_2$–cocycles of $(X, \mathcal{B}, \mu)$ i.e. $\varphi \in K$ if $\varphi \colon X \longrightarrow \mathbf{Z}_2$ is measurable. We endow K with the natural topology given by the metric

$$\varsigma(\varphi, \varphi') = \mu(\varphi^{-1}(0) \triangle \varphi'^{-1}(0)) + \mu(\varphi^{-1}(1) \triangle \varphi'^{-1}(1)).$$

With this metric K becomes a complete, separable metric space. Having fixed $T \colon (X, \mathcal{B}, \mu) \circlearrowleft$ we define the class K_T of all $\mathbf{Z}_2$–extensions T_φ, $\varphi \in K$, of T, i.e.

$$T_\varphi \colon (X \times \mathbf{Z}_2, \widetilde{\mathcal{B}}, \tilde{\mu}) \circlearrowleft, \quad T_\varphi(x, i) = (Tx, \varphi(x) + i),$$

where $\tilde{\mu}$ is the product measure $\mu \times \nu_2$ $(\nu_2(0) = \nu_2(1) = 1/2)$ and $\widetilde{\mathcal{B}}$ is the corresponding product σ–algebra. A cocycle φ is called *ergodic* whenever T_φ is ergodic. Now, changing T we obtain the class of all ergodic $\mathbf{Z}_2$–extensions of automorphisms with r.p.p.s.

This class is one of more interesting classes in ergodic theory. A great deal of the attention has been devoted to the study of it (for instance [**1**], [**2**], [**6**], [**10**], [**13**], [**14**], [**16**], [**18**]).

Assume (D^{n_t}) is a sequence of T–towers of X arising from (1) and let $f \colon N \longrightarrow R$ be a real function. Following [**7**] we call $\varphi \in K$ *oddly (evenly) approximated* with speed $o(f(n))$ if for some subsequence $\{n_{t_k}\}$ there exists sets F_k consisting of an odd (even) number of members of $D^{n_{t_k}}$ such that

$$\mu(\varphi^{-1}(1) \triangle F_k) = o(f(n_{t_k})). \tag{2}$$

The odd approximation with speed $o(1/n)$ guarantees the ergodicity of φ ([**7**]).

Assume $\varphi \in K$ and (D^{n_t}) is a sequence of T–towers given by (1). Then φ is said to be a *Morse cocycle* if there is a subsequence (n_{t_k}) such that $\varphi \mid D_i^{n_{t_k}}$ is constant $(\varphi \mid D_i^{n_{t_k}} = a_i^t)$ on each level of $D^{n_{t_k}}$ except for $i = n_{t_k} - 1$. The main result of [**14**] was

Representation Theorem. *If φ is oddly approximated with speed $0(\frac{1}{n^2})$ then there is a Morse sequence $x = b^0 \times b^1 \times \ldots$ such that T_φ is isomorphic to the Morse dynamical system determined by x. (We refer to [**8**], [**10**], [**12**] for the definition and properties of Morse sequences).*

In the present paper we stenghten the Representation Theorem proving:

Theorem 1. *If $\varphi \in K$ is ergodic and admits an odd (or even) approximation with speed $0(\frac{1}{n^{1+\epsilon}})$, $\epsilon > 0$, then there exists a Morse cocycle ψ such that T_φ and T_ψ are relatively isomorphic.*

We recall that T_φ and T_ψ are *relatively isomorphic* if there exists a cocycle $f: X \to \mathbf{Z}_2$ such that

$$\varphi(x) + f(Tx) = f(x) + \psi(x). \tag{3}$$

Then the map $I(x,i) = (x, f(x) + i)$ establishes an isomorphism between T_φ and T_ψ.

We have been unable to decide whether or not the Theorem holds for $\epsilon = 0$.

Looking at the proof of the Representation Theorem we see that it does not provide any effective way of a construction of the Morse cocycle ψ (or the Morse sequence x). The proof of Theorem 1 is based on a quite different idea and allows to determine the Morse cocycle in an algorithmic way.

Let G be the group of all n_t-adic integers i.e.

$$G = \{g;\ g = \sum_0^\infty g_t \cdot n_{t-1},\ 0 \le g_t \le \lambda_t - 1,\ n_{-1} = 1\},\ \lambda_t = \frac{n_{t+1}}{n_t},\ t \ge 0,$$

and let T be the translation on the unit element $\hat{1}$. Then (G, T, m), (m is the Haar measure) is an ergodic system having r.p.p.s., $S_p(T) = G\{n_t,\ t \ge 0\}$. Hence if $\varphi: G \to \mathbf{Z}_2$ is a cocycle satisfying the assumptions of Theorem 1 then it can be modyfied by a coboundary cocycle getting ψ, which is measurable with respect to the algebra generated by $\{D^{n_t}\}$, $t \ge 0$. Although ψ cannot be continuous on G (except for some trivial cases), there is a metod making such cocycles continuous. Namely, ψ is the so called *Toeplitz cocycle* in the sense of [**13**] i.e. it is completely determined by some Toeplitz sequence $\eta \in \{0,1\}^Z$. If we take $X = \overline{\theta(\eta)}$ (the closure of the trajectory of η via the shift τ) and the cocycle $\psi': \overline{\theta(\eta)} \longrightarrow \mathbf{Z}_2$, $\psi'(y) = y[0]$, then the automorphisms $\tau_{\psi'}$ and T_ψ are metrically isomorphic ([**13**]). In other words there is some effective way of a construction of an ergodic $\tau_{\psi'}$ (with ψ' to be continuous) which is isomorphic to T_ψ.

Notice that from [**4**] it follows that there is a topological process $(\widetilde{G}, \widetilde{T}, \tilde{\mu})$, a metric isomorphism $\Pi: (\widetilde{G}, \tilde{\mu}) \to (G, \mu)$ $\Pi\widetilde{T} = T\Pi$ and a continuous function $\tilde{\varphi}: \widetilde{G} \to \mathbf{Z}_2$ such that $\tilde{\varphi} = \varphi \circ \Pi$ a.e. This implies that T_φ and $\widetilde{T}_{\tilde{\varphi}}$ are isomorphic. In particular, if φ is a Toeplitz cocycle then $\widetilde{G} = \overline{\theta(\eta)}$, $\widetilde{T} = \tau$. The task arises how to determine $(\widetilde{G}, \widetilde{T})$ (in an effective way) for a general $\varphi \in K$. In particular it would be interesting to know whether given $\varphi: G \to \mathbf{Z}_2$ there exists a *Toeplitz cocycle* ψ such that τ_ψ is isomorphic to T_φ. If this is the case we would have $G = \overline{\theta(\eta)}$. Our paper delivers a construction of such a ψ if φ fullfils the assumptions of Theorem 1.

In the remainder of the paper we consider some problems concerning cocycles admitting a high speed of approximation. For instance, it turns out that for any T

(4) *there is a cocycle φ oddly approximated with an arbitrarily high speed and without roots.*

The next application is connected with the oscillation of the rank power function. Let $U:(Y,\tau,\nu)\circlearrowleft$ be an ergodic automorphism. For the definition of the rank ($\mathrm{rk}(U)$) we refer to **[3]**, **[9]** and to **[19]** for the definition of the maximal spectral multiplicity ($\mathrm{m.s.m.}(U)$). These notions allow to define two functions

$$s\longrightarrow \mathrm{rk}(U^s),\quad s\longrightarrow \mathrm{m.s.m.}(U^s),\quad s=1,2,\ldots$$

called *power rank function* and *power multiplicity function* respectively. They are defined for those s's that U^s is ergodic. In **[9]** J. King raised the question whether the power rank function had to be monotonic (it is rather easy to see that $\mathrm{rk}(U)\le \mathrm{rk}(U^s), s\ge 1$). Then in **[3]** there is an example for which the power rank function is oscillatory. However, $\mathrm{rk}(U)\ge \mathrm{m.s.m.}(U)$ and this example is based on the following facts: for some subsequences $(n_k),(m_k)$ $\mathrm{rk}(U^{n_k})=1$ and $\mathrm{m.s.m.}(U^{m_k})\ge 2$. This note delivers examples of ψ's addmitting an arbitrarily high speed of approximation with

$$\mathrm{rk}((T_\varphi)^s)=\begin{cases}1, & \mathrm{g.c.d.}(s,n_t)=1,\ s-\text{odd}\\ 2, & \mathrm{g.c.d.}(s,n_t)=1,\ s-\text{even}\end{cases}\tag{5}$$

$$\mathrm{m.s.m.}((T_\varphi)^s)=1\ ,\quad \mathrm{g.c.d.}(s,n_t)=1.$$

In **[14]** the author raised the *factors problem* for the class of Morse sequences, i.e. given a Morse cocycle φ we seek all T_φ–invariant sub–σ–algebras $\mathcal{C}\subset\mathcal{B}$. Of course $\mathcal{B}=\{A\times \mathbf{Z}_2, A\in\mathcal{B}\}$ is an example of such a $\mathcal{C}$. The action of T_φ on $\mathcal{B}$ is isomorphic to T. On the other hand the class of ergodic $\mathbf{Z}_2$–extensions of r.p.p.s. automorphisms is closed under taking factors. Is there a $\mathcal{C}\subsetneq\tilde{\mathcal{B}}$ such that action of T_φ on $\mathcal{C}$ (i.e. $T_\varphi:(X\times\mathbf{Z}_2,\mathcal{C},\tilde\mu)\circlearrowleft$) has partly continuous spectrum? We remark the following consequence of coding arguments used in **[14]** (compare it with the analogous result of J. King **[9]** for rank 1 class).

Theorem 2. *If φ is a Morse cocycle then each proper factor of T_φ is rigid. In particular continuous substitutions* **[2]** *on two symbols cannot be factors of Morse dynamical systems.*

The natural reverse problem is the following. Can any rigid Morse dynamical system be extended to a Morse dynamical system with larger group of eigenvalues? There is a positive answer in case of sufficiently high speed of approximation (odd or even), but we have been unable to solve this problem in general. Has a positive answer analogous question for rank 1 class?

Proof of Theorem 1

Assume $T:(X,\mathcal{B},\mu)\circlearrowleft$ with r.p.p.s., $S_p(T)=G\{n_t\colon t\ge 0\}$ and let $\varphi\in K$ admit an odd approximation with speed $0(\frac{1}{n^{1+\epsilon}})$, $\epsilon>0$. Hence there is a subsequence $\{n_{t_k}\}$ satisfying (2). For a simplication of notations we assume that this subsequence is equal to the $\{n_t\}$, i.e.

$$n_t^{1+\epsilon}\mu(\varphi^{-1}(1) \triangle F_t) \underset{t\to\infty}{\longrightarrow} 0, \tag{6}$$

where F_t is a union of an odd number of levels of D^{n_t}. Assume that $\psi \in K$ is another ergodic cocycle. Then it is known that any isomorphism between T_φ and T_ψ is of the form $S_{\psi'}: (X \times \mathbf{Z}_2, \widetilde{\mathcal{B}}, \tilde{\mu}) \circlearrowleft$, where $ST = TS$ and

$$\psi'(Tx) + \psi'(x) = \varphi(x) + \psi(Sx) \quad \text{in } ; \mathbf{Z}_2 \tag{7}$$

where $\psi' \in K$ (see [**17**]). Let us notice that given S, (7) has measurable solution iff $T_{\psi\circ S+\varphi}$ is not ergodic. Indeed, $T_{\psi\circ S+\varphi}$ is ergodic iff there is no measurable solution $\xi: X \longrightarrow S^1$ such that

$$\xi(Tx) = (-1)^{(\psi\circ S+\varphi)(x)} \cdot \xi(x), \tag{8}$$

(see [**7**], [**18**]). If $T_{\psi\circ S+\varphi}$ is not ergodic then we get a measurable solution of (8). Thus, the function $\xi(x) = \xi^2(x)$ is T-invariant, so by the ergodicity of T it is constant. Therefore without loss of generality we can assume that $\xi(x) = \pm 1$. Then the function $\psi' = 1_A$, $A = \xi^{-1}(-1)$ is a solution of (7). On the other hand if ψ' is a solution of (7) then the function $\xi(x) = (-1)^{\psi'(x)}$ satisfies (8).

The following simple lemma will be useful in the proof of Theorem 1.

Lemma 1. *Let $U: (Y, C, \nu) \circlearrowleft$ be an automorphism of a Lebesgue space. Assume that there is a sequence $\{A_n\}$, $A_n \in C$ such that*

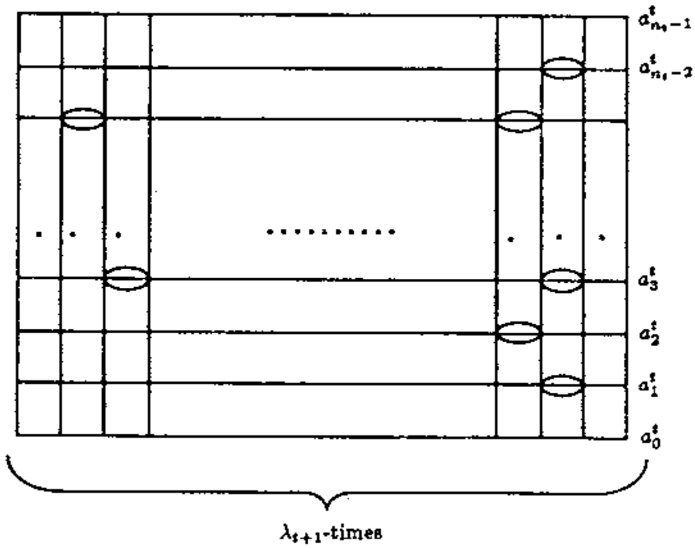
$$\nu(A_{n+1} \triangle A_n) < \epsilon_n, \quad \sum_1^\infty \epsilon_n < \infty \tag{9}$$

$$\nu(UA_n \triangle A_n) \underset{n}{\longrightarrow} 0, \tag{10}$$

$$\exists \delta > 0 \qquad 1 - \delta > \nu(A_n) > \delta, \qquad n = 1, 2, \ldots \tag{11}$$

Then U is not ergodic.

Now, we turn back to the cocycle φ. Fix $t \geq 0$ and consider the tower D^{n_t}.



Picture 1

Then the speed of approximation (which means that (6) holds) says that the function φ restricted to each level $D^{n_t}_i$, $i = 0, 1, \ldots, n_t - 1$, is "almost" constant, i.e.

$$\varphi \mid D^{n_t}_i = a^t_i$$

except for a part of D^{n_t} with measure $\leq \frac{\epsilon_t}{n_t^{1+\epsilon}}$, $\epsilon_t \longrightarrow 0$. Because of the odd approximation,

$$\sum_{i=0}^{n_t-1} a^t_i = 1. \tag{12}$$

To construct $D^{n_{t+1}}$ we divide the tower D^{n_t} into λ_{t+1} columns with the same measure $1/\lambda_{t+1}$ (Picture 1), $\lambda_{t+1} = n_{t+1}/n_t$. The n_t pieces of the s-th column ($s = 0, 1, \ldots, \lambda_{t+1} - 1$) are the levels of $D^{n_{t+1}}$ assigned to the numbers $s \cdot n_t$, $s \cdot n_t + 1, \ldots, s \cdot n_t + n_t - 1$.

We say that there is an error in $D^{n_{t+1}}_j$, $j = i + s \cdot n_t$, $0 \leq i \leq n_t - 1$, $0 \leq s \leq \lambda_{t+1} - 1$ if

$$a^t_i \neq a^{t+1}_j.$$

In Picture 1 such levels are marked by the sign "0 ". Denote by m_t the number of all columns with some errors. The measure μ of such a column is equal to $1/\lambda_{t+1}$. Therefore the measure of all columns with errors is equal to

$$CE^{t+1} = \frac{m_t}{\lambda_{t+1}}. \tag{13}$$

Moreover, the measure μ of any error is equal to $1/n_{t+1}$, so in view of (6)

$$n_t^{1+\varepsilon} \cdot \frac{k_t}{n_{t+1}} \underset{t}{\longrightarrow} 0, \tag{14}$$

where k_t is the number of all errors. But $m_t \le k_t$ and the combination of (13) and (14) implies

$$n_t^{\varepsilon} \cdot CE^{t+1} \underset{t}{\longrightarrow} 0$$

and consequently

$$\sum_{t=0}^{\infty} CE^{t+1} < \infty \tag{15}$$

since $\varepsilon > 0$.

Construction of a Morse cocycle. At each stage t our cocycle ψ will be constant on each level $D_i^{n_t}$, $i = 0, 1, \ldots, n_t - 2$, and will not be defined on $D_{n_t-1}^{n_t}$. We define ψ on D^{n_0} in an arbitrary way and assume that ψ is given on levels of D^{n_t}, $t \ge 0$, i.e.

$$\psi \mid_{D_i^{n_t}} = b_i^t, \quad i = 0, 1, \ldots, n_t - 2.$$

First of all we define $b_{n_t-1}^t$ so that

$$\sum_{i=0}^{n_t-1} b_i^t = 1. \tag{16}$$

We do not change the function ψ on the levels $D_{j\cdot n_t+i}^{n_t}$, $j = 0, 1, \ldots, \lambda_{t+1} - 1$, $i = 0, \ldots, n_t - 2$, i.e. we put

$$\psi \mid_{D_{j\cdot n_t+i}^{n_{t+1}}} = b_i^t.$$

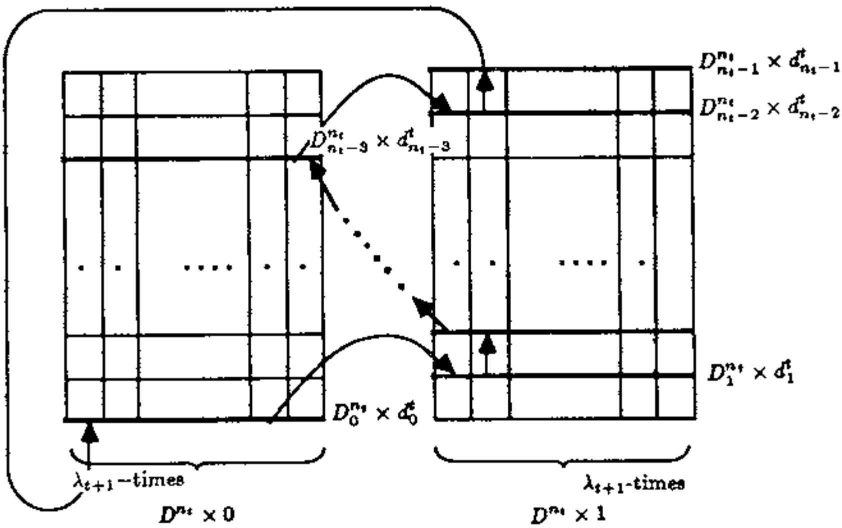
Next we should define the function ψ on the levels $D_{j\cdot n_t+n_t-1}^{n_{t+1}}$, $j = 0, 1, \ldots, \lambda_{t+2} - 2$. To do this we look at the number of errors for φ in the j–th column. If the number of errors is even (in particular if there are no errors) then we put

$$\psi \mid_{D_{j\cdot n_t+n_t-1}^{n_t}} = b_{n_t-1}^t$$

and $1 + b_{n_t-1}^t$ otherwise.

Of course this procedure leads to the definition of a Morse cocycle. Moreover ψ admits an odd approximation with speed $0(1/n^{1+\varepsilon})$, by (16) and (14).

It remains to prove that the cocycle $\varphi+\psi$ is not ergodic, i.e. that $T_{\varphi+\psi}$ is not ergodic. To this end we will define a sequence of sets $\{A_t\}$, $A_t \subset X \times \mathbf{Z}_2$ satisfying the assumptions of Lemma 1. For every $t \geq 0$ we take $D^{n_t} \times 0$ and $D^{n_t} \times 1$.



Picture 2

Put

$$c_i^t = a_i^t + b_i^t\ , \quad i = 0, 1, \ldots, n_t - 1, \quad t \geq 0.$$

Then by (16) and (12)

$$\sum_{i=0}^{n_t-1} c_i^t = 0. \tag{17}$$

We define the sets A_t , $t \geq 1$, putting

$$A_t = \bigcup_{i=0}^{n_t-1} (D_i^{n_t} \times d_i^t),$$

where $d_0^t = 0$ and $d_i^t = c_0^t + \cdots + c_{i-1}^t$, $i = 1, 2, \ldots, n_t$. It is clear that

$$\tilde{\mu}(A_t) = \frac{1}{2}. \tag{18}$$

Now we show that

$$\sum_{t=1}^{\infty} \tilde{\mu}(A_{t+1} \triangle A_t) < \infty.$$

Remark that

$$\tilde{\mu}(A_{t+1} \triangle A_t) = \frac{1}{n_t}.$$

(The number of all levels $D_k^{n_{t+1}}$, $k = j \cdot n_t + i$, $j = 0, \ldots, \lambda_{t+1} - 1$, $i = 0, \ldots, n_t - 1$ such that $d_k^{t+1} \neq d_i^t$). It follows from the definitions of a_i^t and b_i^t that $c_k^{t+1} = c_i^t$ whenever the j–th column contains no errors (with respect to φ) and each of the remaining columns contains an even number of the levels $D_k^{n_{t+1}}$ such that $c_k^{t+1} \neq c_i^t$. Because of (17) we conclude that $d_k^{t+1} = d_i^t$ whenever the j–th column contains no errors. The above considerations show that

$$\tilde{\mu}(A_{t+1} \triangle A_t) \leq \frac{m_t}{\lambda_{t+1}}$$

and then (13) and (15) imply

$$\sum_{t=1}^{\infty} \tilde{\mu}(A_{t+1} \triangle A_t) < \infty. \tag{19}$$

Now, we intend to estimate $\tilde{\mu}(T_{\varphi+\psi} A_t \triangle A_t)$. To this end, let us observe that:

$$T^i_{\varphi+\psi}(x, i) = (T^i x, (\varphi + \psi)(x) + (\varphi + \psi)(Tx) + \cdots + (\varphi + \psi)(T^{i-1} x).$$

In other words, if $\varphi + \psi$ were constant on all levels of D^{n_t} (and equal to c_i^t respectively) then we would get

$$T^i_{\varphi+\psi}(D_0^{n_t} \times 0) = D_i^{n_t} \times (c_0^t + \cdots + c_{i-1}^t) = D_i^{n_t} \times d_i^t, \ i = 1, 2, \ldots, n_t - 1$$

and by (17)

$$T^{n_t}_{\varphi+\psi}(D_0^{n_t} \times 0) = D_0^{n_t} \times 0$$

(see picture 2). So, the above would mean $T_{\varphi+\psi}(A_t) = A_t$. However, $\varphi + \psi$ is not constant on the levels of D^{n_t}. By the argument we have just used it is easy to show that

$$\tilde{\mu}(T_{\varphi+\psi} A_t \triangle A_t) = \frac{\varepsilon_t}{n_t^{\varepsilon}}, \quad \varepsilon_t \longrightarrow 0. \tag{20}$$

Combining (18), (19) and (20) we see that the assumptions of Lemma 1 are satisfied and we conclude that $T_{\varphi+\psi}$ cannot be ergodic.

If we assume that φ admits an even approximation with speed $0(\frac{1}{n^{1+\varepsilon}})$ and that φ is ergodic then we can repeat the foregoing proof with the only change that $b_{n_t-1}^t$ is defined so that

$$\sum_{i=0}^{n_t-1} b_i^t = 0.$$

Therefore the proof of Theorem 1 is complete. ■

Factors of $\mathbf{Z}_2$–extensions given by Morse cocycles

We start with the definition of rigidity. Let $U\colon(Y,\mathcal{B},\nu)\circlearrowleft$ be an ergodic transformation. U is said to be rigid if there exists a sequence of positive integers $\{n_i\}$, $n_i\to\infty$ such that $U^{n_i}\to id$ (the identity) in the weak topology, i.e. $\nu(U^{n_i}(A)\triangle A)\underset{i}{\to}0$ for every $A\in\mathcal{B}$.

Proof of Theorem 2: (In the proof we use notations from [**14**]). Let $x=b^0\times b^1\times\ldots$ be a Morse sequence and let $(\tau,\mathcal{W},\nu)$ be a proper factor of (τ,O_x,μ_x).

Let $\psi\colon(\tau,O_x,\mu_x)\longrightarrow(\tau,\mathcal{W},\nu)$, $\mathcal{W}\subset\{0,1\}^{\mathbf{Z}}$, establish a homomorphism. In order to prove that $\tau\colon\mathcal{W}\circlearrowleft$ is rigid it is enough to show that there is a generic point $w\in\mathcal{W}$ such that for every $\varepsilon>0$ there is $s\in\mathbf{Z}$ such that

$$\bar d(\tau^s w,w)<\varepsilon, \tag{21}$$

where $\bar d(u,u')=\lim_m\inf\frac{1}{m}\mathrm{card}\{1\le i\le m,\ u[i]\ne u'[i]\}$, $u,u'\in O_x$. Fix an $\varepsilon>0$. Then by the Birkhoff Theorem there is a code $\varphi_\varepsilon\colon O_x\longrightarrow\{0,1\}^{\mathbf{Z}}$ (i.e. $\varphi_\varepsilon\tau=\tau\varphi_\varepsilon$, φ_ε is measurable, $z[-k,k]=z'[-k,k]$ implies $(\varphi_\varepsilon z)[0]=(\varphi_\varepsilon z')[0]$, where $k=|\varphi_\varepsilon|$ is the length of the code) such that

$$\bar d(\psi z,\varphi_\varepsilon z)<\frac{\varepsilon}{3}\ \text{for a.e.}\quad z\in O_x. \tag{22}$$

Then take δ, $0<\delta<\varepsilon/300(2\,|\varphi_\varepsilon|+1)$ and fix $w\in\mathcal{W}$. Since ψ cannot be one–to–one, there are $z,z'\in O_x$, $z\ne z'$ such that

$$\psi(z)=\psi(z')=w.$$

Then choosing a code φ_δ we can repeat the proof of Theorem 2 in [**14**] saying that there is an s such that either

$$\bar d(\tau^s z\,,\,z')<100\,\delta \tag{23}$$

or

$$\bar d(\tau^s z\,,\,\tilde z')<100\,\delta. \tag{24}$$

All we have to prove that both (23) and (24) imply (21). First of all Remark the following property of codes.

Lemma 2. *If $u,u'\in O_x$ then*

$$d(\varphi_\varepsilon u\,,\,\varphi_\varepsilon u')\le(2\,|\varphi_\varepsilon|+1)\cdot\bar d(u,u'). \tag{25}$$

Proof: If $u[t-k,t+k]=u'[t-k,t+k]$ then $(\varphi_\varepsilon u)[t]=(\varphi_\varepsilon u')[t]$. Hence if $(\varphi_\varepsilon u)[t]\ne(\varphi_\varepsilon u')[t]$ then it delivers at most $2\,|\varphi_\varepsilon|+1$ places where u and u' are different. ∎

Now

$$\begin{aligned}\bar{d}(r^s w, w) &= \bar{d}(r^s \psi(z), \psi(z')) = \bar{d}(\psi(r^s z), \psi(z')) \\ &\leq \bar{d}(\psi(r^s z), \psi_\varepsilon(r^s z)) + \bar{d}(\varphi_\varepsilon(r^s z), \varphi_\varepsilon(z')) \\ &+ \bar{d}(\varphi_\varepsilon(z'), \psi(z')) \overset{(22)}{\leq} \frac{2\varepsilon}{3} + \bar{d}(\varphi_\varepsilon(r^s z), \varphi_\varepsilon(z')) \overset{(25)}{\leq} \frac{2\varepsilon}{3} \\ &+ (2 \mid \varphi_\varepsilon \mid +1) \cdot \bar{d}(r^s z, z') < \varepsilon\end{aligned}$$

if (23) holds. If (24) holds then using (22) and (25) again we obtain

$$\bar{d}(r^s w, \tilde{w}) < \varepsilon.$$

In both cases we can find a sequence m_s such that either $r^{m_s} \to id$ or $r^{m_s} \to \sigma, \quad (\sigma(u) = \tilde{u})$. In the latter case $r^{2m_s} \to id$. This completes the proof of Theorem 2. ■

Now applying the construction used in the proof of Theorem 1 we are able to indicate some factors of $\mathbf{Z}_2$-extensions determined by Morse cocycles. These factors will have a continuous part of the spectrum.

Assume that $T: (X, \mathcal{B}, \mu) \circlearrowleft$ has r.p.p.s., $S_p(T) = G\{n_t, t \geq 0\}$ and let $\varphi: X \longrightarrow \mathbf{Z}_2$ be a Morse cocycle given by

$$\varphi \mid_{D_i^{n_t}} = a_i^t, \quad i = 0, 1, \ldots, n_t - 2.$$

We can define a sequence of blocks $\{\alpha^t\}, \quad \mid \alpha^t \mid = \lambda_t - 1$ in the following way:

$$\alpha^0 = (a^0[0], \ldots, a^0[\lambda_0 - 2]),$$

$$\alpha^{t+1}[j] = a^{t+1}[j \cdot n_t + n_t - 1], \quad j = 0, 1, \ldots, \lambda_{t+1} - 2.$$

The sequence of blocks $\{\alpha^t\}$ determines the Morse cocycle φ completely. Now, let $s \geq 1$ be such that

$$(s, n_t) = 1, \quad t \geq 0. \tag{26}$$

Consider $\zeta_s: (\mathbf{Z}_s, \nu_s) \circlearrowleft$ the cyclic rotation on $\mathbf{Z}_s = \{0, 1, \ldots, s-1\}$ with the uniform measure. In view of (26)

$$T' = T \times \zeta_s: X \times \mathbf{Z}_s \longrightarrow X \times \mathbf{Z}_s$$

is ergodic and has r.p.p.s., $S_p(T') = G\{s \cdot n_t, t \geq 0\}$. Define a function $\tilde{\varphi}: X \times \mathbf{Z}_s \longrightarrow \mathbf{Z}_2$ by

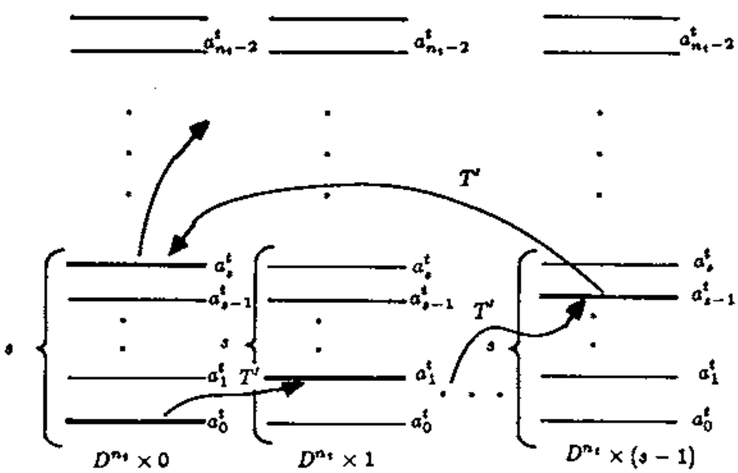
$$\tilde{\varphi}(x, i) = \varphi(x), \quad x \in X, \quad i \in \mathbf{Z}_s.$$

Then we have a $\mathbf{Z}_2$-extensions $(T')_{\tilde{\varphi}} = T_\varphi \times \zeta_s$ of T'. Assume that φ is oddly (or evenly) approximated with speed $0(\frac{1}{n_t^{1+\varepsilon}})$. Then $\tilde{\varphi}$ is approximated with the same speed. If follows from Theorem 1 that $\tilde{\varphi}$ can be modified by a coboundary cocycle i.e. by $\psi' + \psi' \circ T'$ to get a Morse cocycle ψ. Then T_φ is a factor of T'_ψ. We intend to describe this passage from T_φ to T'_ψ in a combinatorial language.

It is easy to see that the condition of the approximation of φ with speed $0(\frac{1}{n_t^{1+\varepsilon}})$ implies

$$\frac{1}{\lambda_{t+1}}\min(fr(0,\alpha^{t+1}), fr(1,\alpha^{t+1})) = 0(\frac{1}{n_t^{\varepsilon}}) \tag{27}$$

where $fr(i,\alpha^{t+1} = \text{card}\{j;\ 0 \le j \le| \alpha^{t+1} |;\ \alpha^{t+1}[j] = i\}$. For every $t \ge 0$ we take the sets $D^{n_t} \times 0$, $D^{n_t} \times 1, \ldots, D^{n_t} \times (s-1)$. Then we can construct a T'–tower of height $m_t = s \cdot n_t$



Picture 3.

We have

$$(T \times \zeta_s)(D_j^{n_t} \times i) = D_{j+1}^{n_t} \times (i+1)\,,$$

where the additions are taken mod n_t and mod s respectively. Let

$$D_k^{m_t} = (T')^k(D_0^{n_t} \times 0)\,, \quad k = 0, 1, \ldots, m_t - 1.$$

Then (26) implies that each $D_k^{m_t}$ coincides with a level $D_j^{n_t} \times i$ for some i, j, $0 \le j \le n_t - 1$, $0 \le i \le s-1$, and this correspondence is one-to-one. Thus

$$D_0^{m_t} \xrightarrow{T'} D_1^{m_t} \xrightarrow{T'} \ldots \xrightarrow{T'} D_{m_t-1}^{m_t} \xrightarrow{T'} D_0^{m_t},$$

so D^{m_t} is a T'–tower of height m_t. If follows from the definition of $\tilde{\varphi}$ (see Picture 3) that

$$\tilde{\psi} \mid_{D_k^{m_t}} = \tilde{a}_k^t = a_i^t \tag{28}$$

if $k = l \cdot n_t + i$ and $i = 0, 1, \ldots, n_t - 2, \quad l = 0, \ldots, s - 1$.

According to (27)

$$\tilde{\psi} \mid_{D_k^{m_t}} = \tilde{a}_k^t = a_{n_t-1}^t \tag{29}$$

if $k = l \cdot n_t + n_t - 1, \quad l = 0, \ldots, s - 1$, where $a_{n_t-1}^t = \tilde{\alpha}_{t+1}$ appears at α^{t+1} with frequency $> 1 - \frac{\varepsilon_t}{n_t^2}$, $\varepsilon_t \to 0$. The symbols $\tilde{a}_u^{t+1}, \quad 0 \le u \le m_{t+1} - 1$ are the following. We write u in the form

$$u = l' \cdot n_{t+1} + v \cdot n_t + r,$$

where $l' = 0, 1, \ldots, s - 1$, $v = 0, \ldots, \lambda_{t+1} - 1$, $r = 0, \ldots, n_t - 1$. Then

$$\begin{cases} \tilde{a}_u^{t+1} = a_i^t, & \text{if} \quad r < n_t - 1, \\ \tilde{a}_u^{t+1} = \alpha^{t+1}[v], & \text{if} \quad v < \lambda_{t+1} - 1 \quad \text{and} \quad r = n_t - 1 \\ \tilde{a}_u^{t+1} = a_{n_{t+1}-1}^{t+1} = \tilde{\alpha}^{t+2}, & \text{if} \quad v = \lambda_{t+1} - 1 \quad \text{and} \quad r = n_t - 1. \end{cases} \tag{30}$$

We define ψ on D^{m_0} in an arbitrary way and suppose that ψ have been defined on D^{m_t} except for the level $D_{m_t-1}^{m_t}$. We should calculate the number of errors in each column of D^{m_t}. Comparing the symbols $\tilde{a}_k^t, 0 \le k \le m_t - 1$ and $\tilde{a}_u^{t+1}, 0 \le u \le m_{t+1} - 1$ (see Picture 4) from (28), (29) and (30) we can formulate the following algorithm of a construction of ψ on $D_{m_t-1}^{m_t}$.

A) Define $\tilde{b}_{m_t-1}^t = \tilde{b}_t$ according to (16), i.e. if $\psi \mid_{D_k^{m_t}} = \tilde{b}_k^t, 0 \le k \le m_t - 1$ we chose $\tilde{b}_t$ in such a way that

$$\sum_{k=0}^{m_t-1} \tilde{b}_k^t = \begin{cases} 1, & \text{if the approximation is odd} \\ 0, & \text{if it is even.} \end{cases}$$

B) Write the block B^t of length $s \cdot \lambda_{t+1}$ as follows

$$B^t = \underbrace{\underbrace{\alpha^{t+1}[0] \ldots \alpha^{t+1}[\lambda - 2]\tilde{\alpha}_{t+2}} \; \underbrace{\alpha^{t+1}[0] \ldots \alpha^{t+1}[\lambda - 2]\tilde{\alpha}_{t+2}} \ldots}_{s-\text{times}}$$
$$\ldots \underbrace{\alpha^{t+1}[0] \ldots \alpha^{t+1}[\lambda - 2]\tilde{\alpha}_{t+2}},$$

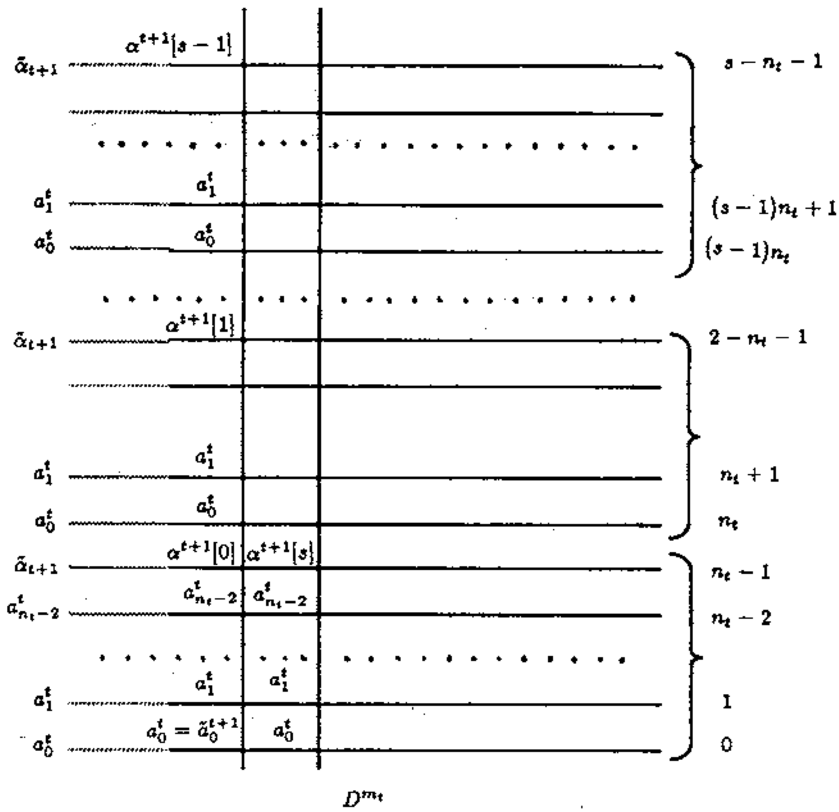
where $\lambda = \lambda_{t+1}$.

C) Divide the block B^t into λ_{t+1} equal parts, i.e.

$$B^t = B_0 B_1 \dots B_{\lambda_{t+1}-1}, \quad |B_j| = s, \quad j = 0, 1, \dots, \lambda_{t+1} - 1$$

and define ψ on $D_u^{m_{t+1}}$, $u = l' \cdot m_t + m_t - 1$, $l' = 0, 1, \dots, \lambda_{t+1} - 2$ putting

$$\psi \mid D_u^{m_{t+1}} = \begin{cases} \tilde{b}_t & \text{if the number of the symbol } (1 + \tilde{\alpha}_{t+1}) \text{ in } B_1 \text{ is even} \\ \tilde{b}_t + 1, & \text{otherwise} \end{cases}$$



Picture 4

According to Theorem 1, T'_φ is metrically isomorphic to $T_\varphi \times \zeta_s$.

The rank power function

Now, we intend to prove the existence of ergodic cocycles satisfying (5). In [11], Kwiatkowski and Rojet discovered a trichotomy concerning Morse shifts. Namely either

$$C(T\varphi) = \{T_\varphi^i \circ \sigma^j\}, \quad i \in \mathbf{Z}, \quad j = 0, 1, \quad \sigma(x, i) = (x, i + 1)$$

or T_φ is rigid. If the latter holds then either

$$C(T_\varphi) = \text{weak closure of } \{(T_\varphi)^n, n \in \mathbf{Z}\}, \tag{31}$$

or

$$C(T_\varphi)/\text{weak closure of } \{(T_\varphi)^n, n \in \mathbf{Z}\} = \{\text{id}, \sigma\}, \tag{32}$$

where $C(T_\varphi)$ is the centralizer of T, i.e. the set of all automorphisms of $(X \times \mathbf{Z}_2, \tilde{\mathcal{B}}, \tilde{\mu})$ commuting with T_φ. No one of these possibilities is vacuous. Combining the Weak Closure Theorem [9] with the fact that the rank of Morse dynamical system is at most 2 one can easily prove that

$$\text{rk}(T_\varphi) = 2 \quad \text{iff} \quad \sigma \notin \text{weak closure}\{(T_\varphi)^n, n \in \mathbf{Z}\}, \tag{33}$$

as soon as φ is a Morse cocycle.

Given $\varphi \in K$ we call the cocycle $\varphi + 1$ (or $T_{\varphi+1} = T_\varphi \circ \sigma$) the completion of φ. The idea to prove (5) consists in the simple observation that a cocycle and its completion can have quite different properties. Take a Morse cocycle φ such that φ is evenly approximated with a given speed (at least $0(\frac{1}{n})$) and moreover that

$$\text{rk}(T_\varphi) = 2.$$

We recall that the paper [**11**] assures an abundance of such cocycles. We assume here that $S_p(T) = G\{n_t, t \geq 0\}, n_t$ are odd. It is an easy observation that the completion of the φ admits then an odd approximation with the same speed.

Therefore

$$\text{rk}(T_{\varphi+1}) = 1$$

and

$$\text{rk}(T_{\varphi+1})^2 = \text{rk}(T_\varphi)^2 \geq \text{rk}(T_\varphi) = 2.$$

Now, $(T_\varphi)^2 = T^2_{\varphi+\varphi\circ T}$ is ergodic and $\varphi + \varphi \circ T$ admits an even approximation with the same speed as φ. Because this speed is at least $0(\frac{1}{n})$ the rank of T_φ is at most 2. Similar reasoning shows that

$$\text{rk}(T_\varphi)^s = \begin{cases} 1, & (s, n_t) = 1, s \text{ is odd} \\ 2, & (s, n_t) = 1, s \text{ is even.} \end{cases}$$

The proof that $m.s.m.(T_\varphi)^s = 1$, $(s, n_t) = 1$ will follow from the following.

Lemma 3. *Let T be an ergodic automorphism with r.p.p.s., $S_p(T) = G\{n_t, t \geq 0\}$. Assume that φ admits an odd or even approximation with speed $0(\frac{1}{n})$ and φ is ergodic and nonconstant. Then T_φ has simple spectrum.*

Proof: The proof follows from the considerations from [**7**]. ■

Remark. If $\mathrm{rk}(T_\varphi) = 2$ and φ admits an even approximation with speed $o(\frac{1}{n^{1+\epsilon}})$ then, of course, T_φ and $T_{\varphi+1}$ cannot be isomorphic ($\mathrm{rk}(T_{\varphi+1}) = 1$). Actually, they cannot be spectrally isomorphic. Indeed, $(T_{\varphi+1})^{m_t} \longrightarrow \sigma$ for a sequence $\{m_t\}$ [9]. If an infinity of the m_t's were even then $(T_\varphi)^{m_t} = (T_{\varphi+1})^{m_t} \to \sigma$ and from (33) T_φ would have rank 1. To avoid the contradiction we are forced to assume that the m_t's are odd. But then

$$(T_\varphi)^{m_t} \to \mathrm{id} \quad \text{and} \quad (T_{\varphi+1})^{m_t} \nrightarrow \mathrm{id}.$$

So T_φ and $T_{\varphi+1}$ cannot be spectrally isomorphic.

Remark. It is interesting to know whether the oscilation of the rank power function is typical for $\mathbf{Z}$–cocycles. But it is not the case, as the following shows. First of all we notice that

$$\mathrm{rk}(T_\varphi)^2 = 2 \quad \text{iff} \quad \mathrm{rk}(T_{\varphi+1}) = 2, \tag{34}$$

as soon as φ admits an odd approximation with speed $o(\frac{1}{n^{1+\epsilon}})$. Indeed, if $\mathrm{rk}(T_\varphi) = 1 = \mathrm{rk}(T_{\varphi+1})$, then

$$\exists m_t \quad (T_\varphi)^{m_t} \longrightarrow \sigma$$

and

$$\exists m_t' \quad (T_{\varphi+1})^{m_t'} \longrightarrow \sigma.$$

Since (33) holds, it is enough to prove that m_t or m_t' are even for an infinity of t's. If this is not the case, then m_t, m_t' are odd. We can also assume $m_t - m_t' \nearrow \infty$ by passing to subsequences. Then

$$(T_\varphi)^{m_t} \circ (T_{\varphi+1})^{-m_t'} \longrightarrow \sigma \circ \sigma^{-1}$$

so

$$(T_\varphi)^{m_t} \circ (T_\varphi)^{-m_t'} \circ \sigma \longrightarrow \sigma \circ \sigma^{-1}$$

which implies

$$(T_\varphi)^{m_t - m_t'} \longrightarrow \sigma$$

and

$$(T_\varphi)^{2s_t} \longrightarrow \sigma\,, \text{ where } \quad 2s_t = m_t - m_t'.$$

If $\mathrm{rk}(T_{\varphi+1}) = 2$, then

$$2 \geq \mathrm{rk}((T_\varphi)^2) = \mathrm{rk}((T_{\varphi+1})^2) \geq \mathrm{rk}(T_{\varphi+1}) = 2.$$

Now the set

$$\mathcal{A} = \{\varphi \in \mathcal{K},\, \mathrm{rk}(T_\varphi) = 1\}$$

is residual because it contains all cocycles oddly approximated with speed $o(\frac{1}{n})$. Therefore the set $\mathcal{A}+1=\{\varphi+1, \varphi\in\mathcal{A}\}$ is residual. Whence

$$\mathcal{A}\cap(\mathcal{A}+1)=\{\varphi\in K, \operatorname{rk}(T_\varphi)=1=\operatorname{rk}(T_{\varphi+1})\}$$

is residual. This fact and (34) give that the rank power function being constant (and equal 1) is a typical property for K.

Example. The next problem we intend to deal with here is the problem of lifting roots. It is well-know how to calculate roots for T with r.p.p.s., $S_p(T)=G\{n_t;\ t\geq 0\}$. It $(s,n_t)=1,\ t\geq 0$ then T^s and T are isomorphic because they have r.p.p.s. and they have the common sequence $(D^{n_t})\nearrow \mathcal{B}$. It would be interesting to know whether a sufficiently high speed of approximation of φ assures the existence of some roots. However the results of [**11**] show that this supposition cannot be true.

We take the Morse sequence

$$x=b^0\times b^1\times\dots$$

where

$$b^t=\overbrace{0101\dots01}^{\mu_t}\ \overbrace{1010\dots101}^{\mu_t+1},\ t\geq 0.$$

Assume that

$$\sum_0^\infty\frac{1}{\mu_t}<\infty.$$

Put

$$\lambda_t=2\mu_t+1,\quad n_t=\lambda_0\cdot\ldots\cdot\lambda_t,\quad c_t=b^0\times\cdots\times b^t,\ t\geq 0$$

and define blocks $\hat{c}_t$, $|\hat{c}_t|=n_t-1$ putting

$$\hat{c}_t[i]=c_t[i+1]+c_t[i]\quad\text{in }\mathbf{Z}_2,\ i=0,1,\dots,\lambda_t-2.$$

Let $\triangle$ be the group of all n_t-adic numbers i.e.

$$\triangle=\{g;\ g=\sum_0^\infty g_t\cdot n_{t-1},\ 0\leq g_t\leq\lambda_t-1,\ n_{-1}=1\},$$

and let μ be the Haar measure of $\triangle$ and T be the rotation on $\hat{1}=(1,0,0,\dots)$. For every $t\geq 0$ we have a T-power D^{n_t} in $\triangle$ such that

$$D_i^{n_t}=\{g;\ \sum_{u=0}^t g_u\cdot n_{u-1}=1\},\quad i=0,1,\dots,n_t-1.$$

The sequence x determines a Morse cocycle φ on $\triangle$ by

$$\varphi\mid_{D_i^{n_t}}=\hat{c}_t[i],\quad i=0,1,\dots,n_t-2.$$

In [**11**] the centralizer of T_φ has been described. Each $S \in C(T_\varphi)$ has a form

$$S(g,i) = (g + g_0\,,\, i + f(g)),$$

where $g_0 \in \triangle$ and $f: \triangle \longrightarrow \mathbf{Z}_2$ satisfy the condition

$$\varphi(g + g_0) + f(g) = f(g + \hat{1}) + \varphi(g). \tag{35}$$

We will show that for every $k > 1$ there exists no $S \in C(T_\varphi)$ such that $S^k = T_\varphi$. Remark that

$$S^k(g,i) = (g + k \cdot g_0\,,\, i + f(g) + \cdots + f(g + (k-1)g_0)).$$

Then $S^k = T_\varphi$ iff $k \cdot g_0 = \hat{1}$ and

$$f(g) + \cdots + f(g + (k-1)g_0) = \varphi(g), \tag{36}$$

where g_0 and f satisfy (35).

Notice that there exists $g_0 \in \triangle$ with $k \cdot g_0 = \hat{1}$ iff $(k, n_t) = 1$. We will denote such g_0 by $\hat{1}/k$. Now we prove that for $k > 2$, $(k, n_t) = 1$, $\hat{1}/k$ does not satisfy (35) and $\hat{1}/2$ satisfies (35) but the corresponding function f does not satisfy (36).

Now, let $k > 2$, $(k, n_t) = 1$, $t = 0, 1, \ldots$. If $\hat{1}/k = (l_t)_0^\infty$, $0 \le l_t \le n_t - 1$, $l_{t+1} \equiv l_t \pmod{n_t}$ then l_t is one of the numbers

$$\frac{n_t + 1}{k}, \frac{2n_t + 1}{k}, \ldots, \frac{(k-1)n_t + 1}{k}.$$

For such a number l_t we have

$$\frac{l_t}{n_t} \ge \frac{1}{k}\,,\, 1 - \frac{l_t}{n_t} \ge \frac{1}{2k} \text{ and } \left|\frac{l_t}{n_t} - \frac{1}{2}\right| \ge \frac{1}{3k}$$

for t large enough.

If $\hat{1}/k$ is represented as a series $\sum_0^\infty \bar{g}_t \cdot n_{t-1}$, then the inequalities

$$\left| \frac{l_t}{n_t} - \frac{\bar{g}_t}{\lambda_t} \right| \le \frac{1}{\lambda_t} - \frac{1}{n_t} \le \frac{1}{2\lambda_t}$$

imply

$$\frac{\bar{g}_t}{\lambda_t} \ge \frac{1}{4k}, \quad 1 - \frac{\bar{g}_t}{\lambda_t} \ge \frac{1}{4k}, \quad \left| \frac{g_t}{\lambda_t} - \frac{1}{2} \right| \ge \frac{1}{4k} \tag{37}$$

for t large enough. On the other hand it is easy to remark that the inequalities

$$\frac{g_i}{\lambda_i} > \delta\,,\, 1 - \frac{g_i}{\lambda_i} > \delta \quad \text{and } \left| \frac{g_i}{\lambda_i} - \frac{1}{2} \right| > \delta\,,\, 0 \le g_i \le \lambda_i - 1$$

imply

$$d\big(b^t b^t [g_t, g_t + \lambda_t - 1] b^t\big) \geq \delta. \tag{38}$$

Moreover, (38) is valid if we replace $b^t b^t$ by $b^t \tilde{b^t}, \tilde{b^t} b^t$ and $\tilde{b^t}\tilde{b^t}$. In view of Theorem 1 in [**11**], (37) and (38) $g_0 = \hat{1}/k$ does not satisfy (35).

Now take $\hat{1}/k$. Then $\frac{\hat{1}}{k} = (\frac{n_t+1}{2})_0^\infty = (\mu_0 + 1) + \sum_{t=1}^\infty \mu_t \cdot n_{t-1}$. It is clear that

$$d\big(b^t \tilde{b^t} [\mu_t + 1, \mu_t + \lambda_t], b^t\big) = \sum_0^\infty \frac{1}{\lambda_t} < \infty.$$

Applying again [**11**] we obtain that (35) is satisfied with $g_0 = \hat{1}/2$ and the function $f = \lim f_t$, where

$$f_t(g) = c_t \tilde{c}_t \left[\frac{n_t + 1}{2} + j_t\right] + c_t [j_t] \right] , \ g = (j_t), \ 0 \leq j_t \leq n_t - 1.$$

Next we have

$$f_t(g) + f_t(g + \hat{1}/2) = \hat{c}_t [j_t + 1] + c_t \left[\frac{n_t + 1}{2} + j_t\right] + c_t \left[\frac{n_t + 1}{2} + j_t\right] + c_t [j_t] =$$

$$= \overbrace{c_t [j_t + 1] + c_t [j_t] + 1}^{\varphi(g)}, \qquad \text{for } g = (j_t) \text{ and } j_t \leq \frac{n_t - 3}{2}.$$

We conclude that (36) does not hold and T_φ has no roots of any degree. At the same time the function φ admits an odd approximation with the speed $o(\frac{1}{\lambda_{t+1}} \cdot \frac{1}{n_t})$.

Taking suitable λ_t's we can get Morse cocycles φ's admitting an approximation with an arbitrarily high speed.

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Rebut el 26 de Maig de 1987