FIBRING THE COMPLEMENT OF THE FENN-ROLFSEN LINK

ROGER FENN

Abstract

In this note it is shown that the complement of the singular linked spheres in four dimensions defined by Fenn and Rolfsen can be fibred by tori.

Also a symmetry between the two components is revealed which shows that the image provides an example of a Spanier–Whitehead duality. This provides an immediate proof that the α-invariant is non zero.

1. Introduction

Define a *link map* of two spheres to be a map

$$f : S_1^p \cup S_1^q \longrightarrow \mathbf{R}^m \text{ (or } S^m)$$

of disjoint spheres of dimension p and q into a Euclidean space or sphere of the same dimension m, such that the images of the two components are disjoint.

$$\text{So } f(S_1^p) \cap f(S_2^q) = \phi.$$

Let $f_1 = f|S_1^p$ and $f_2 = f|S_2^q$. By an abuse of notation the suffixes 1 and 2 will often be dropped when no confusion can arise.

In this paper only low dimensions will be considered: in particular the classical case $p = q = 1$, $m = 3$ and the next interesting case $p = q = 2$, $m = 4$.

For higher dimensions see [K] or [F, H].

Two link maps are said to be *link-homotopic* if they are homotopic through link maps.

Given a link map f as above there is a map

$$\varphi : S^p \times S^q \longrightarrow S^{m-1}$$

defined by

$$\varphi(x,y) = \frac{f(x) - f(y)}{\|f(x) - f(y)\|}$$

The homotopy class of φ written $\alpha(f)$, may be regarded as lying in $\pi_{p+q}(S^{m-1})$ if $p, q \leq m - 2$ and is a link homotopy invariant.

In the classical case $\alpha(f)$ is an integer and corresponds to the linking number. Moreover $\alpha(f)$ in this dimension classifies links up to homotopy.

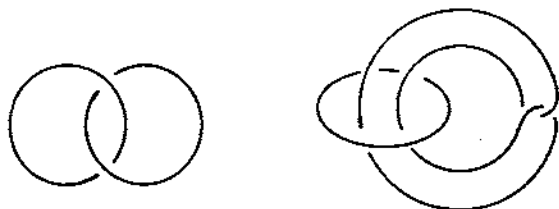As examples consider the Hopf link and the whitehead link illustrated in figure 1.

Figure 1   Hopf link   Whitehead link

The Hopf link has linking number ± 1 and so is not null homotopic (homotopic to a pair of points).

However the Whitehead link is nullhomotopic as can easily be seen. Actually it will be shown that the Whitehead link is nullhomotopic in a rather stronger sense which depends on its symmetry.

For the case of link maps $f : S^2 \cup S^2 \to \mathbf{R}^4$ the α–invariant lies in $\pi_4 S^3 \cong \mathbf{Z}_2$. It is known that α is not a complete invariant in this case, see [K].

2. Constructing linked 2–spheres from linked circles

A link map $f : S^2 \cup S^2 \to \mathbf{R}^3$ is said to be *doubly null homotopic* if each component is null homotopic in the complement of the other. For linked circles with unknotted components being doubly nullhomotopic is the same as having linking number zero although it is not often easy to see.

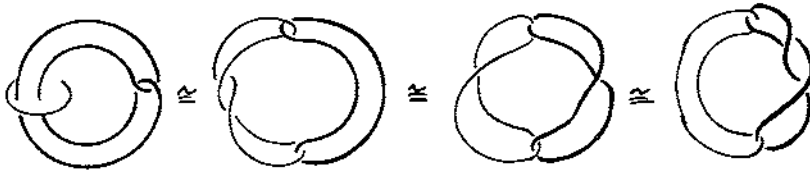Figure 2 illustrates the fact that the Whitehead link is doubly null homotopic.

Figure 2. Symmetry of the Whitehead link

Let $f_{1t}, f_{2t}, 0 \leq t \leq 1$ denote the homotopies of a doubly nullhonotopic link. So $f = f_{10} \cup f_{20}$ whilst f_{11} and f_{21} are constant maps. Moreover $f_{1t} \cup f_{20}$ and $f_{10} \cup f_{2t}$ are link–homotopies which shrink one component while fixing the other.

Let (x, t), $x \in S^1$ and $-1 \leq t \leq 1$, denote cylindrical coordinate on the sphere S^2.

Define $\hat{f} : S_1^2 \cup S_2^2 \to \mathbf{R}^4$ by

$$\hat{f}(x_1, t) = \begin{cases} (1 - t)(f_{1t}(x_1), t), & 0 \leq t \leq 1, x_1 \in S_1^1 \\ (1 + t)(f_{10}(x_1), t), & -1 \leq t \leq 0, x_1 \in S_1^1 \end{cases}$$

$$\hat{f}(x_2, t) = \begin{cases} (1 - t)(f_{20}(x_2), t), & 0 \leq t \leq 1, x_2 \in S_2^1 \\ (1 + t)(f_{2-t}(x_2), t), & -1 \leq t \leq 0, x_2 \in S_2^1 \end{cases}$$
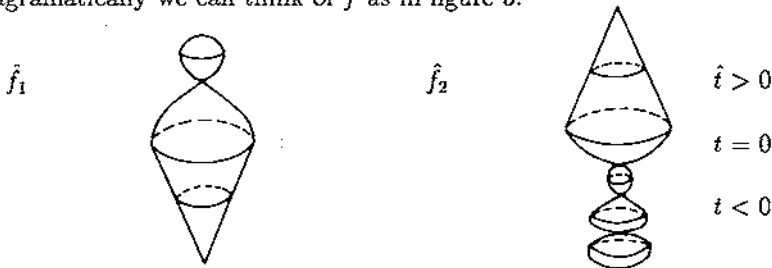
Diagramatically we can think of $\hat{f}$ as in figure 3.

Figure 3. Constructing $\hat{f}$

The correspondence $f \to \hat{f}$ is well defined up to link homotopy because of the asphericity of knots.

If f is the Whitehead link, $\hat{f}$ is the Fenn-Rolfsen link. This case is illustrated in figure 4. Sections are drawn according to approximate choices of the coordinate t.
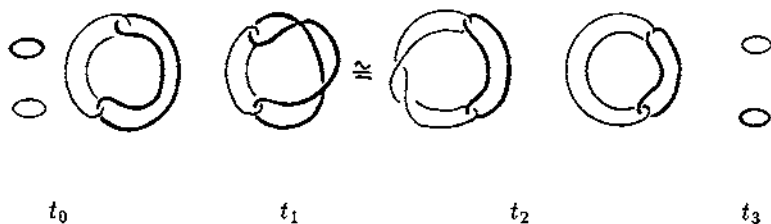
t_0 t_1 t_2 t_3

Figure 4. Sections of the Fenn–Rolfsen link

In figure 4 critical occurences happen at times t_0, t_1, t_2, t_3. They are described in the following list:

$$t_0 : \text{minima of } f_1 \text{ and } f_2$$
$$t_1 : \text{crossing point of } f_1$$
$$t_2 : \text{crossing point of } f_2$$
$$t_3 : \text{maxima of } f_1 \text{ and } f_2$$

Between t_1 and t_2 is an isotopy of the Whitehead link.

3. The double Hopf map

Let $h : S^3 \to S^2$ denote the Hopf map. If $x \in S^2$ then $h^{-1}x$ is an unknotted circle and for two distinct points $x, y \in S^2$ $h^{-1}x, h^{-1}y$ is a Hopf link. As x varies over S^2 the circles $h^{-1}x$ form the Hopf fibration of S^3.

Consider the suspension of $h; \Sigma h : \Sigma S^3 \to \Sigma S^2$.

Identify ΣS^3 with S^4 and ΣS^2 with S^3. Under the identification of ΣS^2 with S^3 let the suspension points lie in $h^{-1}x_+$ and $h^{-1}x_-$. The composition of Σh with h gives a map $S^4 \xrightarrow{g} S^2$.

$$S^4 \cong \Sigma S^3 \xrightarrow{\Sigma h} \Sigma S^2 \cong S^3 \xrightarrow{h} S^2$$

Consider now the inverse images $g^{-1}x$ as x varies over S^2. If $x \neq x_+$ then $g^{-1}x := T_x$ in a torus. If $x_+ \neq x_-$, which can be assumed by slightly changing the identification of ΣS^2 with S^3 if necessary, $g^{-1}x_+ = T_+$ and $g^{-1}x_- = T_-$ are pinched tori. That is a torus with one meridian identified to a point.

Figure 5

It is interesting to note that if $x_+ = x_-$, i.e. the suspension points both lie in the same fibre of S^3, then $T_+ = T_-$ is a Montesinos twin, that is a pair of spheres meeting in two points. See [Mo] and the discussion in [Ma].

Theorem. *With the notation above the pinched tori $T_\pm \subset S^4$ may be identified with the image of the Fenn-Rolfsen link.*

Proof: Generic cross sections of the tori $T_\pm$ look as follows: The torus T_+ is indirected by a heavier line than T_-.

t_0 t_1 t_2 t_3 t_4 t_5 t_6 t_7

Figure 6. Cross sections of the pinched tori $T_\pm$

In figure 6 critical ocurrences happen at times $t_0, \ldots, t_7$. They are described in the following list:

t_0 – two T_+ minima

t_1 – T_+ crossing (pinching of a meridian)

t_2 – T_- minimum

t_3 – T_- saddle

t_4 – T_+ saddle

t_5 – T_+ maximum

t_6 – T_- crossing (pinching of a meridian)

t_7 – two T_- maxima

Figure 6 is shown to represent the same two pinched tori as figure 4 by cancelling a saddle with a minimum on T_+ and a saddle with a maximum on T_- as illustrated in figure 7 on the component T_+. This is achieved by pushing over a 3- ball (shown shaded).
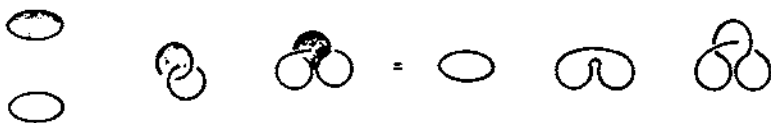

Figure 7. Cancelling a saddle with a minimum

This process applied to T_+ and T_- results in the cross sections of figure 8 which can be readly seen to be the same as figure 4.
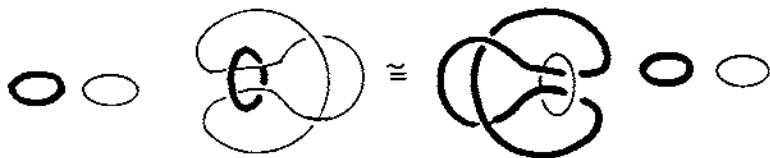

Figure 8

4. Spanier–Whitehead duality

The sphere S^2 has a *singular fibration* by circles in which $c_\pm = x_\pm$ are the two poles and $\{c_\lambda\}$ are the latitude circles, $-1 < \lambda < 1$.

Consider the following subspaces of S^4

$$N_+ = g^{-1} \bigcup_{[0,1]} c_\lambda \qquad N_- = g^{-1} \bigcup_{[-1,0]} c_\lambda$$

$$M = g^{-1} c_0$$

The *halfway* manifold M is a torus bundle over the circle with bundle map $g|M$ and holonomy given by the shear matrix $\begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$.

The manifolds $N_\pm$ are regular neighbourhoods of $T_\pm$ and $S^4 = N_+ \cup N_-$, $N_+ \cap N_- = \partial N_+ = \partial N_- = M$.

So $T_\pm$ is a deformation retract of $S^4 - T_\mp$

Details of the following can be found in [S, W].

Let A, B be spaces with base points a_0, b_0 respectively. Let $A \vee B \subset A \times B$ be the one point union.

$$A \vee B = A \times b_0 \cup B \times a_0$$

and let $A \wedge B$ be the smash product

$$A \wedge B = A \times B / A \vee B$$

An n-duality is a map $u : A \wedge B \to S^n$ which induces an isomorphism between stable homotopy and cohomotopy groups $\pi_k^S A \to \pi_S^{n-k} B$ under the correspondence

$$\{ S^k \xrightarrow{f} A \} \longrightarrow \{ S^k \wedge B \xrightarrow{f \wedge 1} A \wedge B \xrightarrow{u} S^n \}$$

For example if $A, B \subset S^{n+1}$ and $A \cap B = \phi$, A is a deformation retract of $S^{n+1} - B$ and conversely, then the map $u : A \wedge B \to S^n$ given by

$$u(a, b) := \frac{a - b}{\|a - b\|}$$

is an n-duality.

We call the pair A, B in this case an n-duality pair.

It is clear from the discussion above that T_+, T_- is a 3-duality pair. The map φ which defines the α-invariant for f factors as follows:

$$\begin{array}{ccc} & T_+ \wedge T_- & \\ \Psi = \hat{f}_1 \times \hat{f}_2 \nearrow & & \searrow u \\ S^2 \times S^2 & \xrightarrow[\varphi]{} & S^3 \end{array}$$

The map Ψ is of degree 1 and u is a 3-duality so clearly the class of the composite $[\varphi] = \alpha$ cannot be zero.

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Mathematics Division
Sussex University
Mathematics and Physics Building
Brighton BNI 9QH
ENGLAND