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THE WORK OF JOSE LUIS RUBIO DE FRANCIA I

The aim of these pages is to give the reader an idea about the first part of the mathematical life of José Luis Rubio de Francia.

José Luis was an undergraduate student at the University of Zaragoza from 1966 to 1971, and a graduate student from 1971 to 1974. His advisor was professor Luis Vigil, who introduced José Luis to Harmonic Analysis. He began studying Fourier Analysis on groups. In fact, the goal of his thesis was to study in the Abstract Harmonic Analysis context Vigil's unpublished monograph "Series de Fourier en medida", Beca March 1957, see **[V]**.

José Luis took two starting points:

The first one, due to Kolmogorov, states that given a function f in $L^1([0,1])$, the functions

$$f_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} f(x+\frac{j}{n}),\ \ n = 1,2,3,...$$

converge in L^1, to $I = \int_0^1 f(x)dx$, see **[Z]**.

More precisely, if w_p denotes the modulus of continuity in L^p, one finds (see **[V]**)

$$\|f_n - I\|_{L^p} \leq w_p(f;\frac{1}{2n}),\ \ 1 \leq p \leq \infty.$$

The second starting point was due to Vigil.

Given a function f in $L^1([0,1])$, we consider the functions

$$c_k(x) = f(x)e^{-2\pi ikx}$$

and

$$c_{k,n}(x) = \frac{1}{n}\sum_{j=0}^{n-1} c_k(x+\frac{j}{n}).$$

The following identity holds

$$f(x) = \sum_{k=-n}^{n} c_{k,2n+1}(x)e^{2\pi ikx},$$

and we have convergence, in the L^1-norm, of $c_{k,n}$ to the k-th Fourier coeficient of f, namely

$$\int_0^1 f(x)e^{-2\pi ikx}dx.$$

Thus one could have, at least formally, some estimates for the convergence of Fourier Series.

José Luis made already the observation that with these techniques in abstract groups, he could only obtain criteria of Dini-Lipschitz type for the convergence of Fourier Series, although he became an expert in Abstract Harmonic Analysis with the development of his doctoral dissertation, see [**R de F1**].

Now I shall comment on one of the results he proved in this subjet, see [**R de F1**], [**R de F2**].

G will denote a locally compact group with identity e and left Haar measure m. A normal closed subgroup of G will be written H, m_H will be a left Haar measure for H and given a function f defined on G

$$f_H(x) = \int_H f(xt)dm_H(t)$$

will be defined whenever the right-hand side exists (a.e.). The function $\tilde{f}(\pi(x)) = f_H(x)$ is then well defined on $G/_H$, and there is a left Haar measure $\tilde{m}$ on $G/_H$ such that Weil's identity holds:

$$\int_G f dm = \int_{G/_H} \tilde{f} d\tilde{m} \ \ (f \in L^1(G)).$$

Theorem 1. *Let V be a relatively compact open neighbourhood of the identity, such that $VH = G$. Let $f \in L^1 \cap L^p(G)$, with integral $I = \int f dm$. Then*

(1.1) if $1 \leq p < \infty$, $(\int_{G/_H} |\tilde{f} - I|^p d\tilde{m})^{1/p} \leq w_p(f; V)$,

(1.2) if $1 \leq p < \infty$ and G is compact

$$(\int_G |f_H - I|^p dm)^{1/p} \leq w_p(f; V)$$

and

(1.3) if $p = \infty$ and $supp(f) = K$ compact

$$\sup_{x \in G} |f_H(x) - I| \leq m(VK) w_\infty(f, V).$$

As I said before, while he was doing his thesis, José Luis became an expert in Abstract Harmonic Analysis. But not only on that. In his thesis one of the modes of convergence studied was the convergence in measure and again the treatement of the subject was very deep, in fact he made some significant contributions to the theory.

He studied the convergence in measure in a general measure space (X, m) as follows, see [**M, R de F**], [**R de F3**]:

Define $S(m)$ as the set of measurable functions which are zero outside a set of finite measure, and $L^0(m) = L^\infty(m) + S(m)$. Let $\| \cdot \|_0$ be the functional in $L^0(m)$ given by

$$\|f\|_0 = \inf\{s > 0 : \ m(\{x \in X : \ |f(x)| > s\}) < s\}$$

then we have the following.

Theorem 2. *$L^0(m)$ is the space of measurable functions f such that $\|f\|_0 < +\infty$. $\|\cdot\|_0$ is a (F)-norm in $L^0(m)$ and with this (F)-norm $L^0(m)$ is a complete metric space. The convergence in $(L^0(m), \|\cdot\|_0)$ coincides with the convergence in measure.*

He analysed the dual of $L^0(m)$. It is well known that if m is finite and non-atomic then the dual of $L^0(m)$ is $\{0\}$. But when this is not the case he proved the existence of a functional $L \in (L^0(m))^*$ such that $L(f) \neq 0$ for some $f \in L^0(m)$.

In order to prove the last assertion he introduced the following quasinorm in $L^0(m)$, see **[R de F3]**:

$$q(f) = \inf\{\sup_{x\in E} |f(x)| : \ m(E) < +\infty\}$$

and he observed the following facts:

(i) $q(f) < +\infty$ if and only if f belongs to $L^0(m)$,

(ii) $q(f) \leq \|f\|_0$

(iii) The induced (F)-norms in $L^0(m)/_{q^{-1}(0)}$ by $\|\cdot\|_0$ and $q(\cdot)$ are the same.

(iv) $\overline{S(m)} = q^{-1}(0)$.

His deep knowledge of the space $L^0(m)$ was complemented later on when he had contact with Nikishin's Theorem. I would like to suggest to the reader to have a look to the interesting and elegant proof of the Nikishin's Theorem given in section VI.2 of [GC, **R de F**].

From 1974 to 1976 he was a Visiting Fellow in the Institute for Advanced Study of Princeton (USA). In 1976 he got a position as full professor at the Universidad Complutense de Madrid.

In the autumn of 1977 he went back to Universidad de Zaragoza, remaining there until the autumn of 1981.

That was the "gold mathematical period" at Universidad de Zaragoza. He had 7 students, he was the leader of the mathematical community and moreover he was the usual reference to be consulted in any mathematical or human problem.

His mathematical production around 1979 was related with the problem of the vector-valued extension of operators. The problem can be stated as follows:

Let (X, μ) and (Y, ν) be σ-finite measure spaces. Given a bounded linear operator T from $L^p(\mu)$ to $L^0(\nu)$, and a Banach space B, the operator

$$T^B = T \otimes 1_B : \ \sum b_i f_i(x) \to \sum b_i T f_i(y) \ (b_i \in B; \ f_i \in L^p(\mu))$$

is defined a priori on $L^p(\mu) \otimes B$. If T is of weak or strong type (p, q), or simply continuouss in measure, one can ask if the corresponding continuity condition holds for T^B, in which case, it can be uniquely extended to $L^p_B(\mu)$.

In fact José Luis proposed the following question. See [**R de F4**]:

(Q) Let T be an operator of (weak or strong) type (p,q), where $0 < p,\ q \leq \infty$, with norm $\|T\|$. It is true that T^B is also of (weak or strong) type (p,q) with $\|T^B\| \leq M_{p,q}\|T\|$?.

He gave the following partial answer, see [**R de F4**].

Theorem 3. *Question* (Q) *has affirmative answer in the following cases:*

(i) *When T is a positive operator of weak or strong type (p,q), $0 < p,\ q \leq \infty$.*
(ii) *When T is an operator of weak or strong type (p,q); $0 < p < \infty$, $0 < q \leq \infty$ and B is a Hilbert space.*
(iii) *When T is an operator of strong type* $(1,1)$.
(iv) *When T is an operator of weak type (p,q), $1 \leq p < q \leq \infty$, and B is a p-space.*
(v) *When T is a singular integral operator (bounded from $L^p(\mathbf{R}^n)$ to itself, $1 < p < \infty$) and $B = \ell^r$, $1 < r < \infty$.*

In order to build the proof of this Theorem, José Luis handled the tools listed below:

(A) The well known Theorem of Marcinkiewicz and Zygmmund, see [**M,Z**], which states that if T is a bounded operator from $L^p(\mu)$ into $L^q(\nu)$, $0 < p,\ q < \infty$, and H is a Hilbert space then T^H is of strong type (p,q).

(B) Cotlar's inequality, see [**Co**], relating the weak L^q-norm with the strong L^r-norm $(0 < r < q)$ of the restriction to sets of finite measure, namely

$$\|f\|_{q,\infty} \leq N_{q,r}(f) \leq \left(\frac{q}{q-r}\right)^{1/r} \|f\|_{q,\infty}$$

where $N_{q,r}(f) = \sup\limits_{\nu(E)<+\infty} \nu(E)^{1/q-1/r}\|f\mathcal{X}_E\|_r$.

(C) Interpolation theory.

(D) When T is positive (i.e. $f \geq 0 \Rightarrow Tf \geq 0$), then for any Banach space B, we have

$$\|T^B f(y)\| \leq T(\|f\|)(y) \quad (f \in B \otimes L^p(\mu)).$$

(E) The notion of p-space, see [**H**]. That means the Banach spaces B such that for any T which maps $L^p(\mu)$ into $L^p(\nu)$ then T^B maps $L^p_B(\mu)$ into $L^p_B(\nu)$ and $\|T^B\| = \|T\|$.

(F) Theorem of Nikishin-Stein see [**GC, R de F**].

Let $T:\ L^p(\mu) \to L^0(\nu)$ be a continuous sublinear operator, with $0 < p < \infty$. Then, there exists $w(x) > 0 \quad \nu$ a.e. such that

$$\int_{\{x:\ |Tf(x)|>\lambda\}} w(x)d\nu(x) \leq \left(\frac{\|f\|_p}{\lambda}\right)^q \qquad (f \in L^p(\mu),\ \lambda > 0)$$

where $q = \inf(p, 2)$. Moreover, if T is positive than we can take $q = p$.

Now, I shall mention two applications of Theorem 3, see [**R de F4**].

Application 1. Consider linear operators $T, T_n : L^p(\mu) \to L^0(\nu)$, $0 < p < \infty$, and a Banach space B; assume that one of the following conditions holds:

(a) the operators T, T_n are positive.

(b) B is a Hilbert space.

Then

(i) if $T : L^p(\mu) \to L^0(\nu)$, $0 < p < \infty$, is continuous in measure, so is the operator $T^B : L^p_B(\mu) \to L^0_B(\nu)$.

(ii) if $T, T_n : L^p(\mu) \to L^0(\nu)$, $0 < p < \infty$, are continuous in measure ($n \in \mathbf{N}$) and $Tg(y) = \lim_n T_n g(y)$, ν a.e. y, for every $g \in L^p(\mu)$, then

$$\lim_n T_n^B f(y) = T^B f(y), \qquad \nu \quad \text{a.e. } y \quad (f \in L^p_B(\mu)).$$

Application 2. Let G be a compact connected abelian group, with dual group $\hat{G} = \Gamma$. Let $(I_j)_{j \in \mathbf{N}}$ be intervals in Γ, and let $(f_j)_{j \in \mathbf{N}}$ be functions in $L^1(G)$. Then, there is a constant C, depending only on the group G, such that

$$\mu(\{x \in G : (\sum_{j=1}^{\infty} |S_{I_j} f_j(x)|^2)^{1/2} > t\}) \leq \frac{C}{t} \int_G (\sum_{j=1}^{\infty} |f_j|^2)^{1/2} d\mu$$

where μ is the Haar measure on G and $S_I : L^1(G) \to L^0(G)$ is defined for a trigonometric polynomial g by $(S_I g)\hat{} = \hat{g}\mathcal{X}_I$, and extended by continuity.

One of the main goals in the work of José Luis, was to understand the problem of the almost everywhere convergence of the Fourier Series. Using the thecniques that I have discussed above, he obtained the following result for double Fourier Series, see [**R de F5**].

Theorem 4. *Let $G = T \cong [0, 1)$ be the torus, and let*

$$f(x, y) \sim \sum_{j,k} c_{jk} e^{2\pi i(jx+ky)}$$

be the Fourier Series of a function $f \in L^p(T^2)$, $1 < p < \infty$.

Then

$$\lim_{n,m} \int_T |\sum_{|j| \leq n} \sum_{|k| \leq m} c_{j,k} e^{2\pi i(jx+ky)} - f(x, y)|^p dx = 0$$

(a.e. $y \in T$).

This can be consider as an intermediate result between the convergence in norm

$$\lim_{n,m} \int_{T^2} |\sum_{|j| \leq n} \sum_{|k| \leq m} c_{j,k} e^{2\pi i(jx+ky)} - f(x, y)|^p dx dy = 0$$

and the negative result for almost every convergence, that is, the assertion that

$$\lim_{n,m} \sum_{|j|\leq n} \sum_{|k|\leq m} c_{j,k} e^{2\pi i(jx+ky)} = f(x,y) \text{ a.e.}$$

can be false even for a continuous function, see [**F**].

The human and mathematical contact that is usually established between a student and his advisor was, in the case of the students of professor Rubio de Francia, very close. He used to have weekly personal meetings with them and the contact was continued after the student had finished his Ph.D.

In May 1982, F. Ruiz Blasco and I, both from Universidad de Zaragoza, made one of those periodical visits to José Luis who was then already at the Universidad Autónoma de Madrid. In that visit, we were at the blackboard of the Seminar-Room of the Mathematics Department. We were discussing how the Calderón-Zygmund decomposition could be applied to operators bounded a priori in $L^\infty(\mathbf{R})$, namely if T is an operator bounded from $L^\infty(\mathbf{R})$ into $L^\infty(\mathbf{R})$ and g is the "good" part of the Calderón-Zygmund decomposition of a function f in $L^1 \cap L^\infty(\mathbf{R})$, then as $\|Tg\|_\infty \leq C\|g\|_\infty$ and $\|f\|_\infty \leq 2\lambda$, we have

$$|\{x : \ |Tg(x)| > 2C\lambda\}| = 0$$

and therefore in order to obtain that T maps $L^1(\mathbf{R})$ into $L^1(\mathbf{R})$-weak it is enough to estimate the measure of the set

$$\{x : \ |Tb(x)| > \lambda\}$$

where b is the "bad" part of the Calderón-Zygmund decomposition of the function.

We said that this remark was a nice but useless observation.

At some point José Luis wrote on the balckboard the following equality

$$\sup_n |f * k_n(x)| = \|\{f * k_n(x)\}_n\|_{\ell^\infty} = \|f * \{k_n\}(x)\|_{\ell^\infty}.$$

He said that with this point of view the Hardy-Littlewood maximal operator was essencially a convolution operator, bounded a priori from $L^\infty(\mathbf{R}^n)$ into $L^\infty_{\ell^\infty}(\mathbf{R}^n)$ and with kernel

$$K(x) = \{k_r(x)\} = \left\{ \frac{1}{|Q_r|} \mathcal{X}_{Q_r}(x) \right\}_r$$

where Q_r is the cube centered the origin and radius r. Therefore the remark above could be applied to the Hardy-Littlewood maximal operator if the kernel were smooth. He said that even though the kernel was not smooth the operator could be majorized by one operator with smooth kernel

$$\Phi(x) = \left\{ \frac{1}{r^n} \varphi\left(\frac{x}{r}\right) \right\}_r$$

where $\varphi: \mathbf{R}^n \to \mathbf{R}$ smooth.

We had a break, and we went to have lunch. The lunch was delightful not for the food but because of the conversation with José Luis.

After lunch we went back to the Seminar Room and we realized that we could develop a general technique of vector-valued operators that cover the Hardy-Littlewood maximal operator. The theory was influenced by [**B,C,P**] and I would like to present here the main ideas, see [**R de F,R,T**].

Given a Banach space, we denote by $L^p_B = L^p_B(\mathbf{R}^n)$, $1 \le p \le \infty$, the usual Bochner-Lebesgue space. We shall write $L^\infty_{c,B}$ for the space of all compactly supported members of L^∞_B.

Definition 1. For a locally integrable B-valued function f, we define the maximal functions

$$M_r f(x) = \sup_{x\in Q} \left\{ \frac{1}{|Q|} \int \|f(y)\|^r_B dy \right\}^{1/r} \qquad (1 \le r < \infty)$$

and

$$f^\#(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q \|f(y) - f_Q\|_B dy$$

where Q stands for an arbitrary cube in $\mathbf{R}^n$ and f_Q is the average of f over Q.

In terms of $(\cdot)^\#$, we define the space

$$BMO_B = \{ f \in L^1_{loc,B} : \|f\|_{BMO_B} = \|f^\#\|_\infty < +\infty \}.$$

Definition 2. A B-atom is a function $a \in L^\infty_B$ supported in a cube Q and such that

$$\|a(x)\|_B \le \frac{1}{|Q|}, \quad \int_Q a(x)dx = 0.$$

The space $H^1_B(\mathbf{R}^n)$ is, as usual, the subspace of $L^1_B(\mathbf{R}^n)$ formed for all functions $f(x) = \sum_j \lambda_j a_j(x)$; $\lambda_j \in \ell^1$, a_j B-atoms, with $\|f\|_{H^1_B} = \inf \sum_j |\lambda_j|$.

The kernels are strongly measurable functions $K(x)$ defined in $\mathbf{R}^n$ and with values in the space $\mathcal{L}(A,B)$ of all bounded linear operators from the Banach space A into the Banach space B, provided with the operator norm. We suppose that $\|K(x)\|$ is locally integrable away from the origine.

Definition 3. Given $1 \le r \le \infty$, we say that K satisfies the condition (D_r), and write $K \in (D_r)$, if there exists a sequence $\{c_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} c_k = D_r(K) < +\infty$ and for all $k \ge 1$ and $y \in \mathbf{R}^n$,

$$\left\{ \int_{S_k(|y|)} \|K(x-y) - K(x)\|^r dx \right\}^{1/r} \le c_k |S_k(|y|)|^{-1/r'},$$

where $S_k(|y|)$ denotes the spherical shell $2^k|y| < |x| \le 2^{k+1}|y|$.

When $r = \infty$, this must be understood in the usual way, and it is easy to check that $K \in (D_\infty)$ if $\|K(x-y) - K(x)\| \le C|y||x|^{-n-1}$, whenever $|x| > 2|y|$. On the other hand, $K \in (D_1)$ is the familiar Hörmander condition

$$\int_{|x|>2|y|} \|K(x-y) - K(x)\| dx \le D_1(K) < +\infty \quad (y \in \mathbf{R}^n).$$

Definition 4. A linear operator T mapping A-valued functions into B-valued functions is called a singular integral operator (of convolution type) if the following two conditions are fulfilled:

(i) T is a bounded operator from $L^q_A(\mathbf{R}^n)$ to $L^q_B(\mathbf{R}^n)$ for same q, $1 \le q \le \infty$.

(ii) There exists a kernel $K \in (D_1)$ such that

$$Tf(x) = \int K(x-y) f(y) dy$$

for every $f \in L^1_A$ with compact support and for a.e. $x \notin supp(f)$.

Theorem 5. *Let T be a singular integral operator mapping A-valued functions into B-valued ones. Then T can be extended to an operator defined in all L^p_A, $1 \le p < \infty$, and satisfying*

(a) $\|Tf\|_{L^p_B} \le C_p \|f\|_{L^p_A} \quad (1 < p < \infty)$.

(b) $\|Tf\|_{L^1_B-weak} \le C\|f\|_{L^1_A}$.

(c) $\|Tf\|_{L^1_B} \le C\|f\|_{H^1_A}$.

(d) $\|Tf\|_{BMO_B} \le C\|f\|_{L^\infty_A} \quad (f \in L^\infty_{c,A})$.

Moreover, if the kernel of T satisfies (D_r), $1 \le r < \infty$, then

(e) $(Tf)^\#(x) \le C_r M_{r'} f(x) \quad (f \in L^\infty_{c,A}$ *and as a consequence*

$$\|Tf\|_{L^p_B(w)} \le C_p(w) \|f\|_{L^p_A(w)}$$

hold if $w \in A_{p/r'}$ (Muckenhoupt's class, see [M]*), $r' \le p < \infty$.*

If the kernel satisfies (D_∞) then

(f) $(Tf)^\#(x) \le C_\varepsilon M_{1+\varepsilon} f(x)$ *for arbitrary $\varepsilon > 0$, and therefore*

$$\|Tf\|_{L^1_B(w)-weak} \le C(w)\|f\|_{L^1_A(w)}, \quad (w \in A_1).$$

Given a singular integral operator T, a new operator $\tilde{T}$ mapping $\ell^s(A)$-valued functions into $\ell^s(B)$-valued functions (where s is fixed, $1 < s < \infty$) can be defined as

$$\tilde{T}(f_1, f_2, ..., f_j, ...) = (Tf_1, Tf_2, ..., Tf_j, ...).$$

Then $\tilde{T}$ is bounded from $L^s_{\ell^s(A)}$ into $L^s_{\ell^s(B)}$ and the kernel associated to it is $\tilde{K}(x) = K(x) \otimes Id$, so that $\|\tilde{K}(x)\| = \|K(x)\|$ and $D_r(\tilde{K}) = D_r(K)$. In particular, $\tilde{T}$ is a new singular integral operator, and we have.

Corollary 1. *Let T be a singular integral operator mapping A-valued functions into B-valued ones. Then conclusions (a) to (f) of the Theorem 5 are valid if we replace A and B by $\ell^s(A)$ and $\ell^s(B)$, $1 < s < \infty$.*

The proofs of these results are adaptations of the corresponding results for scalar functions, see Part IV of [**R de F,R,T**].

Now I shall mention some applications. (see [**R de F,R,T**] for a detailed list of applications).

Given a function $\varphi \in L^1(\mathbf{R}^n)$, we consider the approximation of the identity $(\varphi_t)_{t>0}$ where $\varphi_t(x) = t^{-n}\varphi(x/t)$, and the associated maximal operator

$$M_\varphi f(x) = \sup_{t>0} |\varphi_t * f(x)|$$

M_φ can be viewed as a linear operator mapping the complex-valued function $f(x)$ into the ℓ^∞-valued function $(\varphi_t * f(x))_{t>0}$. Such an operator certainly satisfies part (i) of Definition 4 with $q = \infty$, while for its kernel $K(x) = (\varphi_t(x))_{t>0}$ to satisfy (D_1), it is necessary and suficient that

$$\text{(Z)} \quad \int_{|x|>2|y|} \sup_{t>0} |\varphi_t(x-y) - \varphi_t(x)| dx \le C.$$

This is the condition of F. Zo, see [**Zo**], and we can state.

Theorem 6. *If $\varphi \in L^1(\mathbf{R}^n)$ satisfies (Z), then M_φ satisfies the following vector-valued inequalities.*

$$\text{(a)} \qquad \|(\sum_j (M_\varphi f_j)^s)^{1/s}\|_p \le C_{p,s} \|(\sum_j |f_j|^s)^{1/s}\|_p, \quad (1 < p, s < \infty)$$

$$\text{(b)} \qquad |\{x : \sum_j M_\varphi f_j(x)^s > \lambda^s\}| \le C_s \lambda^{-1} \|(\sum_j |f_j|^s)^{1/s}\|_1, \quad (1 < s < \infty)$$

$$\text{(c)} \qquad \|(\sum_j (M_\varphi f_j)^s)^{1/s}\|_{BMO} \le C_s \|(\sum_j |f_j|^s)^{1/s}\|_\infty, \quad (1 < s < \infty)$$

Moreover if φ *verifies*

$$|\varphi(x-y)-\varphi(x)| \leq C|y||x|^{-n-1} \quad \textit{when } |x|>2|y|$$

then the measure dx *in (a) can be replaced by* $w(x)dx$, *with* $w \in A_p$*: and in (b) the measure* dx *can be replaced by* $w(x)dx$ *with* $w \in A_1$.

Remark 1. It is clear that φ can be choosen in such a way that $Mf(x) \leq M_\varphi f(x)$ and therefore inequalities (a), (b) and their corresponding weighted versions hold for the Hardy-Littlewood maximal operator.

Theorem 6 and Remark 1 unify and generalize various known results, see **[F,S]**, **[A,J]**, **[Zo]**.

Once we saw that the vector-valued Calderón-Zygmund theory was good to deal with maximal operators, José Luis said: "Let us see what happens with "The Operator" ". For him "The Operator" was the Carleson maximal operator of the convergence of Fourier Series

$$S^*f(x)=\sup_n |p.v. \int \frac{e^{-iny}}{x-y} f(y)dy|.$$

The operator S^*, as before M_φ, can be viewed as a linear operator mapping complex-valued functions $f(x)$ into l^∞-valued functions

$$\left(\int \frac{e^{-iny}}{x-y} f(y)dy\right)_n .$$

This was not a convolution operator but an operator with variable kernel, then we developed a theory based on **[C,M]** and **[J1]**.

We consider, kernels $k(x,y)$ with values in $\mathcal{L}(A,B)$ such that for every $x \in \mathbf{R}^n$, the function $\|k(x,\cdot)\|$ is locally integrable away from x, and therefore, the function

$$(1) \qquad Tf(x)=\int k(x,y)f(y)dy$$

is well defined for every compactly supported $f \in L^1_A(\mathbf{R}^n)$ and a.e.$x \notin \operatorname{supp}(f)$.

Definition 5. Given $1 \leq r \leq \infty$, and a kernel $K(x,y)$, we say that K satisfies (Dr) if there exits a sequence $\{c_k\}_{k=1}^\infty \in \ell^1$ such that

$$\{\int_{x\in S_k(y,z)} \|K(x,y)-K(x,z)\|^r dx\}^{1/r} \leq c_k |S_k(y,z)|^{-1/r'}$$

for all $k \geq 1$ and $y,z \in \mathbf{R}^n$, where

$$S_k(y,z)=\{x : 2^k|y-z|<|x-z|\leq 2^{k+1}|y-z|\}.$$

We say that K satisfies (Dr') if $K'(x,y)=K(y,x)$ satisfies (Dr).

In the next two theorems we assume that we are given a bounded linear operator $T: L^q_A(\mathbf{R}^n) \to L^q_B(\mathbf{R}^n)$, for some fixed q, $1 \leq q \leq \infty$, with a kernel $K(x,y)$ satisfying (1).

Theorem 7. *If K satisfies (D_1), then T can be extended to an operator defined in L^p_A, $1 \le p \le q$, and satisfying*

(a) $$\|Tf\|_{L^p_B} \le C_p \|f\|_{L^p_A} \qquad (1 < p \le q)$$

(b) $$\|Tf\|_{L^1_B-\text{weak}} \le C\|f\|_{L^1_A}$$

(c) $$\|Tf\|_{L^1_B} \le C\|f\|_{H^1_A}$$

Moreover, if K satisfies (Dr) with $1 < r < \infty$, there the weak type inequality

(d) $$w(\{x : \|Tf(x)\|_B > \lambda\}) \le C(w)\lambda^{-1} \int \|f(x)\|_A w(x)dx$$

holds for $w(x)^{\alpha'} \in A_1$, $\alpha = \min(q, r)$.

Theorem 8. *If K satisfies (D_1'), then T can be extended to an operator defined in L^p_A, $q \le p < \infty$ and satisfying*

(a) $$\|Tf\|_{L^p_B} \le C_p \|f\|_{L^p_A} \quad (q \le p < \infty)$$

(b) $$\|Tf\|_{BMO_B} \le C\|f\|_{L^\infty_A} \quad (f \in L^\infty_{C,A}).$$

Moreover, if $K \in (D_r')$ with $1 \le r < \infty$, and $\beta = \max(q, r')$ then

(c) $$(Tf)^{\#}(x) \le CM_\beta f(x) \quad (f \in L^\infty_{C,A})$$

and as a consequence, T verifies the weighted inequality

$$\int \|Tf(x)\|^p_B w(x)dx \le C_{p,w} \int \|f(x)\|^p_A w(x)dx$$

$w \in A_{p/\beta}$, $\beta < p < \infty$.

It is clear that one can consider some ℓ^s-extensions $\tilde{T}$ of the operator T and we can obtain for example the following.

Corollary 2. *If K satisfies (D_1'), then the following inequalities hold:*

$$\|(\sum_j \|Tf_j\|^s_B)^{1/s}\|_p \le C_{s,p} \|(\sum_j \|f_j\|^s_A)^{1/s}\|_p$$

for $q \le s \le p < \infty$.

Now I shall show how we applied the last results to the Carleson operator.

Consider a homogeneous function of degree O in $\mathbf{R}^n, \Omega(x) = \Omega\left(\frac{x}{|x|}\right)$. Assume that Ω is of class $C^{(1}$ outside the origin and satisfies the cancellation property $\int_{|x'|=1} \Omega(x')d\sigma(x') = 0$. For each $\xi \in \mathbf{R}^n$, we define the kernel $k_\xi(y) = e^{2\pi i\xi\cdot y}\Omega(y')|y|^{-n}$, and the corresponding operator

$$\begin{aligned} T_\xi f(x) &= p.v. \int k_\xi(x-y)f(y)dy \\ &= e^{2\pi i\xi\cdot x}p.v. \int \Omega((x-y)')|x-y|^{-n}e^{-2\pi i\xi\cdot y}f(y)dy. \end{aligned}$$

It is known, see [**Sj**], that

$$\|T^* f\|_p \leq C_p\|f\|_p \quad (1 < p < \infty)$$

where

$$T^* f(x) = \sup_\xi |T_\xi f(x)|.$$

When $n = 1$ and $\Omega(y) = \frac{1}{\pi} sign(y), T^*$ is Carleson's maximal operator, see [**C**], [**Ht**]. Then the following Theorem is a consequence of Theorem 8.

Theorem 9. *T^* is a bounded linear operator from L_c^∞ to BMO, and more precisely, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

(a) $$(T^* f)^\#(x) \leq C_\varepsilon\ M_{1+\varepsilon}f(x) \quad (f \in L_c^\infty).$$

Moreover, for all p with $1 < p < \infty$ and all $w \in A_p$

(b) $$\int_{\mathbf{R}^n} T^* f(x)^p w(x)dx \leq C_{p,w} \int |f(x)|^p w(x)dx.$$

For the proof it suffices to consider $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ with rational coordinates. Then, the operator

$$U : L^p(\mathbf{R}^n) \to L^p_{\ell^\infty}(\mathbf{R}^n)$$

defined by the kernel

$$K(x,y) = \left(e^{-2\pi i\xi\cdot y}\frac{\Omega((x-y)')}{|x-y|^n}\right)_{\xi\in\mathbf{Q}^n} \in \ell^\infty$$

satisfies $\|Uf(x)\|_{\ell^\infty} = T^* f(x)$, so that it is bounded in all $L^p, 1 < p < \infty$. On the other hand if $|x-z| > 2|y-z|$ then

$$\begin{aligned} \|k(y,x) - k(z,x)\|_{\ell^\infty} &= \left|\frac{\Omega((y-x)')}{|y-x|^n} - \frac{\Omega((z-x)')}{|z-x|^n}\right| \\ &\leq C|y-z||x-z|^{-n-1}. \end{aligned}$$

Thus K satisfies (D'_∞) and Theorem 8 applies for arbitrary $\beta > 1$. Since $(T^*f)^\# \leq 2(Uf)^\#$, the estimate (a) follows. If $1 < p < \infty$, inequality (b) also follows for every $\omega \in \cup_{\beta>1} A_{p/\beta} = A_p$.

Remark 2. Inequality (a) in Theorem 9 was new, and it would be very interesting to have a different proof of it without using the Carleson-Hunt-Sjolin result.

As I said before the theory that we developed of vector-valued singular integrals was an updated review of [**B,C,P**]. As in [**B,C,P**] our theory could be applied to the operators that usually appear in Littlewood-Paley theory.

Given an interval I in $\mathbf{R}$, define in $L^2(\mathbf{R})$ the operator $S_I f$ by

$$(S_I f)\hat{}\,(\xi) = \chi_I(\xi)\hat{f}(\xi) \quad (\xi \in \mathbf{R}).$$

Given a sequence $\{I_k\}$ of disjoint intervals, we form the quadratic expression

$$\Delta f(x) = (\sum_k |S_{I_k} f(x)|^2)^{1/2} \tag{2}$$

Simple examples show that inequality

$$\|\Delta f\|_p \leq C_p \|f\|_p \tag{3}$$

for $p < 2$ is false. However for $2 \leq p < \infty$ it was an open problem to determine whether or not (3) held. In 1983, José Luis proved the following see [**R de F 6**]

Theorem 10. *For every p with $2 \leq p < \infty$, there exists $C_p > 0$ such that, for every sequence $\{I_k\}$ of disjoint intervals, the operator Δ defined by (2) satisfies*

$$\|\Delta f\|_p \leq C_p \|f\|_p \quad (f \in L^p(\mathbf{R})).$$

I shall give now an idea of his proof.

Definition 6. A sequence of intervals $\{I_k\}$ is call well distributed if the doubles of the intervals have bounded overlapping i.e. $\sum_k \chi_{2I_k}(x) \leq C$.

Define the Whitney decomposition $W(I)$ of an interval I as follows. First of all, the definition is invariant under translations and dilations, and if $I = [0,1]$, then $W(I)$ consists of the intervals:

$$\{[a_{k+1}, a_k]\}_{k=0}^\infty;\ \left[\frac{1}{3}, \frac{2}{3}\right],\ \{[1-a_k, 1-a_{k+1}]\}_{k=0}^\infty$$

where $a_k = 2^{-k}/3$.

Then the following lemma can be proved

Lemma 1. *Given disjoint intervals $\{I_k\}$, let $\Delta f(x)$ be defined as in (2), and let*

$$\Delta_k f(x) = \Big(\sum_{H\in W(I_k)} |S_H f(x)|^2\Big)^{1/2}$$

Then for all $1 < p < \infty$, we have the equivalence

$$\|\Delta f\|_p \sim \|(\sum_k |\Delta_k f|^2)^{1/2}\|_p.$$

Therefore as the sequence $\{H : H \in W(I_k) \quad \text{for some } k\}$ is well distributed, we have the following

Lemma 2. *In proving Theorem 10 for every sequence of disjoint intervals, it is no restriction to assume that the given sequence of intervals $\{I_k\}$ is well distributed.*

He made a second reduction as follows:

We start with a well distributed sequence, and we divide each interval I into seven consecutive intervals of equal length

$$I = I^{(1)} \cup I^{(2)} \cup \ldots \cup I^{(7)}, \quad I^{(i)} = |I|/7$$

so that $8I^{(i)} \subset 2I$. It suffices to prove the theorem for each one of the families $\{I^{(i)} | I \in \quad \text{initial sequence}\}$. Therefore, we can assume from the beginning that we are given a sequence $\mathbf{I}$ of disjoint intervals such that

$$\sum_{I\in\mathbf{I}} \chi_{8I}(x) \le C \qquad (x \in \mathbf{R}).$$

He labelled the intervals of the sequence according to their length. Thus, for each integer k, let

$$\{I_k^j\}_j = \{I \in \mathbf{I} | 2^k \le |I| < 2^{k+1}\}$$

For every k, j let n_k^j be the first integer such that $n_k^j 2^k \in I_k^j$, and fix a Schwartz function $\varphi(x)$ whose Fourier transform satisfies

$$\chi_{[-2,2]} \le \hat{\varphi} \le \chi_{[-3,3]},$$

then we define

$$\varphi_k^j(x) = 2^k \varphi(2^k x) \exp(2\pi i n_k^j 2^k x)$$

so that the Fourier transform of φ_k^j is adapted to I_k^j, i.e.

$$(\varphi_k^j)\hat{}(\chi) = \hat{\varphi}(2^{-k}\chi - n_k^j) = \begin{cases} 1 & \text{if} \quad \chi \in I_k^j \\ 0 & \text{if} \quad \chi \notin 8I_k^j \end{cases}$$

Now define the smooth operator G by

$$Gf(x) = (\sum_{k\in\mathbf{Z}}\sum_j |\varphi_k^j \star f(x)|^2)^{1/2}$$

$$= \{\sum_{k,j} |\int 2^k\varphi(2^k\varphi(2^k(x-y))\exp(-2\pi i n_k^j 2^k y)f(y)dy|^2\}^{1/2}.$$

Since $\sum_{k,j} |(\varphi_k^j)\hat{}(\chi)|^2 \le C$, by Plancherel's Theorem we have that Gf is well defined in $L^2(\mathbf{R})$ and satisfies

$$\|Gf\|_2 \le C\|f\|_2.$$

After this, he proved the following pointwise estimate

$$(Gf)^{\#}(x) \le CM_2f(x), \quad (f\in L_c^\infty,\, x\in\mathbf{R}).$$

This completed the proof of Theorem 10, since for all $f \in L_c^\infty$ and $2 < p < \infty$ we have

$$\|(\sum_{k,j} |S_{I_k^j} f|^2)^{1/2}\|_p \le C_p\|Gf\|_p \le C_p'\|(Gf)^{\#}\|_p$$

$$\le Cc_p'\|M_2 f\|_p \le C_p''\|f\|_p$$

(the first inequality follows by the usual truncation argument which can be seen in [Z],[S]).

The proof of estimate (4) is based in the following generalization of Definition 5 and Theorem 8.

Lemma 3. *We consider a kernel $K(x,y)$ with values in a separable Hilbert space $H \cong \mathcal{L}(\mathbf{C},H)$, such that K satisfies, for some $L>0, \alpha>1$, the condition $((D_2')$-weak$)$*

$$\int_{S_k(x,z)} |< K(x,y)-K(z,y),\lambda >|^2 dy \le L^2\frac{2^{-\alpha k}\|\lambda\|_H^2}{|x-z|}$$

for every $x,z\in\mathbf{R}$, $\lambda\in H$ and $k\ge 1$. Then for the operator $Gf(x)=\|Tf(x)\|_H$ we have the estimate

$$(Gf)^{\#}(x) \le C(L,\alpha)M_2f(x) \quad (f\in L_c^\infty).$$

The proof is a repetition of the proof of the part (c) in the Theorem 8.

It is clear now that in order to show estimate (4), it is enough to show that the ℓ^2-valued kernel

$$K(x,y) = \{2^k\varphi(2^kx-2^ky)\exp(-2\pi i n_k^j 2^k y)\}_{k,j}$$

satisfies the condition D_2'-weak. José Luis showed this with $\alpha = 5/3$, see [**R de F 6**] and then Theorem 10 is proved.

Remark 3. I should mention that was not the first contribution of José Luis to the Littlewood-Paley theory. He proved, see [**R de F 7**], that is the case of the family $I_k = [k, k+1]$, $k \in \mathbf{Z}$, the operator G satisfies the rather sharp inequality

$$Gf(x) \le CM_2f(x) \qquad (f \in L^1 + L^\infty,\ x \in \mathbf{R}).$$

This inequality holds even in $\mathbf{R}^n$, with the obvious modifications, see [**R de F 7**].

For an $\mathbf{R}^n$-version of Theorem 10, see [**J2**].

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THE WORK OF JOSÉ LUIS RUBIO DE FRANCIA II

I am going to discurs the work José Luis Rubio did on *weighted norm inequalities*. Most of it is in the book we wrote together on the subject [**12**].

His main contributions are:

1) The equivalence between vector-valued inequalities and weithted norm inequalities.
2) The precise formulation of the general principle that *the boundedness properties of a linear operator depend only on the weighted L^2 inequalities that it satisfies.*
3) A very simple construction, sometimes called *the Rubio de Francia (R. de F.) algorithm*, which allows one to pass from weighted inequalities with different weights to inequalities with only one weight.
4) As a consequence of 2) and 3) he formulated the beautiful *extrapolation theorem*, which we may choose as the most representative result in this circle of ideas.

I shall try to present these results as they were discovered in order to give a feeling of the way José Luis worked.

1. The equivalence between vector-valued inequalities and weighted norm inequalities

This result came from a deep understanding of the theory of B. Maurey of factorization of operators (see [**23**], [**24**] and [**25**]). José Luis became familiar with this theory at the Williamstown Conference in 1978, where John Gilbert [**13**] gave a talk on some applications of Maurey's theory to Fourier Analysis, mainly Nikishin's theorem. We attended that Conference together, and I remember how this subject aroused a tremendous interest in José Luis. He reformulated Maurey's theory, so as to adapt it to Fourier Analysis. The result is chapter VI of our book. I shall make a short presentation of these results so that it becomes clear how they lead to the equivalence between vector-valued inequalities and weighted norm inequalities. Suppose we have a Banach (or τ-Banach) space B and also a σ-finite measure space (X, dm). $L^0(m)$ will be the space of measurable functions finite a.e. with the topology of local convergence in measure.

Definition 1.1. We say that the operator $T : B \to L^0(m)$ sublinear and continuous in measure factors through some space $L \subset L^0(m)$ if there exists a function $g(x) > 0$ a.e. and a continuous operator $T_0 : B \to L$ such that the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{T} & L^0(m) \\ T_0 \searrow & & \nearrow M_g \\ & L & \end{array}$$

where $M_g f = g \cdot f$, the multiplication operator.

To understand the connection with weighted inequalities, suppose $L = L^p(m)$. Then we must have:

$$\int |T_0 f(x)|^p \, dm(x) \leqq C\|f\|_B^p$$

but, since $T_0 f(x) = \frac{Tf(x)}{g(x)}$ we get the weighted inequality

$$\int |Tf(x)|^p w(x) \, dm(x) \leqq C\|f\|_B^p \text{ with } w(x) = g(x)^{-p}.$$

Now Maurey's theory gives conditions for factorization (through L_*^p, L^p, etc.) in terms of the vector extension $\tilde{T}$ of the operator T. $\tilde{T}$ is the operator sending a sequence (f_j) of vectors in B to the sequence of functions (Tf_j). It is now clear how this theory can cast some light on the equivalence between vector-valued inequalities and weighted norm inequalities.

We shall give two theorems. One for factorization through $L_*^p(m)$= weak L^p = the Lorentz space $L^{(p,\infty)}(m)$. The other for factorization through $L^p(m)$.

Theorem 1.2. (Factorization through $L_*^p(m)$).

Let $T : B \to L^0(m)$ be a continuous sublinear operator and let $0 < p < \infty$. The following conditions are equivalent:

a) *T factors through $L_*^p(m)$.*

b) *There exists $w(x) > 0$ a.e. such that:*

$$\int_{\{x \in X : |Tf(x)| > \lambda\}} w(x) \, dm(x) \leqq \left(\frac{\|f\|_B}{\lambda}\right)^p$$

c) *(for $m(X) < \infty$). For every $\varepsilon > 0$, there exist $E_\varepsilon \subset X$ and $C_\varepsilon > 0$ such that $m(X \backslash E_\varepsilon) < \varepsilon$ and*

$$m(\{x \in E_\varepsilon : |Tf(x)| > \lambda\}) \leqq C_\varepsilon \left(\frac{\|f\|_B}{\lambda}\right)^p$$

d)

$$\begin{array}{l} \tilde{T} : l_B^p \longrightarrow L_{l^\infty}^0(m) \text{ is continuous} \\ (f_j)_j \longrightarrow (Tf_j)_j \end{array}$$

When $m(X) < \infty$, d) means that there exists $C(\lambda) \to 0$ for $\lambda \to \infty$ such that if $\sum_j \|f_j\|_B^p \leqq 1$

$$m\left(\{x \in X : \sup_j |Tf_j(x)| > \lambda\}\right) \leqq C(\lambda)$$

Nikishin's theorem is a consequence of this result. Let us comment briefly how one derives it.

Definition 1.3. Recall that the space B is of Rademacher type p where $0 < p \leqq 2$ if and only if there is a constant C such that

$$\int_0^1 \| \sum_j r_j(t) f_j \|_B^p \, dt \leqq C \sum_j \|f_j\|_B^p$$

where $r_j(t)$ are the Rademacher functions (see [**12**, Appendix A]).

It turns out that when B is of type p, Maurey's condition d) on theorem 1.2. can be easily established, so that every operator $T : B \to L^0(m)$ factors through L^p_*. Since $L^p(\mu)$ is of type $q = \min(2, p)$, we get, as a corollary

Theorem 1.4. (Nikishin) *Let (Y, μ) be an arbitrary measure space and let $T : L^p(\mu) \to L^0(m)$ be a continuous sublinear operator with $0 < p < \infty$. Then, there exists $w(x) > 0$ a.e. such that:*

$$\int_{\{x : |Tf(x)| > \lambda\}} w(x) \, dm(x) \leqq \left(\frac{\|f\|_p}{\lambda}\right)^q$$

for every $f \in L^p(\mu)$ and every $\lambda > 0$, where $q = \min(p, 2)$. If T is positive, we can take $q = p$.

The observation about positive operators is trivial since for $\sum \|f_j\|_p^p \leqq 1$ we have $|Tf_j(x)| \leqq Tf(x)$ where $f(x) = \left(\sum |f_j(x)|^p\right)^{1/p}$ has $\|f\|_p^p = \sum \|f_j\|_p^p \leqq 1$. Thus condition d) is immediately checked.

When the operator commutes with translations we get

Theorem 1.5. (E.M. Stein [**32**]) *Let G be a locally compact group with left Haar measure m, and let $T : L^p(G) \to L^0(G)$, $0 < p < \infty$ be continuous in measure, sublinear and invariant under left translations ($T(f_y) = (T(f))_y$ where $f_y(x) = f(yx)$). Then for every compact set $K \subset G$, there exists a constant $C_K > 0$ such that*

$$m\left(\{x \in K : |Tf(x)| > \lambda\}\right) \leqq C_K \left(\frac{\|f\|_p}{\lambda}\right)^q$$

with $q = \min(p, 2)$, or $q = p$ if T is positive. In particular if G is a compact group, T is of weak-type (p, q).

Proof: Nikishin's theorem gives

$$\int_{\{x \in G: |Tf(x)| > \lambda\}} w(x)\, dm(x) \leqq \left(\frac{\|f\|_p}{\lambda}\right)^q$$

Since for every $y \in G$ $\|f_y\|_p = \|f\|$ and T is invariant, by applying the previous inequality to f_y in place of f, we get

$$\int_{\{x \in G: |Tf(x)| > \lambda\}} w(y^{-1}x)\, dm(x) \leqq \left(\frac{\|f\|_p}{\lambda}\right)^q$$

We can obviously assume $w \in L^\infty$. Then take $h \in L^1_+(G)$ with $\|h\|_1 = 1$ and integrate the last inequality against $h(y)$. We get

$$\int_{\{x \in G: |Tf(x)| > \lambda\}} w * h(x)\, dm(x) \leqq \left(\frac{\|f\|_p}{\lambda}\right)^q.$$

Since $w * h$ is continuous, it will have a minimum $\delta > 0$ over the compact K. Then

$$m(\{x \in K : |Tf(x)| > \lambda\}) \leqq \delta^{-1} \left(\frac{\|f\|_p}{\lambda}\right)^q. \quad \blacksquare$$

Corollary 1.6. *Let $0 < p \leqq 2$. Every sublinear operator $T : L^p(\mathbb{R}^n) \to L^0(\mathbb{R}^n)$ which is continuous in measure and invariant under translations and dilations is of weak-type (p, p). If T is positive, the result is valid for $0 < p < \infty$.*

Proof: As before we obtain

$$\int_{\{x \in \mathbb{R}^n: |Tf(x)| > \lambda\}} w(x)\, dx \leqq \left(\frac{\|f\|_p}{\lambda}\right)^p$$

with w continuous and everywhere > 0. The invariance under dilations yields

$$\int_{\{x \in \mathbb{R}^n: |Tf(x)| > \lambda\}} w(\delta^{-1}y)\, dy \leqq \left(\frac{\|f\|_p}{\lambda}\right)^p$$

Letting $\delta \to \infty$, we obtain:

$$m(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leqq w(0)^{-1} \left(\frac{\|f\|_p}{\lambda}\right)^p. \quad \blacksquare$$

Example. As an application of these principles, let us record briefly, how the weak-type 1,1 of the maximal conjugate function (Kolmogorov's inequality) is a mere consequence of its existence: For $f \in L^1(\mathbb{T})$, where $\mathbb{T}$ is the torus, let $u(\tau e^{it}) = P_\tau * f(t)$, the Poisson integral of f and let v be the harmonic conjugate of u such that $v(0) = 0$.

Define $Tf(t) = \sup_{0 \leqq \tau < 1} |v(\tau e^{it})|$.

That $Tf(t) < \infty$ a.e. is a simple consequence of Faton's theorem (take $f \geqq 0$ and consider the bounded holomorphic function $e^{-(u+iv)}$). By Banach principle $T : L^1(\Pi) \to L^0(\Pi)$ is continuous (Banach principle is simply an application of the closed graph theorem to the operator

$$\begin{aligned} L^1(\Pi) &\longrightarrow L^0_{L^\infty}(\Pi) \\ f &\longrightarrow \{v(\tau e^{i\cdot})\}_{0<\tau<1} \end{aligned})$$

Since T commutes with translations and Π is compact, Stein's theorem gives $T : L^1(\Pi) \to L^1_*(\Pi)$, which is Kolmogorov's result.

Theorem 1.7. (Factorization through $L^p(m)$).

Let $T : B \to L^0(m)$ be a continuous sublinear operator and let $0 < p < \infty$. The following conditions are equivalent:

a) *T factors through $L^p(m)$.*

b) *There exists $w(x) > 0$ a.e., such that:*

$$\int_X |Tf(x)|^p w(x)\, dm(x) \leqq \|f\|_B^p$$

c) *(for $m(X) < \infty$) For every $\varepsilon > 0$, there exist $E_\varepsilon \subset X$ and $C_\varepsilon > 0$ such that $m(X \setminus E_\varepsilon) < \varepsilon$ and*

$$\int_{E_\varepsilon} |Tf(x)|^p\, dm(x) \leqq C_\varepsilon \|f\|_B^p$$

d)

$$\begin{aligned} \tilde{T} :\ & l^p_B \longrightarrow L^0_{l^p}(m) \text{ is continuous} \\ & (f_j)_j \longrightarrow (Tf_j)_j \end{aligned}$$

When $m(X) < \infty$, d) means that there exists $C(\lambda) \to 0$ for $\lambda \to \infty$ such that if $\sum_j \|f_j\|_B^p \leqq 1$

$$m\left(\left\{x \in X : \left(\sum_j |Tf_j(x)|^p\right)^{1/p} > \lambda\right\}\right) \leqq C(\lambda).$$

Now, let us concentrate our attention on operators mapping into some space smaller than $L^0(m)$, for example, in $L^q(m)$ for some $0 < q < \infty$. For such an operator $T : B \to L^q(m)$ sublinear and continuous, factorization through $L^p(m)$ means, as before, the existence of $T_0 : B \to L^p(m)$ continuous and $g(x) > 0$ a.e. such that we have the commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{T} & L^q(m) \\ T_0 \searrow & & \nearrow M_g \\ & L^p(m) & \end{array}$$

with M_g being continuous.

Two comments are relevant here:

1) When $0 < p \leqq q$, factorization always occurs, since we have $L^q(m) \subset L^p(w\,dm)$ simply by Hölder's inequality if we take $w \in L^1 \cap L^\infty$. Thus, only factorization through $L^p(m)$ with $p > q$ is interesting now.
2) In order for M_g to be continuous, we must have $g \in L^r(m)$ where $\frac{1}{p}+\frac{1}{r} = \frac{1}{q}$. Now factorization is equivalent to the weighted inequality

$$\int_X |Tf(x)|^p w(x)\,dm(x) \leqq \|f\|_B^p$$

where $w(x) = g(x)^{-p}$. Thus, we must have $w^{-1} \in L^{r/p}(m)$.

In this case, the condition for factorization is the most natural counterpart of condition d) in theorem 1.7., namely that

$$\begin{aligned} \tilde{T} : l_B^p &\longrightarrow L_{l^p}^q(m) \text{ be continuous} \\ (f_j)_j &\longrightarrow (Tf_j)_j \end{aligned}$$

This condition takes now the form of a vector-valued inequality

$$\left\| \left(\sum_j |Tf_j(x)|^p \right)^{1/p} \right\|_q \leqq C \left(\sum_j \|f_j\|_B^p \right)^{1/p}$$

We shall state the theorem for a slightly more general situation, dealing with a family of operators, instead of just one operator. The simultaneous factorization of the operators in the family, or what is the same, their uniform boundedness from B to $L^p(w)$ with the same w, is equivalent to a vector-valued inequality.

Theorem 1.8. *Let $\mathcal{T}$ be a family of continuous sublinear operators $T : B \to L^q(m)$, $0 < q < \infty$, and let $q < p < \infty$. Call $\frac{1}{r} = \frac{1}{q} - \frac{1}{p} > 0$. Then the inequality*

$$\left\| \left(\sum_j |T_j f_j|^p \right)^{1/p} \right\|_q \leqq C \left(\sum_j \|f_j\|_B^p \right)^{1/p} \quad ; f_j \in B,\ T_j \in \mathcal{T}$$

holds if and only if there exists $w(x) > 0$ a.e. such that $\|w^{-1}\|_{r/p} \leqq 1$ and

$$\|Tf\|_{L^p(w)} \leqq C\|f\|_B;\ f \in B,\ T \in \mathcal{T}.$$

We shall also consider the dual problem of factoring a sublinear operator $T : L^q(m) \to B$. Now factorization means the existence of $T_0 : L^p(m) \to B$ continuous and $g(x) > 0$ a.e. such that we have a commutative diagram:

$$\begin{array}{ccc} L^q(m) & \xrightarrow{T} & B \\ M_g \searrow & & \nearrow T_0 \\ & L^p(m) & \end{array}$$

Since $T_0(f) = T(f/g)$, the boundedness of T_0 means

$$\|T(h)\|_B^p \leqq C^p \int_X |h(x)|^p g(x)^p\, dm(x)$$

or $\|T(h)\|_B \leqq C\|h\|_{L^p(w)}$ where $w = g^p$. Two comments must be made, as before:

1) Now, if $p > q$, Hölder's inequality implies that $L^p(w) \hookrightarrow L^q(m)$ for some w. Thus, only the case $0 < p < q$ is interesting.
2) On the other hand, for the continuity of M_g, we must have $g \in L^r(m)$ where $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ or, what is the same, $w \in L^{r/p}(m)$.

   Here also factorization is equivalent to the boundedness of

$$\begin{aligned} \tilde{T} : L^q_{l^p}(m) &\longrightarrow l^p_B \\ (f_j)_j &\longrightarrow (Tf_j)_j \end{aligned}$$

   or, in other words, the vector-valued inequality:

$$\left(\sum_j \|Tf_j\|_B^p\right)^{1/p} \leqq C \left\|\left(\sum_j |f_j|^p\right)^{1/p}\right\|_q$$

   We also formulate a general result valid for a family of operators.

Theorem 1.9. *Let $\mathcal{T}$ be a family of operators sublinear and bounded $T : L^q(m) \to B$, $0 < q < \infty$. We assume $0 < p < q$ and let r be defined by $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Then the inequality*

$$\left(\sum_j \|T_j f_j\|_B^p\right)^{1/p} \leqq C \left\|\left(\sum_j |f_j|^p\right)^{1/p}\right\|_q,\ T_j \in \mathcal{T},\ f_j \in L^q(m)$$

holds if and only if there exists $w(x) > 0$ such that $\|w\|_{r/p} \leqq 1$ and

$$\|Tf\|_B \leqq C\|f\|_{L^p(w)},\ f \in L^q,\ T \in \mathcal{T}$$

After the brief presentation we have made of B. Maurey's theory of factorization of operators, we are in a position to apply it to derive the first important theorem of José Luis Rubio we are going to discurs: *the equivalence between vector-valued inequalities and weighted-norm inequalities* [**27**].

Suppose we have a family $\mathcal{T}$ of sublinear operators $T : L^q(m) \to L^s(\mu)$ which are uniformly bounded.

We are interested in knowing when the following vector-valued inequality holds:

$$\left\| \left(\sum_j |T_j f_j|^p \right)^{1/p} \right\|_s \leqq C \left\| \left(\sum_j |f_j|^p \right)^{1/p} \right\|_q \tag{1.10}$$

We shall deal with the cases $p > q$, s and $p < q$, s. Let us start with the second case:

$$\left\| \left(\sum_j |T_j f_j|^p \right)^{1/p} \right\|_s^p = \left(\int_Y \left(\sum_j |T_j f_j(y)|^p \right)^{s/p} d\mu(y) \right)^{p/s} =$$

$$\int_Y \sum_j |T_j f_j(y)|^p u(y)\, d\mu(y) \text{ for some } u \in L_+^{(s/p)'}(\mu) \text{ with } \|u\| = 1$$

If given $u \in L_+^{(s/p)'}(\mu)$ we could find $v \in L_+^{(q/p)'}(m)$ such taht $\|v\|_{(q/p)'} \leqq \|u\|_{(s/p)'}$ and

$$\int_Y |T_j f_j(y)|^p u(y)\, d\mu(y) \leqq C \int_X |f_j(x)|^p v(x)\, dm(x)$$

with a uniform constant C, we could continue writing

$$C \int_X \sum_j |f_j(x)|^p v(x)\, dm(x) \leqq$$

$$\leqq C \left(\int_X \left(\sum_j |f_j(x)|^p \right)^{q/p} dm(x) \right)^{p/q} \left(\int_X v(x)^{(q/p)'} dm(x) \right)^{\frac{1}{(q/p)'}} \leqq$$

$$\leqq C \left\| \left(\sum_j |f_j(x)|^p \right)^{1/p} \right\|_q^p$$

It turns out, as we shall see, that this condition is also necessary. Note that $\frac{1}{(s/p)'} = 1 - \frac{p}{s}$ and $\frac{1}{(q/p)'} = 1 - \frac{p}{q}$.

The case $p > q$, s is also easy. We start at the other end

$$\left\| \left(\sum_j |f_j(x)|^p \right)^{1/p} \right\|_q^p = \left(\int_X \left(\sum_j |f_j(x)|^p \right)^{q/p} dm(x) \right)^{p/q} =$$

$$\int_X \sum_j |f_j(x)|^p v(x)^{-1}\, dm(x) \cdot \left(\int_X v(x)^{\frac{q}{p}\left(\frac{p}{q}\right)'} dm(x) \right)^{\frac{p/q}{(p/q)'}} =$$

$$\int_X \sum_j |f_j(x)|^p v(x)^{-1}\, dm(x) \text{ for some } v > 0 \text{ such that } \int_X v(x)^{\frac{q}{p}\left(\frac{p}{q}\right)'} dm(x) = 1.$$

Note $\frac{1}{\frac{q}{p}\cdot\left(\frac{p}{q}\right)'} = \frac{p}{q}\left(1 - \frac{q}{p}\right) = \frac{p}{q} - 1 \equiv \frac{1}{\beta}$ say. Now if for every $v \in L_+^\beta(m)$ there exists $u \in L_+^\alpha(\mu)$, where $\frac{1}{\alpha} = \frac{p}{s} - 1$, such that $\|u\|_\alpha \leqq \|v\|_\beta$ and

$$\int_Y |Tf(y)|^p u(y)^{-1}\, d\mu(y) \leqq C^p \int_X |f(x)|^p v(x)^{-1}\, dm(x)$$

we can continue like this:

$$\geqq C^{-p} \int_Y \sum_j |T_j f_j(y)|^p u(y)^{-1}\, d\mu(y) \geqq C \left\| \left(\sum_j |T_j f_j|^p \right)^{1/p} \right\|_s$$

It turns out that this condition is also necessary. We treat both cases together in the following theorem.

Theorem 1.11. *Let $0 < p, q, s < \infty$, and define α and β by: $\frac{1}{\alpha} = \left|1 - \frac{p}{s}\right|$, $\frac{1}{\beta} = \left|1 - \frac{p}{q}\right|$.*

1) *If $p < q, s$, then (1.10) holds if and only if for every $u \in L_+^\alpha(\mu)$, there exists $v \in L_+^\beta(m)$ such that $\|v\|_\beta \leqq \|u\|_\alpha$ and*

$$\int_Y |Tf(y)|^p u(y)\, d\mu(y) \leqq C^p \int_X |f(x)|^p v(x)\, dm(x) \quad T \in \mathcal{T}.$$

2) *If $p > q, s$, then (1.10) holds if and only if for every $v \in L_+^\beta(m)$, there exists $u \in L_+^\alpha(\mu)$ such that $\|u\|_\alpha \leqq \|v\|_\beta$ and*

$$\int_Y |Tf(y)|^p u(y)^{-1}\, d\mu(y) \leqq C^p \int_X |f(x)|^p v(x)^{-1}\, dm(x) \quad T \in \mathcal{T}.$$

Proof:

1) If $u \in L^\alpha_+(\mu)$ with $\|u\|_\alpha = 1$, then $L^s(\mu) \subset L^p(u\,d\mu) \equiv B$ with norm 1, by Hölder's inequality. Consequently (1.10) implies

$$\left(\sum_j \|T_j f_j\|_B^p\right)^{1/p} = \left\|\left(\sum_j |T_j f_j|^p\right)^{1/p}\right\|_{L^p(u\,d\mu)} \leqq C\left\|\left(\sum_j |f_j|^p\right)^{1/p}\right\|_q$$

Now theorem 1.9. applies, giving us $w > 0$ with $\|w\|_{r/p} \leqq 1$ and $\|Tf\|_B \leqq C\|f\|_{L^p(w)}$.

But $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, so that $\frac{1}{r/p} = 1 - \frac{p}{q} = \frac{1}{\beta}$.

Thus $w \in L^\beta_+(m)$ with $\|w\|_\beta \leqq 1$ and

$$\int_Y |Tf(y)|^p u(y)\,d\mu(y) \leqq C^p \int_X |f(x)|^p w(x)\,dm(x)$$

We take $v = w$ and 1) is proved.

2) Given $v \in L^\beta_+(m)$ with $\|v\|_\beta = 1$ we have $B \equiv L^p(v^{-1}dm) \subset L^q(m)$ with norm 1. Thus 1.10. implies

$$\left\|\left(\sum_j |T_j f_j|^p\right)^{1/p}\right\|_s \leqq C\left\|\left(\sum_j |f_j|^p\right)^{1/p}\right\|_{L^p(v^{-1}\,dm)} = \left(\sum_j \|f_j\|_B^p\right)^{1/p}.$$

Theorem 1.8 gives us $w > 0$ such that $\|w^{-1}\|_{r/p} \leqq 1$ and $\|Tf\|_{L^p(w\,d\mu)} \leqq$ $\leqq C\|f\|_B$, where now $\frac{1}{r} = \frac{1}{s} - \frac{1}{p}$, so that $\frac{1}{r/p} = \frac{p}{s} - 1 = \frac{1}{\alpha}$.

If we take $u = w^{-1} \in L^\alpha_+(\mu)$ we have

$$\int_Y |Tf(y)|^p u(y)^{-1}\,d\mu(y) \leqq C \int_X |f(x)|^p v(x)^{-1}\,dm(x) \quad \blacksquare$$

When we are dealing with a family of operators bounded on the same space, say $L^p(m)$, theorem 1.11. can be improved. We shall obtain an equivalence between the vector-valued inequality and weighted inequalites with the same weight on both sides. This *unification of the weight*, as José Luis Rubio liked to call it, is achieved by means of a simple iterative process, which is nowadays called the *Rubio de Francia algorithm.*

Let us start by reformulating theorem 1.11. for a family of operators bounded in $L^p(m)$.

We shall change notation just for psichological reasons.

Theorem 1.12. *Let $\mathcal{T}$ be a family of sublinear operators uniformly bounded in $L^p(m)$ where $0 < p < \infty$. Suppose $o < \tau < \infty$.*

We want to find conditions under which the following vector-valued inequality holds:

$$\left\| \left(\sum_j |T_j f_j|^\tau \right)^{1/\tau} \right\|_p \leqq C \left\| \left(\sum_j |f_j|^\tau \right)^{1/\tau} \right\|_p \tag{1.13}$$

We let α be given by $\frac{1}{\alpha} = \left| 1 - \frac{\tau}{p} \right|$.

1) *If $\tau < p$, then (1.13) holds if and only if for every $u \in L^\alpha_+(m)$ there exists $v \in L^\alpha_+(m)$ such that $\|v\|_\alpha \leqq \|u\|_\alpha$ and*

$$\int_X |Tf(x)|^\tau u(x)\, dm(x) \leqq C^\tau \int_X |f(x)|^\tau v(x)\, dm(x),\ T \in \mathcal{T}.$$

2) *If $\tau > p$, then (1.13) holds if and only if for every $v \in L^\alpha_+(m)$ there exists $u \in L^\alpha_+(m)$ such that $\|u\|_\alpha \leqq \|v\|_\alpha$ and*

$$\int_X |Tf(x)|^\tau u(x)^{-1}\, dm(x) \leqq C^\tau \int_X |f(x)|^\tau v(x)^{-1}\, dm(x),\ T \in \mathcal{T}.$$

In case $\tau < p$, the weight can be unified. Let us see how.

Theorem 1.14. *Let $0 < \tau < p$ and suppose that (1.13) holds. Let $\frac{1}{\alpha} = 1 - \frac{\tau}{p} (\alpha = (p/\tau)')$. Then, for every $u \in L^\alpha_+(m)$, there exists $w \in L^\alpha_+(m)$ such that:*

i) $u(x) \leq w(x)$ *a.e.*

ii) $\|w\|_\alpha \leqq 2\|u\|_\alpha$ *and*

iii) $\int_X |Tf(x)|^\tau w(x)\, dm(x) \leq 4C^\tau \int_X |f(x)|^\tau w(x)\, dm(x)$

Moreover if Q is a sublinear contraction in $L^\alpha(m)$ then w can be chosen so that $|Qw(x)| \leqq 4w(x)$.

Proof: From theorem 1.12., to each $u \in L^\alpha_+(m)$ we can associate $V \in L^\alpha_+(m)$ such that $\|V\|_\alpha \leqq \|u\|_\alpha$ and

$$\int_X |Tf(x)|^\tau u(x)\, dm(x) \leqq C^\tau \int_X |f(x)|^\tau V(x)\, dm(x)$$

Now, given $u \in L^\alpha_+(m)$, we define a sequence u_j inductively by:

$$u_0 = u, u_1 = \tfrac{V + |Qu|}{2}, \cdots$$
$$\cdots, u_{j+1} = \tfrac{V_j + |Qu_j|}{2}, \cdots$$

Then we have:

$$|Qu_j| \leqq 2u_{j+1}, \quad \|u_{j+1}\|_\alpha \leqq \|u_j\|_\alpha$$

$$\int_X |Tf(x)|^r u_j(x)\,dm(x) \leqq C^r \int_X |f(x)|^r V_j(x)\,dm(x) \leqq$$

$$\leqq 2C^r \int_X |f(x)|^r u_{j+1}(x)\,dm(x)$$

If we take $w(x) = \sum_{j=0}^{\infty} \frac{u_j(x)}{2^j}$, we have:

$$w(x) \geqq u_0(x) = u(x) \text{ a.e.}$$

$$\|w\|_\alpha \leqq \sum_{j=0}^{\infty} 2^{-j}\|u_j\|_\alpha \leqq 2\|u\|_\alpha$$

$$|Qw(x)| \leqq \sum_{j=0}^{\infty} 2^{-j}|Qu_j(x)| \leqq 2\sum_{j=0}^{\infty} 2^{-j}u_{j+1}(x) \leqq 4w(x)$$

Finally

$$\int_X |Tf(x)|^r w(x)\,dm(x) \leqq 2C^r \int_X |f(x)|^r \sum_{j=0}^{\infty} \frac{u_{j+1}(x)}{2^j}\,dm(x) \leqq$$

$$\leqq 4C^r \int_X |f(x)|^r w(x)\,dm(x)$$

as we wanted to prove. ■

2. The boundedness principle

Now the general result we have established can be combined with a theorem of Marcinkiewicz and Zygmund which says that every linear operator bounded between Lebesgue spaces admits a bounded l^2-valued extension. It is in this way that we obtain the first precise formulation of the fact that *all the information that one may wish concerning the boundedness properties of a linear operator, is contained in the weighted-L^2 inequalities that this operator satisfies.*

Theorem 2.1. (Marcinkiewicz and Zygmund). *Let $T : L^p(m) \to L^q(\mu)$ be a bounded linear operator, $0 < p,q < \infty$ with "norm" $\|T\|$. Then T has an l^2-valued extension, and more precisely:*

$$\left\| \left(\sum_j |Tf_j|^2 \right)^{1/2} \right\|_q \leqq C_{p,q}\|T\| \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_p$$

Proof: Consider first the case $q \leqq p$. Take a Gaussian sequence (Z_j) in some probability space (Ω, P) and recall the following basic fact: For every $0 < \tau < \infty$

$$\left\| \sum_j \lambda_j Z_j \right\|_\tau = b_\tau \left(\sum_j |\lambda_j|^2 \right)^{1/2}$$

with some constant $b_\tau < \infty$ (see [**12**, chapter V]). Then

$$\left\| \left(\sum_j |Tf_j|^2 \right)^{1/2} \right\|_q^q = b_q^{-q} \int_Y d\mu(y) \int_\Omega \left| \sum_j Tf_j(x) Z_j(\omega) \right|^q dP(\omega) =$$

$$b_q^{-q} \int_\Omega \left\| T \left(\sum_j Z_j(\omega) f_j \right) \right\|_q^q \leqq b_q^{-q} \|T\|^q \int_\Omega \left\| \sum_j Z_j(\omega) f_j \right\|_p^q dP(\omega) \leqq$$

$$b_q^{-q} \|T\|^q \left(\int_\Omega \left\| \sum_j Z_j(\omega) f_j \right\|_p^p dP(\omega) \right)^{q/p} = (b_p/b_q)^q \|T\|^q \left\| (\sum_j |f_j|^2)^{1/2} \right\|_p^q$$

and the theorem is proved with $C_{p,q} = b_p/b_q$. In particular $C_{p,p} = 1$.

Now we consider the case $p < q$. Let $s = q/p$. For every $u(y) \geqq 0$ with $\|u\|_{s'} \leqq 1$, the operator $T_u f(y) = u(y)^{1/p} Tf(y)$ satisfies

$$\|T_u f\|_p \leqq \|Tf\|_q \leqq \|T\| \|f\|_p$$

and by the case already proved, we have:

$$\left\| \left(\sum_j |Tf_j|^2 \right)^{1/2} \right\|_q = \sup_u \left\{ \int_Y \left(\sum_j |Tf_j|^2 \right)^{p/2} u \, d\mu \right\}^{1/p} =$$

$$= \sup_u \left\| \left(\sum_j |T_u f_j|^2 \right)^{1/2} \right\|_p \leqq \|T\| \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_p \quad \blacksquare$$

Theorem 2.2. *Let* $1 < p < \infty$ *and* $\frac{1}{\alpha} = \left| 1 - \frac{2}{p} \right|$.

A linear operator T is bounded in $L^p(m)$ if and only if, for every $u \in L^\alpha_+(m)$ there exists $w \in L^\alpha_+(m)$ such that

$$u(x) \leqq w(x) \text{ a.e.}$$
$$\|w\|_\alpha \leqq 2\|u\|_\alpha \text{ and}$$

T is bounded in $L^2(w^\sigma)$ (where $\sigma = 1$ when $2 \leqq p$ and $\sigma = -1$ when $p < 2$) with norm independent of u.

Proof: The *if* part is in both cases a trivial consequence of Hölder's inequality. When $p > 2$

$$\|Tf\|_p^2 = \||Tf|^2\|_{p/2} = \int_X |Tf(x)|^2 u(x)\, dm(x)$$

where $u \in L_+^{(p/2)'}(m)$ with norm 1, note that $(p/2)' = \alpha$. Given u we have w, so that:

$$\|Tf\|_p^2 \leqq \int_X |Tf(x)|^2 w(x)\, dm(x) \leqq C \int_X |f(x)|^2 w(x)\, dm(x) \leqq$$
$$\leqq C\|f\|_p^2 \|w\|_\alpha \leqq 2C\|f\|_p^2$$

When $p < 2$

$$\|f\|_p^2 = \int_X |f(x)|^2 u(x)^{-1}\, dm(x)$$

for some $u \in L_+^{\frac{p}{2}\left(\frac{2}{p}\right)'}(m)$ with norm 1. Note that $\frac{1}{\frac{p}{2}\left(\frac{2}{p}\right)'} = \frac{2}{p}\left(1 - \frac{p}{2}\right) = \frac{2}{p} - 1 = \frac{1}{\alpha}$. Thus we have w and we can continue

$$\geqq \int_X |f(x)|^2 w(x)^{-1}\, dm(x) \geqq C^{-1} \int_X |Tf(x)|^2 w(x)^{-1}\, dm(x) \geqq (2C)^{-1} \|Tf\|_p^2$$

To prove the *only if* part of the statement consider first the case $2 < p$.

Since T is bounded in $L^p(m)$, the Marcinkiewicz and Zygmund theorem implies

$$\left\| \left(\sum_j |Tf_j|^2 \right)^{1/2} \right\|_p \leqq \|T\| \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_p$$

Then theorem 1.14. gives us what we want. Now if $p < 2$, we consider the adjoint operator $T^* : L^{p'}(m) \to L^{p'}(m)$ and apply the previous case taking into account that $\frac{1}{\alpha} = \left|1 - \frac{2}{p}\right| = \left|1 - \frac{2}{p'}\right|$ and that the inequalities

$$\int_X |T^* f(x)|^2 w(x)\, dm(x) \leqq C \int_X |f(x)|^2 w(x)\, dm(x)$$

and

$$\int_X |Tf(x)|^2 w(x)^{-1}\, dm(x) \leqq C \int_X |f(x)|^2 w(x)^{-1}\, dm(x)$$

are equivalent. ■

In 3 we shall present some versions of theorem 2.2. valid for a larger class of spaces. But first we shall indicate briefly how José Luis Rubio used theorem 1.14. to give a very simple proof of the factorization theorem for weights in the classes A_p of Muckenhoupt. This theorem had been discovered previously by Peter Jones [**20**] with a much more complicated proof.

Given a σ-finite measure space (X, dx) and a family $(E_i)_{i\in I}$ of positive linear operators, we form the maximal operator

$$Mf(x) = \sup_{i\in I} |E_i f(x)|$$

We assume that M is bounded in $L^p(dx)$ $1 < p < \infty$ and denote by $W_p = W_p(M)$ the class of all measurable functions $w(x) \geqq 0$ which are finite a.e. and verify

$$\int_X Mf(x)^p w(x)\,dx \leqq C \int_X |f(x)|^p w(x)\,dx$$

for all suitable f, where $C = C(p, w)$ is independent of f. For $p = 1$, we define $W_1 = W_1(M)$ as the class of those $w \geqq 0$, finite a.e. and satisfying $Mw(x) \leqq Cw(x)$ a.e. for a certain C. Then we have the following

Theorem 2.3. (Factorization theorem for weights). *Let $1 < p < \infty$, and suppose that $w \in W_p$ and $w^{-p'/p} \in W_{p'}$. Then, there exist $w_0, w_1 \in W_1$ such that $w(x) = w_0(x)w_1(x)^{1-p}$.*

Proof: Let $\mathcal{T}$ denote the family of linear operators of the form $Tf(x) = \sum_{i\in I} a_i(x)E_i f(x)$ where $(a_i(x))_{i\in I}$ are measurable functions $a_i(x) \geqq 0$ and $\sum_i a_i(x) = 1$. The family $\mathcal{T}$ will be used as a linearization of M. Denoting by T^* the adjoint of T (relative to the measure dx) we have:

$$\left\| \left(\sum_j |T_j f_j|^r \right)^{1/r} \right\|_{L^{p'}(w^{-p'/p})} \leqq C \left\| \left(\sum_j |f_j|^r \right)^{1/r} \right\|_{L^{p'}(w^{-p'/p})} \qquad p' \leqq r \leqq \infty$$

and

$$\left\| \left(\sum_j |T_j^* f_j|^r \right)^{1/r} \right\|_{L^p(w)} \leqq C \left\| \left(\sum_j |f_j|^r \right)^{1/r} \right\|_{L^p(w)} \qquad 1 \leqq r \leqq p$$

where $T_j \in \mathcal{T}$.

The first inequality is proved by interpolation in r.

For $\tau = p'$ it is trivial:

$$\left\| \left(\sum_j |T_j f_j|^{p'} \right)^{1/p'} \right\|_{L^{p'}(w^{-p'/p})} = \int_X \sum_j |T_j f_j|^{p'} w^{-p'/p} \leqq$$

$$\leqq C^{p'} \int_X \sum_j |f_j|^{p'} w^{-p'/p} = C^{p'} \left\| \left(\sum_j |f_j|^{p'} \right)^{1/p'} \right\|_{L^{p'}(w^{-p'/p})}$$

just because $w^{-p'/p} \in W_{p'}$.

For $\tau = \infty$ it is also easy because $\sup_j |T_j f_j| \leqq M(\sup_j |f_j|)$, so that

$$\| \sup_j |T_j f_j| \|_{L^{p'}(w^{-p'/p})} \leqq \| M(\sup_j |f_j|) \|_{L^{p'}(w^{-p'/p})} \leqq C \| \sup_j |f_j| \|_{L^{p'}(w^{-p'/p})}$$

The second inequality follows by duality. In order to prove the theorem, note that if it is true for some p, it is also true for p', so we may assume $1 < p \leqq 2$.

We use the inequality

$$\left\| \sum_j |T_j^* f_j| \right\|_{L^p(w)} \leqq C \left\| \sum_j |f_j| \right\|_{L^p(w)}$$

Applying theorem 1.14. (now $\alpha = p'$) we know that for every $u \in L_+^{p'}(w)$ there exists $v \in L_+^{p'}(w)$ such that $u \leqq v$, $\|v\|_{L^{p'}(w)} \leqq 2\|u\|_{L^{p'}(w)}$ and

$$\int_X |T^* f(x)| v(x) w(x)\, dx \leqq 4C \int_X |f(x)| v(x) w(x)\, dx$$

This is equivalent to $T(vw) \leqq 4Cvw$ that is: $vw = w_0 \in W_1$. We also need $w = w_0 v^{-1} = w_0 w_1^{1-p}$ i.e. $w_1 = v^{\frac{1}{p-1}} \in W_1$ or, in other words

$$M(v^{p'/p}) \leqq C v^{p'/p}$$

This can also be achieved, since $v \to C^{-1} M(v^{p'/p})^{p/p'}$ is a sublinear contraction in $L^{p'}(w)$.

It is sublinear because $p'/p > 1$ and also

$$\int |M(v^{p'/p})^{p/p'}|^{p'} w = \int M(v^{p'/p})^p w \leqq C \int v^{p'} w. \quad \blacksquare$$

Corollary 2.4. *Suppose that*

$$(2.5)\quad w \in W_p \text{ if and only if } \sup_{i\in I} E_i w(x)\left(E_i(w^{-p'/p})(x)\right)^{p/p'} \leqq C,\ 1 < p < \infty$$

Then $W_p = \{w_0 w_1^{1-p} : w_0, w_1 \in W_1\}$.

Proof: First of all, it follows from (2.5) that $w \in W_p$ if and only if $w^{-p'/p} \in W_{p'}$, thus theorem 2.3. gives us the factorization of every $w \in W_p$. On the other hand, if $w = w_0 w_1^{1-p}$ with $w_0, w_1 \in W_1$, then:

$$E_i(w)\left(E_i(w^{-p'/p})\right)^{p/p'} = E_i(w_0 w_1^{1-p}) E_i(w_0^{-\frac{1}{p-1}} w_1)^{p-1} \leqq$$
$$\leqq C E_i(w_0) E_i(w_1)^{1-p} E_i(w_0)^{-1} E_i(w_1)^{p-1} = C. \blacksquare$$

(2.5) holds in many interesting cases, which we collect below:

a) Let $\mathcal{B}$ be a basis in $\mathbb{R}^n$, that is, a collection of open subsets $B \subset \mathbb{R}^n$. For each $B \in \mathcal{B}$, we consider

$$E_B(f) = \left(|B|^{-1} \int_B f\right) \chi_B.$$

The corresponding maximal operator will be

$$M_{\mathcal{B}} f(x) = \sup_{B\in\mathcal{B}} |E_B f(x)|$$

We shall also define the classes $A_{p,\mathcal{B}}$ $1 < p < \infty$ in the following way:

$$w \in A_{p,\mathcal{B}} \text{ if and only if } \sup_{B\in\mathcal{B}} (E_B w(x))(E_B(w^{-p/p'})(x))^{p/p'} \leqq C$$

We say that $\mathcal{B}$ is a Muckenhoupt basis if W_p coincides with $A_{p,\mathcal{B}}$ for all $1 < p < \infty$, that is, if we have: $M_{\mathcal{B}}$ is bounded in $L^p(w) \Leftrightarrow w \in A_{p,\mathcal{B}}$, $1 < p < \infty$. For any basis $\mathcal{B}$, the definition of $A_{1,\mathcal{B}}$ as limit of the conditions $A_{p,\mathcal{B}}$, coincides with W_1.

Corollary 2.4. applies to every Muckenhoupt basis, giving

$$A_{P,\mathcal{B}} = \{w_0 w_1^{1-p} : w_0, w_1 \in A_{1,\mathcal{B}}\}.$$

Here are some examples of Muckenhoupt bases:

1) The basis Q formed by the cubes with sides parallel to the coordinate axes $M_Q = M$ = Hardy-Littlewood maximal operator. $A_{P,Q} = A_p$ = the usual Muckenhoupt classes. That Q is a Muckenhoupt basis was shown by Muckenhoupt **[26]**. See also **[5]** and **[12**, chapter IV**]**.
2) The basis R formed by all n-dimensional intervals.

   M_R = strong maximal function $A_{p,R} = A_p^*$ in the notation of **[12**, chapter IV**]**, where it is shown that R is a Muckenhoupt basis.
3) The basis ζ formed by the intervals in $\mathbb{R}^n$ whose sidelenghts are of the form $\{s, t, st\}$. This was shown to be a Muckenhoupt basis by R. Fefferman (see, for example **[9]**).

b) Let $((F_t)_{0<t<\infty}$ be an increasing family of σ-algebras in a probability space (Ω, F, P) such that $\bigcup_t F_t = F$. We denote by E_t the conditional expectation operator with respect to F_t and write $E^* f = \sup_t |E_t f|$. Assume that, for each $f \in L^1_+(\Omega)$ we can define $(f_t)_{t>0}$ so that $f_t = E_t f$ a.e. $(t > 0)$ and $\sup f_{t^-}(w)/f_t(w) \leqq C$ (a.e.)(this happens, for example in Brownian martingales). Then $W_p(E^*) = A_p(\{F_t\})$ is given by condition (2.5), and therefore corollary 2.4. applies (see [7], [34]). The same can be said in the discrete case $(F_n)_{n\in\mathbf{N}}$ where we have to assume that

$$E_n f \leqq C E_{n+1} f \text{ a.e. } (n \in \mathbf{N}, f \in L^1_+).$$

c) Let D be the unit ball in $\mathbf{C}^n$ equipped with the measure

$$dm_\alpha(z) = (1 - |z|^2)^{\alpha-1} dm(z)$$

where $m = m_1$ is Lebesgue measure. If $\mathcal{B}$ is the family of balls touching the boundary ∂D, then the operator

$$M_\alpha f(z) = \sup_{z\in B\in\mathcal{B}} m_\alpha(B)^{-1} \left| \int_B f \, dm_\alpha \right|$$

is bounded in $L^p(w\, dm_\alpha)$ if and only if $w \in W_p(M_\alpha) = B_p^\alpha$ which is defined as in (2.5). Therefore the factorization $B_p^\alpha = B_1^\alpha (B_1^\alpha)^{1-p}$ holds. See [2].

d) Let (Ω, F, P) be a non-atomic probability space, and let T be an ergodic measure preserving bijection in Ω. It is shown in [1] that if

$$E_{m,n} f(x) = (n + m + 1)^{-1} \sum_{j=-n}^{m} f(T^j x),\ n, m \geqq 0$$

then $E^* f = \sup_{n,m} |E_{n,m} f|$ is bounded in $L^p(w\, dP)$, $1 < p < \infty$ if and only if $w \in A_p'$ which means

$$\sup_m E_{0,m} w(x) \cdot \left(E_{0,m}(w^{-p'/p})(x) \right)^{p/p'} < C \text{ a.e.}$$

Therefore, corollary 2.4. also applies in this case.

3. Extrapolation theorems

By using factorization, P. Jones observed the following simple consequence:

Suppose T is a linear operator, which is bounded in $L^{p_0}(w)$ for every $w \in A_{p_0}$ and in $L^{p_1}(w)$ for every $w \in A_{p_1}$ where $1 < p_0 < p_1 < \infty$ and the classes A_p are those of Muckenhoupt (example a) 1 at the end of section 2). Then for every $p_0 < p < p_1$ and every $w \in A_p$, T is bounded in $L^p(w)$. Indeed if $w \in A_p$ we can write $w = w_0 w_1^{1-p}$ with $w_0, w_1 \in A_1$. Then T is bounded in $L^{p_0}(w_0 w_1^{1-p_0})$ and in $L^{p_1}(w_0 w_1^{1-p_1})$. By using the theorem of interpolation with change of measure of Stein-Weiss **[33]** we get that T is bounded in $L^p(w)$.

One of the great discoveries of José Luis Rubio was the extrapolation theorem which says that only the boundedness at $L^{p_0}(w)$ for every $w \in A_{p_0}$ is enough to imply boundedness in $L^p(w)$ for every $w \in A_p$ and every $1 < p < \infty$. We shall not present the extrapolation theorem in the first formulation he gave **[28]**. We shall choose a slightly more general setting following two later papers of José Luis **[29]** and **[30]**.

This will allow us to get a deeper insight into the fact that the boundedness properties of an operator are all contained in the weighted L^2 inequalities it satisfies.

We fix a complete σ-finite measure space (Ω, μ) and consider real Banach lattices $X \subset L^0(\mu)$, i.e. $(X, \|\cdot\|_X)$ is a Banach space whose elements are equivalence classes (modulo equality a.e.) of measurable functions in (Ω, μ) and such that:

$$x \in X \text{ and } |y(\omega)| \leqq |x(\omega)| \text{ a.e.} \Rightarrow y \in X, \|y\|_X \leqq \|x\|_X$$

All the lattices considered will also satisfy the following two properties:

i) There exists $x \in X$ such that $x(\omega) > 0$ a.e.
ii) If $x_n \in X$, $0 \leqq x_1(\omega) \leqq x_2(\omega) \leqq \cdots \leqq x_n(\omega) \leqq \cdots \to x(\omega)$ a.e. and if $\sup_n \|x_n\|_X < \infty$, then $x \in X$ and $\|x\|_X = \lim_n \|x_n\|_X$.

i) is usually refered to by saying that X has a weak unit or that the support of X is all of Ω. ii) is called the Fatou property. Examples of lattices fulfilling these requirements are: $L^p = L^p(\mu)$ and the Lorentz spaces $L(p,q)$, $1 \leqq p, q \leqq \infty$ (except $L(1,\infty)$ which is not normed); Orlicz spaces $\Phi(L)$ where Φ is convex, strictly increasing in $[0,\infty[$ and $\Phi(0) = 0$; the mixed norm spaces $L^{p_1,p_2}(\Omega,\mu)$ if $\Omega = \Omega_1 \times \Omega_2$ and $\mu = \mu_1 \otimes \mu_2$; weighted spaces $L^p(w)$ where w is a weight, that is, a measurable function in Ω such that $0 < w(\omega) < \infty$ a.e. etc. Given a lattice X, we shall use the following notation:

$$X^* = \text{dual of } X$$

It is a lattice but, in general it can not be identified with a lattice of functions in (Ω, μ)

$$X' = \alpha - \text{dual of } X = \{y \in L^0(\mu) : xy \in L^1(\mu) \forall x \in X\}$$

X' is a lattice of functions on (Ω, μ) with the norm $\|y\|_{X'} = \sup\{\|xy\|_1 : \|x\|_X \leqq 1\}$. X' is a closed norming subspace of X^*. Norming means that

$$\|x\|_X = \sup\{\|xy\|_1 : \|y\|_{X'} \leqq 1\}, \text{ for all } x \in X$$

This holds due to the Fatou property

$$X_+ = \{x \in X : x(\omega) > 0 \text{ a.e.}\}$$

For $a > 0$

$$X^a = \{y \in L^o(\mu) : |y| = x^a \text{ for some } x \in X\}$$

If we define $\|y\|_{X^a} = \||y|^{1/a}\|_X^a$, we get a norm if $a \leqq 1$ and (in general) a $(1/a)$-norm if $a > 1$. However for some lattices X^a is a Banach lattice for some $a > 1$. For example if $X = L^p$, $p > 1$, then $X^a = L^{p/a}$ and this is still a Banach lattice for $a \leqq p$, and not only for $a \leqq 1$. This characterizes the p-convex Banach lattices.

Definition 3.1. Let $1 \leqq p, q \leqq \infty$. The lattice X is said to be

a) p-convex if

$$\left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\|_X \leqq M \left(\sum_{i=1}^n \|x_i\|_X^p \right)^{1/p}$$

b) q-concave if

$$\left(\sum_{i=1}^n \|x_i\|_X^q \right)^{1/q} \leqq M \left\| \left(\sum_{i=1}^n |x_i|^q \right)^{1/q} \right\|_X$$

Proposition 3.2. (see [**22**]).

a) *Every Banach lattice is 1-convex and ∞-concave.*
b) *If X is p_0-convex and q_0-concave, then it is also p-convex for $1 \leqq p \leqq p_0$ and q-concave for $q_0 \leqq q \leqq \infty$.*
c) *$X'' = (X')' = X$ (for this it is basic that X satisfies property ii) above).*
d) *X' satisfies i) and ii) also.*
e) *If X is p-convex and q-concave, an equivalent norm can be defined so that inequalities a) and b) in the definition above hold with $M = 1$.*
f) *X is p-convex if and only if X^p is a Banach lattice (with X renormed according to e)).*
g) *X is p-convex (q-concave) if and only if X' is p'-concave (q'-convex).*

When for a single operator T we say that $T : X \to Y$ is bounded and also $T : Z \to W$ is bounded, we are always implicitly assuming that $X \cap Z$ is dense in both X and Z, which justifies the uniqueness of its extension so that we may consider it as the same operator.

Let us start by presenting a version of theorem 1.9. valid for 2-convex lattices. Clearly L^q is q-convex and, consequently 2-convex if $q > 2$. For a 2-convex lattice X we define a new lattice $\tilde{X} = (X^2)'$. For example if $X = L^q$ with $q > 2$ then $X^2 = L^{q/2}$ and $\tilde{X} = (X^2)' = L^{(q/2)'}$.

Here is the version of the factorization theorem we need, also part of Maurey's theory.

Theorem 3.3. *Let X be a 2-convex Banach lattice and B a Banach space. If $T : X \to B$ is a bounded sublinear operator, the following are equivalent:*

a) *T factors through L^2 in this form:*

$$T : X \xrightarrow{Mg} L^2 \xrightarrow{T_0} B$$

where $\|T_0\| \leqq C$, $g \in (\tilde{X})^{1/2}$ with $\|g\|_{\tilde{X}^{1/2}} \leqq 1$.

b) *$\tilde{T} : X(l^2) \to l^2_B$ is bounded or, in other words: for all $x_1, x_2, \ldots, x_n \in X$*

$$\left(\sum_j \|Tx_j\|_B^2\right)^{1/2} \leqq C \left\| \left(\sum_j |x_j|^2\right)^{1/2} \right\|_X .$$

When $X = L^q$, $q > 2$, this is contained in theorem 1.9. Note that the condition $g \in \tilde{X}^{\frac{1}{2}}$ means that $|xg|^2 \in L^1\, \forall x \in X$ and this is exactly what we need for the operator Mg to be bounded from X to L^2. Of course if $X = L^q$, $\tilde{X}^{\frac{1}{2}} = L^{(q/2)' \cdot 2}$ and $\frac{1}{(q/2)' \cdot 2} = \frac{1}{2}\left(1 - \frac{2}{q}\right) = \frac{1}{2} - \frac{1}{q} = \frac{1}{r}$ in theorem 1.9.

We shall use theorem 3.3. in order to derive a boundedness criterion similar to theorem 2.2. but valid for Banach lattices (2-convex or 2-concave). We shall need the following extension of the Marcinkiewicz and Zygmund theorem.

Theorem 3.4. (Grothendieck-Krivine). *Let $T : X \to Y$ be a linear operator bounded from X to Y, both Banach lattices. Then*

$$\left\| \left(\sum_j |Tf_j|^2\right)^{1/2} \right\|_Y \leqq K_G \|T\| \left\| \left(\sum_j |f_j|^2\right)^{1/2} \right\|_X$$

for $f_1, \ldots, f_n \in X$; where K_G is Grothendieck's universal constant, whose exact value is still unknown, although $1 < K_G < 2$.

In other words, the vector extension $\tilde{T}$ is bounded from $X(l^2)$ to $Y(l^2)$ with $\|\tilde{T}\| \leq K_G \|T\|$ (see [21], [22]*).*

Here is our boundedness criterion for 2-convex Banach lattices.

Theorem 3.5. *Let X and Y be 2-convex Banach lattices in (Ω, μ). For a linear operator T, the following statements are equivalent:*

a) *T is a bounded operator from X to Y.*

b) *For every $u \in \tilde{Y}_+$, there exists $v \in \tilde{X}_+$ such that T is a bounded operator from $L^2(v)$ to $L^2(u)$.*

c) *There is a constant $C > 0$ such that b) holds with $\|v\|_{\tilde{X}} \leqq \|u\|_{\tilde{Y}}$ and $\|Tx\|_{L^2(u)} \leqq C\|x\|_{L^2(v)}$ for all $x \in X$.*

Moreover if C is the least constant in c) and $M = \|T\|_{L(X,Y)}$ we have: $M \leqq C \leqq K_G \cdot M$.

Proof: The difficult part is a) $\Rightarrow$ c). The proof is similar to that of part 1) in theorem 1.11. Let T be linear and bounded $T : X \to Y$ with norm $= M$. Given $u \in \tilde{Y}_+$ we assume without loss of generality that $\|u\|_{\tilde{Y}} = 1$. Then we see that $Y \hookrightarrow L^2(u)$. Indeed

$$\int_\Omega |y|^2 u \, d\mu \leqq \|y^2\|_{Y^2} \cdot \|u\|_{\tilde{Y}} = \|y\|_Y^2$$

Putting $L^2(u) \equiv B$, we have, from the Grothendieck-Krivine theorem plus the inclusion $Y \hookrightarrow B$:

$$\left(\sum_j \|Tx_j\|_B^2\right)^{1/2} = \left\|\left(\sum_j |Tx_j|^2\right)^{1/2}\right\|_B \leqq$$
$$\leqq \left\|\left(\sum_j |Tx_j|^2\right)^{1/2}\right\|_Y \leqq K_G \cdot M \left\|\left(\sum_j |x_j|^2\right)^{1/2}\right\|_X$$

According to theorem 3.3. this implies the factorization

$$T : X \xrightarrow{Mg} L^2 \xrightarrow{T_0} B = L^2(u)$$

with $\|T_0\| \leqq C$ and $\|g\|_{\tilde{X}^{1/2}} \leqq 1$.

In other words

$$\|Tx\|_{L^2(u)} \leqq C^2 \int_\Omega |x|^2 g^2 \, d\mu$$

This is precisely c) with $v = g^2 \in \tilde{X}_+$ and $C \leqq K_G \cdot M$.

c) $\Rightarrow$ b) We have $T : X \to L^2(u)$ bounded when we consider on X the norm $\|\cdot\|_{L^2(v)}$.

We know $X \hookrightarrow L^2(v)$. This is proved exactly as the inclusion $Y \hookrightarrow L^2(u)$. It is enough to prove that X is dense in $L^2(v)$. In order to do that we use the Hahn-Banach theorem. Let $g \in L^2(v)^* = L^2(v^{-1})$ and suppose that g vanishes on X as a functional, i.e. $\int xg \, d\mu = 0 \, \forall x \in X$. Now $L^2(v^{-1}) \subset X'$. Thus $\|g\|_{X'} = 0$. But this implies $g = 0$ a.e. This finishes the proof that X is dense in $L^2(v)$.

b) $\Rightarrow$ a) Given an arbitrary $u \in \tilde{Y}_+$, let $v \in \tilde{X}_+$ the function associated to u in b). As we have seen $X \hookrightarrow L^2(v)$. Therefore we can define Tx for every $x \in X$. Besides, this definition is independent of v. We have $T : X \to T(X) \subset \bigcap_{u \in \tilde{Y}_+} L^2(u) = Y$.

The last identity is proved as follows:

If $y \in L^2(u)$ for every $u \in \tilde{Y}_+$, we have $|y|^2 u \in L^1$ for every $u \in (Y^2)'$. Thus $|y|^2 \in (Y^2)'' = Y^2$ since Y^2 satisfies Fatou's property. This is the same as saying $y \in Y$.

Consequently we have a linear operator: $T : X \to Y$. Its boundedness will be a consequence of the closed graph theorem. Suppose $x_n \to 0$ in X and $Tx_n \to y$ in Y. We take u and v such that $T : L^2(v) \to L^2(u)$ is continuous. Since $X \hookrightarrow L^2(v)$ and $Y \hookrightarrow L^2(u)$ we have: $x_n \to 0$ in $L^2(v)$ and $Tx_n \to y$ in $L^2(u)$. Therefore $y = 0$, and we have $T : X \to Y$ continuous.

The equivalence a) $\Leftrightarrow$ b) $\Leftrightarrow$ c) is already established. In order to complete the proof of the theorem, we need to see that $M \leqq C$. But

$$\|Tx\|_Y^2 = \||Tx|^2\|_{Y^2} = \sup\left\{\int |Tx|^2 u\, d\mu : \|u\|_{\tilde{Y}} \leqq 1\right\} \leqq$$

$$\leqq \sup\left\{C^2 \int |x|^2 v\, d\mu : \|v\|_{\tilde{X}} \leqq 1\right\} = C^2 \||x|^2\|_{X^2} = C^2 \|x\|_X^2 \quad \blacksquare$$

Now exactly as theorem 1.12. (part 1) can be improved for operators bounded in a fixed L^p space giving rise to theorems 1.14. and 2.2., theorem 3.5. can also be improved by unifying the weight when we are dealing with operators bounded in a fixed 2-convex Banach lattice. We obtain the following result:

Theorem 3.6. *Let X be a 2-convex Banach lattice in (Ω, μ). For a linear operator T, the following statements are equivalent:*

a) *T is a bounded operator in X.*

b) *For each $u \in \tilde{X}_+$, there exists $w \in \tilde{X}_+$ such that $u \leqq w$ and T is a bounded operator in $L^2(w)$.*

c) *There is a constant $C > 0$ such that b) holds with $\|w\|_{\tilde{X}} \leqq 2\|u\|_{\tilde{X}}$ and $\|Tx\|_{L^2(w)} \leqq C\|x\|_{L^2(w)}$ for every $x \in X$.*

Moreover the least constant C in c) is comparable to $\|T\|_{L(X)}$.

Proof: We prove c) $\Rightarrow$ b) $\Rightarrow$ a) and $M \leqq \sqrt{2}C$ as in the proof of theorem 3.5. Then we use the R. de F. algorithm to prove that a) $\Rightarrow$ c). Given $u \in \tilde{X}_+$ we define $(u_j)_{j=0,1,\dots}$ inductively by: $u_0 = u$, $u_1 = v, \dots, u_{j+1} = v_j, \dots$ where v_j is the weight associated to u_j as in theorem 3.5. This means $\|u_j\|_{\tilde{X}} \leqq \|u\|_{\tilde{X}}$ and

$$\int_\Omega |Tx|^2 u_j\, d\mu \leqq (K_G \cdot M)^2 \int_\Omega |x|^2 u_{j+1}\, d\mu$$

Now we define $w = \sum_{j=0}^\infty 2^{-j} u_j$ and we have:

$$u \leqq w,\ \|w\|_{\tilde{X}} \leqq 2\|u\|_{\tilde{X}} \text{ and } \int_\Omega |Tx|^2 w\, d\mu = \sum_{j=0}^\infty 2^{-j} \int_\Omega |Tx|^2 u_j\, d\mu \leqq$$

$$\leqq (K_G M)^2 \sum_{j=0}^\infty 2^{-j} \int_\Omega |x|^2 u_{j+1}\, d\mu \leqq 2(K_G M)^2 \int_\Omega |x|^2 w\, d\mu$$

This proves c) with $C \leqq \sqrt{2} K_G M$. ■

From theorems 3.5. and 3.6. we shall obtain, by duality, results for 2-concave lattices. For a 2-concave lattice X, we consider:

$$\hat{X} = \{x \in L^0(\mu) : xg \in X^2 \, \forall g \in L^1\}$$

Note that if $X = L^q(\mu)$, $q < 2$, then $\hat{X} = L^r(\mu)$ where $\frac{1}{r} = \frac{2}{q} - 1$.

We define $\|x\|_{\hat{X}} = \sup_{\|g\|_1 \leqq 1} \|xg\|_{X^2}$.

Lemma 3.7. *Let X be a 2-concave Banach lattice satisfying i) and ii). Then:*

a) *$\hat{X}$ is a Banach lattice with the norm $\|\cdot\|_{\hat{X}}$.*
b) *$\hat{X}$ also satisfies i) and ii).*
c) *$(\hat{X})' = X'^2$ with identity of norms.*

Proof: X' is 2-convex, thus X'^2 is a Banach lattice verifying i) and ii). Also $Z = (X'^2)'$ satisfies i) and ii) and $Z' = X'^2$. It is enough to check that $\hat{X} = Z$ with identity of norms. But: $x \in \hat{X} \Leftrightarrow |x|^{1/2} h \in X \, \forall h \in L^2 \Leftrightarrow |x|^{1/2} hy \in L^1, \forall y \in X', h \in L^2 \Leftrightarrow |x|^{1/2} y \in L^2, \forall y \in X' \Leftrightarrow x|y|^2 \in L^1 \, \forall y \in X' \Leftrightarrow x \in (X'^2)' = Z$. ■

Theorem 3.8. *Let X and Y be 2-concave Banach lattices in (Ω, μ). For a linear operator T, the following statements are equivalent:*

a) *T is a bounded operator from X to Y.*
b) *For each $u \in \hat{X}_+$, there exists $v \in \hat{Y}_+$ such that T is a bounded operator from $L^2(u^{-1})$ to $L^2(v^{-1})$.*
c) *There is a constant $C > 0$ such that b) holds with $\|v\|_{\hat{Y}} \leqq \|u\|_{\hat{X}}$ and $\|Tx\|_{L^2(v^{-1})} \leqq C\|x\|_{L^2(u^{-1})} \, \forall x \in L^2(u^{-1})$.*

Moreover $K_G^{-1} C \leqq \|T\|_{L(X,Y)} \leqq C$.

Proof: The proof is based upon the following facts:

1) $X^* = X'$, $Y^* = Y'$ because X and Y are sequentially order complete (since they are 2-concave) and have weak units (see [**22**]).
2) The adjoint operator T^* defined by

$$\int_\Omega (Tx) y' \, d\mu = \int_\Omega (T^* y') x \, d\mu \quad x \in X, \, y' \in Y'$$

is a well defined operator from Y' to X' (which are 2-convex) and $\|T\|_{L(X,Y)} = \|T^*\|_{L(Y',X')}$.
3) $T : L^2(u^{-1}) \to L^2(v^{-1})$ is a bounded operator if and only if $T^* : L^2(v) \to L^2(u)$ is a bounded operator.

Using this 3 facts, we just need to apply theorem 3.5. to the operator $T^* : Y' \to X'$. ■

The version for an operator bounded on a given 2-concave Banach lattice is as follows:

Theorem 3.9. *Let X be a 2-concave Banach lattice in (Ω, μ). For a linear operator T, the following statements are equivalent:*

a) *T is a bounded operator in X.*

b) *For each $u \in \hat{X}_+$, there exists $w \in \hat{X}_+$ such that $u \leqq w$ and T is bounded in $L^2(w^{-1})$.*

c) *There is a constant C, such that b) holds with $\|w\|_{\hat{X}} \leqq 2\|u\|_{\hat{X}}$ and $\|Tx\|_{L^2(w^{-1})} \leqq C\|x\|_{L^2(w^{-1})}$ for all $x \in L^2(w^{-1})$.*

Moreover the least constant C in c) is comparable to $\|T\|_{L(X)}$.

We are going to obtain now, from theorems 3.5., 3.6., 3.8. and 3.9., a qualitative result that will make precise the boundedness principle given as 2, in the introduction, and which will also be a general version of a theorem of extrapolation (from L^2).

We shall be able to handle non-necessarily linear operators.

Definition 3.10. $T : X \to L^0(\mu)$ is linearizable if there exists a Banach space B and a linear operator $T_0 : X \to L^0(\mu, B)$ such that $Tx(\omega) = \|T_0x(\omega)\|_B \, \forall x \in X$.

For example if $\{T_n\}$ are linear operators defined in X, the maximal operator $Mx(\omega) = \sup_n |T_nx(\omega)|$ is linearizable.

We shall use the following notation:

$$\begin{aligned} V(T) = \{(u,v) | u > 0 \text{ a.e.}, v > 0 \text{ a.e. and} \\ T : L^2(v) \longrightarrow L^2(u) \text{ is bounded}\} \\ W(T) = \{w > 0 \text{ a.e.} : T : L^2(w) \longrightarrow L^2(w) \text{ is bounded}\} \end{aligned}$$

Theorem 3.11. *Let S and T be operators such that:*

a) S is linear.

b) T is linearizable.

c) $V(S) \subset V(T)$.

If X and Y are Banach lattices in (Ω, μ) both 2-convex or both 2-concave and if $S : X \to Y$ is a bounded operator, then T is also bounded from X to Y.

In the case $X = Y$, c) can be replaced by the weaker hypothesis

c') $W(S) \subset W(T)$.

Proof: Suppose first that X and Y are 2-convex. If S is bounded from X to Y, then b) of theorem 3.5. holds for S, and because of c), b) of theorem 3.5. also holds for T. Now the implication b) $\Rightarrow$ a) in the proof of theorem

3.5. is equally correct for linearizable operators because we have a well defined operator

$$T_0 : X \longrightarrow Y(B) = \{x \in L^0(\mu, B) : \|x(\cdot)\|_B \in Y\}$$
$$= \cap_{u\in \check{Y}_+} L^2_B(u)$$

and the continuity follows from the closed graph theorem.

When X and Y are 2-concave, we get also $T_0 : X \to Y(B)$ by using $X = \cup_{u\in \check{X}_+} L^2(u^{-1})$ and $Y(B) = \cup_{v\in \check{Y}_+} L^2_B(v^{-1})$. In order to prove the continuity, we consider the family of linear operators

$$T_h x(\omega) = \langle T_0 x(\omega), h(\omega)\rangle$$

associated to $h \in L^\infty_{B^*}$ with $\|h\| \leqq 1$.

Since $Tx(\omega) = \sup_h |T_h x(\omega)|$, it is enough to prove that the T_h are uniformly bounded from X to Y. But this follows from the Banach-Steinhauss theorem, since

a) Every T_h is bounded from X to Y.
b) For each $x \in X$ $\sup_h \|T_h x\|_Y \leqq \|\sup_{h\in H} T_h x(\cdot)\|_Y = \|Tx\|_Y < \infty$. ■

In theorems 3.5., 3.6., 3.8. and 3.9. the exponent 2 was crucial because of the Grothendieck-Krivine inequality, which is false in general for exponents $\tau \neq 2$. However, if for a given operator $T \in L(X, Y)$ we know that

$$\left\|\left(\sum_j |Tx_j|^\tau\right)^{1/\tau}\right\|_Y \leqq C\left\|\left(\sum |x_j|^\tau\right)^{1/\tau}\right\|_X \tag{3.12}$$

the arguments used in the proofs of those theorems work provided that X and Y are both τ-convex or both τ-concave.

We state the analogue of theorem 3.5. in this situation.

Theorem 3.13. *Let X and Y be τ-convex Banach lattices for some $1 < \tau < \infty$. For a linear operator T, the following are equivalent to (3.12):*

a) *For each $u \in (Y^\tau)'_+$, there exists $v \in (X^\tau)'_+$ such that T is a bounded operator from $L^\tau(v)$ to $L^\tau(u)$.*
b) *The preceding statement holds with $\|v\|_{(X^\tau)'} \leqq \|u\|_{(Y^\tau)'}$ and $\|Tx\|_{L^\tau(u)} \leqq \leqq C\|x\|_{L^\tau(v)}$ for all $x \in X$ (C being as in (3.12)).*

We could also formulate results corresponding to theorems 3.6., 3.8. and 3.9.

A particular case in which (3.12) holds with $1 \leqq \tau \leqq \infty$ and $C = \|T\|_{L(X,Y)}$ is when T is a positive operator. Indeed

$$\left(\sum_{j=1}^n |x_j|^\tau\right)^{1/\tau} \geqq \sum_{j=1}^n a_j x_j \text{ whenever } \sum_{j=1}^n |a_j|^{\tau'} \leqq 1$$

Therefore, since T is positive:

$$T\left(\left(\sum_{j=1}^{n}|x_j|^\tau\right)^{1/\tau}\right) \geqq \sum_{j=1}^{n} a_j T x_j$$

Hence

$$\left(\sum_{j=1}^{n}|Tx_j|^\tau\right)^{1/\tau} = \sup\left\{\sum_{j=1}^{n} a_j T x_j : \sum_{j=1}^{n}|a_j|^{p'} \leqq 1\right\} \leqq T\left(\left(\sum_{j=1}^{n}|x_j|^\tau\right)^{1/\tau}\right)$$

and, consequently

$$\left\|\left(\sum_{j=1}^{n}|Tx_j|^\tau\right)^{1/\tau}\right\|_Y \leqq \left\|T\left(\left(\sum_{j=1}^{n}|x_j|^\tau\right)^{1/\tau}\right)\right\|_Y \leqq \|T\| \left\|\left(\sum_{j=1}^{n}|x_j|^\tau\right)^{1/\tau}\right\|_X$$

With the help of this observation we can obtain a counterpart of theorem 3.11. valid for S linear and positive and an exponent $\tau \neq 2$. We shall use the following notation:

$$\begin{aligned} V_p(T) = &\{(u,v) | u > 0 \text{ a.e., } v > 0 \text{ a.e. and} \\ &\quad T : L^p(v) \longrightarrow L^p(u) \text{ is bounded }\} \\ W_p(T) = &\{w > 0 \text{ a.e. } | T : L^p(w) \longrightarrow L^p(w) \text{ is bounded }\}. \end{aligned}$$

Theorem 3.14. *Let $1 < \tau < \infty$ and suppose that S and T are operators such that:*

a) *S is linear and positive.*
b) *T is linearizable.*
c) *$V_\tau(S) \subset V_\tau(T)$.*

If X and Y are Banach lattices in (Ω, μ) which are both τ-convex or both τ-concave and $S : X \to Y$ is bounded, then T is also bounded from X to Y.

In case $X = Y$, condition c) can be replaced by the weaker hypothesis

$$W_\tau(S) \subset W_\tau(T).$$

For every fixed τ and p, $1 < \tau, p < \infty$ and weights $u(x)$, $v(x)$, the lattices $X = L^p(v)$, $Y = L^p(u)$ are either τ-convex if $\tau \leqq p$ or τ-concave if $\tau \geqq p$. If we apply theorems 3.11. and 3.14. to this case, we obtain:

Theorem 3.15. *Let S be a linear operator and T a linearizable operator.*

a) *If $W_2(S) \subset W_2(T)$, then $W_p(S) \subset W_p(T)$, $1 < p < \infty$.*
b) *If $V_2(S) \subset V_2(T)$, then $V_p(S) \subset V_p(T)$, $1 < p < \infty$.*
c) *If S is positive and $W_r(S) \subset W_r(T)$ for some r, then $W_p(S) \subset W_p(T)$, $1 < p < \infty$.*
d) *If S is positive and $V_r(S) \subset V_r(T)$ for some r, then $V_p(S) \subset V_p(T)$, $1 < p < \infty$.*

Theorem 3.15. is a general extrapolation theorem. Part c) was obtained by Jawerth **[19]** with a constructive proof.

By making specific choices of the operator S, we get concrete extrapolation theorems. For example, if $S = R_1 + R_2 + \cdots + R_n$ where R_j $1 \leqq j \leqq n$ are the Riesz transforms in $\mathbb{R}^n$, given as multipliers by

$$(R_j f)^\wedge(\xi) = (-i\xi_j/|\xi|)\hat{f}(\xi)$$

then $W_p(S) = A_p$, the Muckenhoupt class, for $1 < p < \infty$ (see **[12]**).

Likewise, if we take S to be the multiple Hilbert transform in $\mathbb{R}^n$, it is also shown in **[12]** that $W_p(S) = A_p^*$, the class associated to the basis ? of intervals. We can write

Corollary 3.16. *Let T be a linearizable operator which is bounded in $L^2(w)$ for all $w \in A_2$ in $\mathbb{R}^n$ (resp. A_2^*). Then T is bounded in $L^p(w)$ for all $w \in A_p$ (resp. A_p^*) where $1 < p < \infty$.*

This corollary (for the classes A_p) is contained in **[10]** with a constructive proof. See also **[11]**, **[14]** and **[17]**.

4. Applications

The three applications we shall present are all contained in **[29]**.

The first one consists in obtaining *weighted inequalities for singular integrals with non-smooth kernels.*

We shall consider a singular integral

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(y) f(x-y)\, dy$$

with a kernel K satisfying:

i) $K(y) = \frac{\Omega(y')}{|y|^n}$, $y' = \frac{y}{|y|}$, with $\Omega \in L^\infty(\sum_{n-1})$.
ii) $\int_{\sum_{n-1}} K(y)\, d\sigma(y) = 0$ where σ is Lebesgue measure over $\sum_{n-1}$, the unit sphere of $\mathbb{R}^n$.

It is known from **[5]** (see also **[12]**) that when Ω is smooth, T is bounded in $L^p(w)$ for every $w \in A_p$, $1 < p < \infty$. J. Duoandikoetxea and J.L. Rubio **[8]** proved that the result holds without any regularity assumption on Ω. Here is their result

Theorem 4.1. *If T is as above, then T is bounded in $L^p(w)$ for every $w \in$ $\in A_p$, $1 < p < \infty$.*

Proof: For $f \in \mathcal{S}$, the Schwartz class, we decompose

$$Tf(x) = \sum_{-\infty<j}^{\infty} \int_{2^j \leqq |x-y| < 2^{j+1}} K(x-y) \int \hat{f}(\xi) e^{2\pi i y\cdot\xi} d\xi \, dy =$$
$$= \sum_{-\infty<j}^{\infty} \sum_{-\infty<l}^{\infty} \int_{2^j \leqq |x-y| < 2^{j+1}} \int K(x-y) \hat{f}(\xi) \Psi(2^l|\xi|)^2 e^{2\pi i y\cdot\xi} d\xi \, dy$$

where $\Psi \in S$ has support in $[1/2, 2]$ and satisifes $\sum_{-\infty<l}^{\infty} \Psi(2^l t)^2 = 1$ for every $t > 0$.

In the double sum we group terms for which the difference between l and j is constant. For $k \in \mathbb{Z}$ we define

$$T_k f(x) = \sum_{l-j=k} (\dots) = \sum_{-\infty<j}^{\infty} \int K_j(x-y) S_{k+j}(S_{k+j} f)(y) \, dy$$

where $K_j(y) = K(y)\chi_{[2^j, 2^{j+1}[}(|y|)$ and

$$(S_l f)^\wedge(\xi) = \Psi(2^l|\xi|)\hat{f}(\xi).$$

Then $Tf = \sum_{k\in\mathbb{Z}} T_k f$. We shall prove:

$$\|T_k f\|_{L^2(w)} \leqq C\|f\|_{L^2(w)}, \; w \in A_2 \tag{4.2}$$

$$\|T_k f\|_{L^2} \leqq C 2^{-|k|/2} \|f\|_{L^2}. \tag{4.3}$$

This will be enough. Indeed, given $w \in A_2$ we know that $w^{1+\varepsilon} \in A_2$ for some $\varepsilon > 0$ (with another constant C_ε).

Since $w^{\frac{1}{2}} = w^{\frac{1+\varepsilon}{2} \cdot \frac{1}{1+\varepsilon}} \cdot 1^{\frac{\varepsilon}{1+\varepsilon}}$ we get, by interpolation with change of measure **[33]**:

$$\|T_k f\|_{L^2(w)} \leqq C_\varepsilon 2^{-\frac{|k|}{2} \frac{\varepsilon}{1+\varepsilon}} \|f\|_{L^2(w)}$$

enough to conclude that:

$$\|Tf\|_{L^2(w)} \leqq \sum_k \|T_k f\|_{L^2(w)} \leqq C\|f\|_{L^2(w)}.$$

All that remains is to prove (4.2) and (4.3). (4.2) is proved by using the Littlewood-Paley inequality and its dual, which hold in $L^2(w)$ provided w is an

A_2 weight

$$\|T_k f\|^2_{L^2(w)} = \left\| \sum_j S_{k+j}(K_j * S_{k+j} f) \right\|^2_{L^2(w)} \leqq$$
$$\leqq C \int \sum_j |K_j * S_{k+j} f(x)|^2 w(x)\, dx \leqq \text{(by i))} \leqq$$
$$\leqq C' \sum_j \int (M(S_{k+j}f)(x))^2\, w(x)\, dx \leqq$$
$$\leqq C'' \sum_j \int |S_{k+j} f(x)|^2 w(x)\, dx \leqq C''' \|f\|^2_{L^2(w)}$$

In order to prove (4.3), observe that

$$|\hat{K}_0(\xi)| = \left| \int_{1 \leqq |y| < 2} \frac{\Omega(y')}{|y|^n} e^{-2\pi i \xi \cdot y}\, dy \right| =$$
$$= \left| \int_{\Sigma_{n-1}} \Omega(y') \int_1^2 e^{-2\pi i \tau y' \cdot \xi} \frac{d\tau}{\tau}\, d\sigma(y') \right| \leqq$$
$$\leqq C \int_{\Sigma_{n-1}} |I(y' \cdot \xi)|\, d\sigma(y') \text{ where } I(t) = \int_1^2 e^{-2\pi i \tau t} \frac{d\tau}{\tau}$$

From the obvious estimates $|I(t)| \leqq 1$, $|I(t)| \leqq \frac{C}{|t|}$, we get $|I(t)| \leqq C|t|^{-1/2}$. Therefore:

$$|\hat{K}_0(\xi)| \leqq C \int_{\Sigma_{n-1}} |y' \cdot \xi|^{-1/2} d\sigma(y') \leqq C|\xi|^{-1/2}$$

Also $|\hat{K}_0(\xi)| \leqq C|\xi| \leqq C|\xi|^{1/2}$ for $|\xi| \leqq 1$ since $\hat{K}_0$ is C^∞ and $\hat{K}_0(0) = 0$.

Now $\hat{K}_j(\xi) = \hat{K}_0(2^j \xi)$ and

$$\|T_k f\|_2^2 = \int \left| \sum_j \hat{K}_0(2^j\xi) \Psi(2^{k+j}|\xi|)^2 \hat{f}(\xi) \right|^2 d\xi \leqq$$
$$\leqq C \int \left(\sum_j \min(|2^j\xi|^{1/2}, |2^j\xi|^{-1/2}) \Psi(2^{k+j}|\xi|)^2 |\hat{f}(\xi)| \right)^2 d\xi \leqq$$
$$\leqq C \int \left(\min(2^k, 2^{-k})^{1/2} \sum_l \Psi(2^l|\xi|)^2 \right)^2 |\hat{f}(\xi)|^2\, d\xi = C 2^{-|k|} \|f\|_2^2. \blacksquare$$

The second application will deal with *the Hilbert transform for groups and its action on Banach lattices.*

Let G be a compact abelian group with an ordered dual Γ (iff G is connected). In $L^2(G)$ we define the conjugate function or Hilbert transform by:

$$(Hf)^\wedge(\gamma) = \begin{cases} -i\hat{f}(\gamma) & \text{if } \gamma > 0 \\ 0 & \text{if } \gamma = 0 \\ i\hat{f}(\gamma) & \text{if } \gamma < 0 \end{cases}$$

The theorem of M. Riesz extends to this context, namely: H is bounded in $L^p(G)$ $1 < p < \infty$ (see [**31**]). The same happens with the Helson-Szegö theorem characterizing those w such that H is bounded in $L^2(w)$.

Theorem 4.4. (Helson-Szegö). *Given $w \in L^1_+(G)$, the following conditions are equivalent:*

a) *H is bounded in $L^2(w)$.*
b) *$w = e^{u+H(v)}$ with $u, v \in L^\infty$, $\|v\|_\infty < \pi/2$.*
c) *$w \sim w_0$ (i.e. $C^{-1}w \leqq w_0 \leqq Cw$) with $|Hw_0| \leqq K \cdot w_0$ a.e.*

Moreover the constants $\|u\|_\infty$, arctan $\|v\|_\infty$ *in b) and C, K in c) are controlled in terms of the norm of H in $L^2(w)$.*

See [**16**] and [**15**].

Definition 4.5. Given a lattice X, an operator linear $T : X \longrightarrow L^0$ is

i) u-bounded in X if

$$\sup_{\|x\|\leqq 1} \inf_{|x|\leqq y \in X} (\|y\| + \|Ty\|) = C < \infty$$

ii) semi-bounded in X if for every $x \in X$, there exists $y \in X$ such that $|x| \leqq y$ and $|Ty| \leqq$ const. y a.e.

We have T bounded $\Rightarrow$ Tu-bounded $\Rightarrow$ T semi-bounded (To prove the second implication, given $x = x_0 \in X_+$ define $(x_n)(y_n)$ so that $x_n \leqq y_n$

$$\|y_n\| + \|Ty_n\| \leqq C\|x_n\|, \; x_{n+1} = Ty_n$$

and then $y = \sum_{n=0}^{\infty}(2C)^{-n}y_n \in X$ satisfies ii)) c) above is equivalent to the fact that H is u-bounded in $w \cdot L^\infty$.

Since $w \cdot L^\infty = L^1(w)' = L^2(w)^\sim$ the following is a natural extension of the Helson-Szegö theorem to Banach lattices.

Corollary 4.6. *Given a Banach lattice in (G, dx) which is 2-convex (resp. 2-concave), the following conditions are equivalent*

a) *H is a bounded operator in X.*
b) *H is u-bounded in $\tilde{X}$ (resp. in $\hat{X}$).*
c) *H is semi-bounded in $\tilde{X}$ (resp. in $\hat{X}$).*

Proof: Suppose first that X is 2-convex a) $\Rightarrow$ b). By theorem 3.6., given $u \in \tilde{X}$ with $\|u\|_{\tilde{X}} = 1$, there is $w \in \tilde{X}_+ \ni |u| \leqq w$, $\|w\|_{\tilde{X}} \leqq 2$ and H is bounded in $L^2(w)$ with norm $\leqq M$ (independent of u). Therefore c) in the Helson-Szegö theorem holds with some w_0, which satisfies

$$|u| \leqq w \leqq w_0 \leqq C^2 w \quad |Hw_0| \leqq K w_0$$

where C and K are independent of u. In particular

$$\|w_o\|_{\tilde{X}} + \|Hw_0\|_{\tilde{X}} \leqq 2C^2(1+K)$$

This shows that H is u-bounded in $\tilde{X}$.

b) $\Rightarrow$ c) is true in general.

c) $\Rightarrow$ a) Given $u \in \tilde{X}_+$ we know $|Hw| \leqq Cw$ for some $w \in \tilde{X}$, $w \geqq u$. This implies that H is bounded in $L^2(w)$ by the Helson-Szegö theorem. Thus b) in theorem 3.6. holds, showing that H is bounded in X.

The proof for 2-concave lattices is similar using now theorem 3.9 plus the fact that H bounded in $L^2(w) \Leftrightarrow H$ bounded in $L^2(w^{-1})$. ■

When $p > 2$, $(L^p(w))^\sim = L^{p/2}(w)' = L^{(p/2)'}(w^{1-(p/2)'})$. Call $\omega = w^{1-(p/2)'}$ and write $(p/2)' = q$. Then T is u-bounded or semi-bounded in $L^q(\omega)$ if and only if $f \to \omega^{1/q} T(\omega^{-1/q} f)$ satisfies the same property in L^q. Thus we obtain a result of Cotlar and Sadosky [6]:

If $p > 2$, H is bounded in $L^p(w)$ if and only if $Sf = w^{-\frac{2}{p}} H(w^{2/p} f)$ is u-bounded in L^q if and only if S is semi-bounded in L^q.

Corollary 4.7. *Let H be the Hilbert transform in a compact, abelian, connected group G and let $1 < p < \infty$. If $w \in W_p(H)$, then there exists $\varepsilon > 0$ such that $w \in W_{p-\varepsilon}(H)$ and $w^{1+\varepsilon} \in W_p(H)$.*

Proof: We have $w \in W_p \Leftrightarrow w^{1-p'} \in W_{p'}$, $1 < p < \infty$, $w \in W_p \Rightarrow w \in W_q$ $1 < p < q < \infty$.

The first property is proved by duality. Since $H^* = -H$. The second by proving $W_p \subset W_{2p}$ by means of the magic formula

$$(Hf)^2 = f^2 + 2H(fHf)$$

and then using reiteration and interpolation. Now we observe that $w \in W_2$ implies $w^\alpha \in W_2$ for some $\alpha > 1$ due to the Helson-Szegö theorem. Next we prove the same thing for $w \in W_p$, $p > 2$.

The operator $T_w f = w^{1/p} H(w^{-1/p} f)$ is bounded in L^p and theorem 3.6. gives $\forall u \in L^q_+ \exists v \in L^q_+ \ni u \leqq v$, $\|v\|_q \leqq 2\|u\|_q$ and T_w is bounded in $L^2(v)$ with norm $\leqq M$ where $q = (p/2)'$ and M depends only on w (not on u).

The last assertion means that H is bounded in $L^2(w^{\frac{2}{p}} v)$ i.e. $w^{\frac{2}{p}} v \in W_2$. Therefore $v^\alpha w^{\frac{2\alpha}{p}} \in W_2$ for some $\alpha > 1$ (depending only on M).

On the other hand $w \in W_p (p > 2)$ implies $w^{p'-1} \in W_2$. Interpolating we get

$$v^{\alpha\theta} w^{2\alpha\theta/p} w^{(p'-1)(1-\theta)} \in W_2$$

Taking $\theta = \frac{1}{\alpha}$, we get

$$vw^{\frac{2}{p}+(p'-1)(1-\theta)} = vw^{\frac{2}{p}(1+\epsilon)} \in W_2$$

where $\varepsilon = \frac{p}{2}(p'-1)(1-\theta)$.

Thus, again by theorem 3.6., $T_{w^{1+\epsilon}}$ is bounded in L^p, i.e. $w^{1+\epsilon} \in W_p$, as we wanted to prove.

If $1 < p < 2$, we use duality.

Then, to prove that $w \in W_p \Rightarrow w \in W_{p-\varepsilon}$ we proceed as follows: for any $1 < p < \infty$ H is bounded in $L^p(w^{1+\delta})$ and also H is bounded in L^r for $r < p$. By interpolation with change of measure we get that H is bounded in $L^{p-\epsilon}(w)$ for some $\epsilon > 0$. ■

This is a proof of $w \in W_p \Rightarrow w \overset{1+\varepsilon}{\in} W_p$ by complex variable methods. $w \in W_p(H) \Rightarrow w \in W_{p-\varepsilon}(H)$ means that if H is bounded in $X = L^p(w)$, then it is also bounded in $X^\alpha = L^{p/\alpha}(w)$ for some $\alpha > 1$. The same methods yield:

Theorem 4.8. *If X is a 2-convex lattice on (G, dx) and H is bounded in X, then there exists $\alpha > 1$ such that H is bounde in X^α.*

The last application will be to *U.M.D. lattices*. Given a Banach space B, we consider $L^2_B = L^2_B(\mathbf{T})$ the space of functions $f : \mathbf{T} \to B$ strongly measurable and such that $x \to \|f(x)\|_B$ belongs to L^2. For functions $f \in L^2 \otimes B$, that is, for finite linear combinations of functions of the type $\varphi(x) \cdot b (\varphi \in L^2, b \in B)$ the vectorial conjugate operator $\tilde{H} = H \otimes \text{id}$ can be defined by

$$\tilde{H}\left(\sum_{j=1}^{n} \varphi_j b_j\right) = \sum_{j=1}^{n} H(\varphi_j) \cdot b_j$$

The problem to characterize when $\tilde{H}$ can be extended to a bounded operator in L^2_B is solved by the following result (Burkholder [4], Bourgain [3]).

Theorem 4.9. *For a Banach space B, the following conditions are equivalent*

a) $\|\tilde{H} f\|_{L^2_B} \leqq C\|f\|_{L^2_B}$ $(\forall f \in L^2 \otimes B)$.

b) *B is ζ-convex, i.e. $\exists \zeta(x, y)$ real on $B \times B$, convex in each variable $\ni \zeta(0,0) > 0$, $\zeta(x, y) \leqq \|x + y\|_B$, when $\|x\|_B \leqq 1 \leqq \|y\|_B$.*

The same result is valid for L^p_B, $1 < p < \infty$. Thus ζ-convexity is necessary and sufficiente for the partial sums

$$S_n f(x) = \sum_{-n}^{n} \hat{f}(k) e^{2\pi i k x}$$

of the Fourier series of $f \in L^p_B(\mathbb{T})$ to converge in L^p_B $1 < p < \infty$.

It is natural to think that ζ-convexity (or U.M.D. as it is alternatively called) is a sufficient condition to extend to the vectorial case the basic results of Fourier Analysis. In particular, for Carleson's theorem, we get the following partial answer

Theorem 4.10. *Let B be a Banach lattice over (Ω, μ) (with weak unit and Fatou property). Suppose that B is 2-convex or 2-concave. Then the partial sums $S_n f(x) \to f(x)$ a.e. for every $f \in L^2_B$ if and only if B is ζ-convex.*

Proof: In general for any Banach space the convergence in measure of $S_n f$ for every $f \in L^2_B$ implies the boundedness of $\tilde{H}$ in all the L^p_B 's and, consequently, the ζ-convexity.

Suppose, conversely, that B is ζ-convex. Consider $X = L^2_B$. If B is a Banach lattice on (Ω, μ) we can identify $X = L^2_B$ with a lattice over $(\mathbb{T} \times \Omega, (dx) \otimes \mu)$ with

$$\|f\|^2_X = \int_{\mathbb{T}} \|f(x, \cdot)\|^2_B \, dx$$

Then $\tilde{H}$, which is bounded in X, is defined as $\tilde{H} f(x, \omega) = H(f(\cdot, \omega))(x)$. Likewise $S_n f(x, \omega) = S_n(f(\cdot, \omega))(x)$. It is enough to prove that the maximal operator $Mf(x, \omega) = \sup_n |S_n f(x, \omega)|$ is bounded in X; because then

$$\int_{\mathbb{T}} \sup_n \|S_n f(x)\|^2_B \, dx \leqq \int_{\mathbb{T}} \| \sup_n |S_n f(x, \cdot)| \|^2_B \, dx = \int_{\mathbb{T}} \|Mf(x, \cdot)\|^2_B \, dx \leqq C \int_{\mathbb{T}} \|f(x)\|^2_B \, dx.$$

Now if B is 2-convex (resp. 2-concave), X is also 2-convex (resp. 2-concave).

Since M is linearizable, we can use theorem 3.11. It is enough to prove that

$$w \in W(\tilde{H}) \Rightarrow w \in W(M)$$

where now w are $w(x, \omega)$ on $\mathbb{T} \times \Omega$.

Given $w \in W(\tilde{H})$ consider functions $f(x, \omega) = g(x) \chi_E(\omega)$ where g is a trigonometric polynomial with rational coefficients and E is a measurable subset of Ω. Then $\tilde{H} f(x, \omega) = Hg(x) \cdot \chi_E(\omega)$ and the inequality $\|\tilde{H} f\|_{L^2(w)} \leqq C\|f\|_{L^2(w)}$ is now

$$\int_E \int_{\mathbb{T}} |Hg(x)|^2 w(x, \omega) \, dx \, d\mu(\omega) \leqq C^2 \int_E \int_{\mathbb{T}} |g(x)|^2 w(x, \omega) \, dx \, d\mu(\omega)$$

Since this happens for every measurable E, we get

$$\int_{\mathbf{T}} |Hg(x)|^2 w(x,\omega)\,dx \leqq C^2 \int_{\mathbf{T}} |g(x)|^2 w(x,\omega)\,dx \tag{4.11}$$

μ-a.e. for every g, and also for all the g we have considered, since they form a countable collection.

Now if $\omega \in \Omega$ is such that (4.11) holds for every g, then $w(\cdot,\omega) \in A_2$ with A_2 constant $\leqq C$. Thus

$$\sup_I \left(\frac{1}{|I|}\int_I w(x,\omega)\,dx\right)\left(\frac{1}{|I|}\int_I w(x,\omega)^{-1}\,dx\right) \leqq C \tag{4.12}$$

μ-a.e.

We shall see that (4.12) implies $w \in W(M)$. We use the (scalar) result of R. Hunt and W.S. Young [**18**] which says

$$\int_{\mathbf{T}} |S^*\varphi(x)|^2 v(x)\,dx \leqq C(v) \int_{\mathbf{T}} |\varphi(x)|^2 v(x)\,dx$$

where $C(v)$ depends only on the A_2 constant of v and $S^*\varphi(x) = \sup_n |S_n\varphi(x)|$.

Thus (4.12) implies

$$\|Mf\|_{L^2(w)} = \int_\Omega \int_{\mathbf{T}} |S^*(f(\cdot,\omega))(x)|^2 w(x,\omega)\,dx\,d\mu(\omega) \leqq$$
$$\leqq C \int_\Omega \int_{\mathbf{T}} |f(x,\omega)|^2 w(x,\omega)\,dx\,d\mu(\omega) = \|f\|_{L^2(w)} \blacksquare$$

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THE WORK OF JOSE LUIS RUBIO DE FRANCIA III

The aim of this paper is to review a set of articles ([6], [10], [11], [13], [16], [25]) of which José Luis Rubio de Francia was author or co-author written between 1985 and 1987.

I had the luck of being his graduate student around this time so that we collaborated in some of this work. It is hard to say in a few words how was José Luis Rubio but at least I would like to point out that he influenced my career in a decisive way and that he was one of the nicest persons I have ever met.

1. Singular integrals with rough kernels: L^p theory ([16], [6], [13])

In all these papers a common approach in used to study the boundedness of several singular integrals, based on the following idea: decompose the operator T as a sum

$$T = \sum_{k=-\infty}^{\infty} \tilde{T}_k$$

in such a way that

$$\|\tilde{T}_k f\|_2 \leq C 2^{-\alpha|k|} \|f\|_2 \text{ for some } \alpha > 0; \tag{1}$$

if now one of the following inequalities happens

$$\|\tilde{T}_k f\|_1 \leq C \|f\|_{H^1}; \tag{2}$$

$$\|\tilde{T}_k f\|_{p_0} \leq C \|f\|_{p_0} \text{ for some } p_0 \neq 2; \tag{3}$$

$$\|\tilde{T}_k f\|_{L^2(w)} \leq C \|f\|_{L^2(w)} \text{ for some weight } w, \tag{4}$$

interpolation with (1), summing a geometric series and duality give the boundedness of T in L^p, $1 < p < \infty$, in L^p, $p_0 < p < p_0'$ (or $p_0' < p < p_0$) and $L^2(w^\theta)$, $0 \leq \theta < 1$, respectively. Polynomial growth in k can be allowed in inequalities (2), (3) and (4) with the same conclusion.

To get (1) we'll start with a natural descomposition of T as $\sum_{j=-\infty}^{\infty} T_j$ where each T_j is given by convolution with a measure σ_j.

If, for example, one can prove

$$|\hat{\sigma_j}(\xi)| \leq C \min(|2^j\xi|, |2^j\xi|^{-1})^\alpha \text{ for some } \alpha > 0, \tag{5}$$

then we can construct $\tilde{T}_k$ as follows: choose a function $\psi \in C^\infty(\mathbf{R}^n)$, supported in $\frac{1}{2} < |\xi| < 2$ and such that

$$\sum_{j=-\infty}^{\infty} \psi(2^j\xi) = 1 \qquad \forall \xi \neq 0, \tag{6}$$

define S_j as

$$(S_j f)\hat{}(\xi) = \psi(2^j\xi)\hat{f}(\xi)$$

and take

$$\tilde{T}_k = \sum_{j=-\infty}^{\infty} T_j S_{j+k}. \tag{7}$$

Under these circunstances, (1) is easily verified using (5) and Plancherel's theorem.

a) The simplest application will consider the singular integral

$$Tf(x) = p.v. \int_{\mathbf{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y)\, dy \tag{8}$$

where Ω is homogeneous of degree zero, its restriction to the unit sphere has mean value zero and is in $L^q(S^{n-1})$ for some $q > 1$. It is well-known that T is bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, by using the method of rotations but the present method offers an alternative approach. $T_j f$ is the integral restricted to $2^j \leq |y| < 2^{j+1}$ and σ_j is the integrable function given by $\Omega(y)|y|^{-n}\mathcal{X}_{\{2^j \leq w|y| < 2^{j+1}\}}$ where $\mathcal{X}_A$ stands for the characteristic function of the set A. The estimation of an oscillatory integral shows that (5) happens for any $\alpha < 1/q'$.

Theorem 1. *Let $\{\sigma_j\}$ be a sequence of Borel measures supported in $\{x \in \mathbf{R}^n : |x| \leq 2^j\}$, with uniform total variation and integral zero such that*

$$|\hat{\sigma}_j(\xi)| \leq C|2^j\xi|^{-\alpha} \text{ for some } \alpha > 0.$$

Then, $Tf = \sum_j \sigma_j * f$ is bounded in L^p, $1 < p < \infty$.

Defining $\tilde{T}_k$ as above we only need to compute the Hörmander constant of its kernel to show that (2) is verified with constant $C(1 + |k|)$.

As an application, singular integrals of the type (8) are bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$. If one introduces a bounded radial function in the kernel of (8), the method of rotations is not applicable but theorem 1 gives again the L^p-boundedness, $1 < p < \infty$.

Before modifiying some aspects of this theorem, let us state a new one related to maximal operators.

Theorem 2. *Let $\{\mu_j\}$ a sequence of positive Borel measures supported in $\{x \in \mathbf{R}^n : |x| \leq 2^j\}$, with uniform total variation such that*

$$|\hat{\mu}_j(\xi)| \leq C|2^j\xi|^{-\alpha} \text{ for some } \alpha > 0,$$

then, $\mathcal{M}f(x) = \sup_j |\mu_j * f(x)|$ defines a bounded operator in $L^p(\mathbf{R}^n)$, $1 < p \leq \infty$.

To prove this theorem, define $\sigma_j = \mu_j - \hat{\mu}_j(0)\varphi_j$ where $\varphi_j = 2^{-jn}\varphi(2^{-j}\cdot)$ and φ is a $\mathcal{C}^\infty(\mathbf{R}^n)$ function, supported in the unit ball and such that $\hat{\varphi}(0) = 1$. Then, the sequence $\{\sigma_j\}$ satisfies the hypotheses of theorem 1 and the same is true for $\{\varepsilon_j\sigma_j\}$, where $\varepsilon_j = \pm 1$ arbitrarily, with constants independent of the sequence of signs. It is enough to observe that

$$\mathcal{M}f(x) \leq \Big(\sum_j |\sigma_j * f(x)|^2\Big)^{1/2} + cMf(x) \tag{9}$$

(where M stands for the Hardy-Littlewood maximal function) and apply theorem 1 to get the L^p-boundedness of the square function (via the uniform boundedness of $\sum_j \varepsilon_j\sigma_j * f$ and the usual argument with Rademacher functions).

A consequence of theorem 2 is the L^p-boundedness of the lacunary spherical maximal function (take μ_j = normalized Lebesgue measure over the sphere of radius 2^j). Obviously one can substitute the sphere by any other compact surface with enough curvature to ensure the required decay condition of the Fourier transform for the Lebesgue measure carried by it.

b) Given a matrix A whose eigenvalues have nonnegative real part, we can define the associated group of dilations $\{\delta_t\}$ by $\delta_t x = t^A x$ and a "norm" in $\mathbf{R}^n$

such that $\|\delta_t x\| = t\|x\|$, $t > 0$, (see [**26**]). If this norm is used in theorems 1 and 2 instead of the euclidean norm, they are still true. Apart from standard modifications of operators like (8), this provides other interesting results.

Given a curve $\Gamma : t \to \gamma(t)$ in $\mathbf{R}^n$, two operators are usually associated to it: the Hilbert transform along Γ

$$H_\Gamma f(x) = p.v. \int_{-\infty}^{\infty} f(x - \gamma(t)) \frac{dt}{t}$$

and the maximal function along Γ

$$M_\Gamma f(x) = \sup_{h>0} \frac{1}{2h} \left| \int_{-h}^{h} f(x - \gamma(t)) dt \right|.$$

$H_\Gamma f = \sum_j \sigma_j * f$ where σ_j is the measure of size $\dfrac{1}{t}$ over the portion of Γ where $2^j \leq |t| < 2^{j+1}$ and M_Γ is equivalent to $\sup_j |\mu_j * f|$ where μ_j is the measure of size 2^{-j-1} over the same portion of Γ. A homogeneous curve is given by

$$\gamma(t) = t^A u, \quad t > 0, \qquad \gamma(t) = (-t)^A v, \quad t < 0.$$

where $u, v \in S^{n-1}$ and the positive and negative parts of γ generate the same subspace of $\mathbf{R}^n$. The boundedness of the Hilbert transform and the maximal function along a homogeneous curve are now a consequence of the estimates for $\hat{\sigma_j}$ and $\hat{\mu_j}$, which can be found in [**26**]. In that paper L^p-boundedness for $p \neq 2$ is proved via analytic interpolation which we avoid here. The same result holds for well-curved curves, see again [**26**] for the definition and the proof of the key estimate.

c) Inequalities like (3) can be used instead of (2). In order to get them one modifies the choice of ψ above requiring

$$\sum_{j=-\infty}^{\infty} |\psi(2^j \xi)|^2 = 1, \; \forall \xi \neq 0 \tag{10}$$

instead of (6) so that $\sum_j S_j^2 = I$ and Littlewood-Paley type inequalities occur in both senses (see [**17**, chap. V]). The following chain of inequalities can be written

$$\begin{aligned} \|\tilde{T}_k f\|_{p_0} &= \|\sum_j T_j S_{j+k}^2 f\|_{p_0} \leq C \|(\sum_j |T_j S_{j+k} f|^2)^{1/2}\|_{p_0} \leq \\ &\leq C \|(\sum_j |S_{j+k} f|^2)^{1/2}\|_{p_0} \leq C \|f\|_{p_0}, \end{aligned} \tag{11}$$

provided the $\{T_j\}$ satisfy the following vector-value inequality

$$\|(\sum_j |T_j f_j|^2)^{1/2}\|_{p_0} \leq C\|(\sum_j |f_j|^2)^{1/2}\|_{p_0}.$$

This inequality is easily obtained from an uniform weighted inequality

$$\int |T_j f|^2 u \leq C \int |f|^2 Au \tag{12}$$

where A is bounded from L^q to L^q and $q = (\frac{p_0}{2})'$. For convolution operators T_j with kernel σ_j, (12) holds with $Aw = \sigma^*(w) = \sup_j ||\sigma_j| * w|$.

We can state the following theorem:

Theorem 3. *Let $\{\sigma_j\}$ be a sequence of Borel measures in $\mathbf{R}^n$ with uniform total variation, such that*

$$|\hat{\sigma_j}(\xi)| \leq C \min(|2^j\xi|, |2^j\xi|^{-1})^\alpha \textit{ for some } \alpha > 0. \tag{5}$$

If σ^* is bounded in $L^q(\mathbf{R}^n)$ for some $q \geq 1$, then $Tf = \sum_j \sigma_j * f$ is bounded in $L^p(\mathbf{R}^n)$, $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2q}$.

We can avoid the compactness assumption for the support of σ_j but no new interesting results come from this generalization. Its main interest with respect to theorem 1 lies in the modifications to be given below.

Theorem 2 is also obtained from theorem 3 by using a bootstraping argument (again we don't need to assume that supp μ_j is compact but then we have to add $|\hat{\mu_j}(\xi) - \hat{\mu_j}(0)| \leq C|2^j\xi|^\alpha$). As before we define $\sigma_j = \mu_j - \hat{\mu_j}(0)\varphi_j$ and apart from (9) we also have

$$\sigma^*(f) \leq \mathcal{M}f + CMf. \tag{13}$$

This inequality together with theorem 3 and (9) imply: *if $\mathcal{M}$ is bounded in L^q, it is also bounded in L^p, $\frac{1}{p} - \frac{1}{2} < \frac{1}{2q}$, (i.e., $p > \frac{2q}{q+1}$).*

Starting with $q = 2$ where the result is given by the hypotheses on $\hat{\mu_j}(\xi)$, any $p > 1$ is reached after a finite number of steps.

d) Let $\xi = (\xi_1, \xi_2) \in \mathbf{R}^m \times \mathbf{R}^{n-m}$, theorem 3 can be modified in the following way:

Theorem 4. *If in theorem 3 we assume*

$$|\hat{\sigma_j}(\xi)| \leq C \min(|2^j\xi_1|, |2^j\xi_1|^{-1})^\alpha \text{ for some } \alpha > 0, \tag{14}$$

instead of (5), *the same conclusion holds.*

The same proof works after taking operators S_j which act only on the variable ξ_1.

Condition (14) implies $\hat{\sigma_j}(0, \xi_2) = 0, \forall \xi_2 \in \mathbf{R}^{n-m}$ which is usually too strong. It seems better to assume

$$\begin{aligned} &|\hat{\sigma_j}(\xi_1, \xi_2) - \hat{\sigma_j}(0, \xi_2)| \leq C|2^j\xi_1|^\alpha \\ &|\hat{\sigma_j}(\xi)| \leq C|2^j\xi_1|^{-\alpha} \end{aligned} \tag{15}$$

But then one has to make some hypothesis on $\hat{\sigma_j}(0, \xi_2)$, for example,

$$|\hat{\sigma_j}(0, \xi_2)| \leq C \min(|2^j\xi_2|, |2^j\xi_2|^{-1})^\alpha. \tag{16}$$

Writting

$$\hat{\sigma_j}(\xi_1, \xi_2) = [\hat{\sigma_j}(\xi_1, \xi_2) - \hat{\sigma_j}(0, \xi_2)\varphi_j(\xi_1)] + \hat{\sigma_j}(0, \xi_2)\varphi_j(\xi_1)$$

theorem 4 is applied twice.

For the maximal operator conditions like (15) are to be assumed on $\{\mu_j\}$ and an extra hypothesis on the boundedness of the maximal operator associated to $\hat{\mu_j}(0, \xi_2)$. All the technical details can be found in [**16**].

Now we can prove the boundedness of the Hilbert transform and the maximal function along a homogeneous curve with A diagonal by induction without using non-isotropic dilations. If the entries in A are integers, the estimations we need are also much easier. In addition we get a result for flat curves which is not given by theorem 1 and 2:

Corollary 5. *Let* $\Gamma = (t, \varphi(t))$ *be a curve in* $\mathbf{R}^2$ *such that* $\varphi(0) = \varphi'(0) = 0$, $\varphi''(t) > 0$ *and increasing for* $t > 0$, φ *odd or even, then* H_Γ *and* M_Γ *are bounded in* $L^p(\mathbf{R}^2)$, $1 < p < \infty$.

e) If in the hypotheses of corollary 5 we merely assume $\varphi''(t) > 0$ (i.e. not necessarily increasing) the conclusion can be false. In [**21**] the following result was proved: *let* $\Gamma = (t, \varphi(t))$ *be an even convex curve in* $\mathbf{R}^2$, *then* H_Γ *is bounded in* L^2 *if and only if* $\exists C > 1$ *such that*

$$\varphi'(Ct) \geq 2\varphi'(t), \ \forall t > 0. \tag{17}$$

Assuming (17), inequality (14) fails in an angular sector which moves with j. This sequence of sectors is lacunary so that one can apply Littlewood-Paley inequalities associated to them as was proved by Nagel, Stein and Wainger [**22**]. Combining these inequalities with theorem 4, the following is proved in [**6**]:

Theorem 6. *Let $\Gamma = (t, \varphi(t))$ be a curve in $\mathbf{R}^2$, odd or even, $\varphi''(t) > 0$ for $t > 0$, satisfying (17). Then, M_Γ and H_Γ are bounded in $L^p(\mathbf{R}^2)$, $1 < p < \infty$.*

Together with the result in [**21**] this theorem implies that for φ even, (17) is necessary and sufficient for the L^p boundedness.

In [**13**], A. Córdoba and José Luis Rubio de Francia generalized the preceding theorem to the case where the curve is neither odd nor even. The proof works when some balance condition between the positive and negative parts of the curve is assumed. They also proved that the condition is necessary.

Theorem 7. *Let $\Gamma = (t, \varphi(t))$ be a curve in $\mathbf{R}^2$ such that $\varphi(0) = \varphi'(0) = 0$; $|\varphi'(t)|$ increasing if $t > 0$ and decreasing if $t < 0$; $\exists C > 1$ such that $|\varphi'(Ct)| \geq 2|\varphi'(t)|, \forall t \neq 0$ and $\exists k > 1$ s.t. $|\varphi(k^{-1}t)| \leq |\varphi(-t)| \leq |\varphi(kt)|$ for every $t > 0$ (balance condition). Then M_Γ and H_Γ are bounded in $L^p(\mathbf{R}^2)$, $1 < p < \infty$.*

f) Inequality (4) is also useful and gives weighted inequalities for singular integrals with rough kernels. In the chain of inequalities (11), one can use the $L^2(w)$ norm instead of L^{p_0} if $w \in A_2$ and the first and third inequalities still hold for the Littlewood-Paley theory (see [**20**]).

If T is given by (8) with $\Omega \in L^\infty$, $T_j f \leq CMf$ and the vector valued inequality also holds in $L^2(w)$ (see [**17**]). Then

Theorem 8. *Let T be as in (8) with $\Omega \in L^\infty(S^{n-1})$. Then, T is bounded in $L^p(w)$, $\forall w \in A_p$.*

To pass from $L^2(w)$ inequalities to all $L^p(w)$, $w \in A_p$, one uses the extrapolation theorem of Rubio de Francia ([**24**]).

g) For all the singular integrals studied above, including Hilbert transforms along curves, the maximal operator over the truncated integrals is shown to be bounded in the same spaces giving the a.e. convergence of the truncated integrals.

M. Christ used in [**8**] and [**9**] methods similar to those developed here, independently. In fact, theorem 1 is a modification of [**16**] following his ideas. In [**8**], Hilbert transform and maximal functions along homogeneous curves in nilpotent groups are studied.

Extensions of the theory to the multiparameter setting with applications to multiple singular integrals and operators along hypersurfaces are in [**14**], to operators which are not necessarily of convolution type in [3] and [5]. Further results on curves are in [4] (see also Wainger's lecture in this Proceedings) and more weighted inequalities appear in [**15**] and [**27**].

2. Maximal functions with continuous parameter [25]

Maximal functions were studied in the preceding section only when they were controlled by their dyadic version but this is not the general case as the spherical maximal function shows. Moreover, this is also an example where the dyadic maximal function is bounded in a range which is larger than the one for the continuous maximal function. In [**25**], José L. Rubio de Francia gave a simple proof of the theorem of Stein on the boundedness of the spherical maximal function for $p > \frac{n}{n-1}$, $n \geq 3$, (see [**26**] for the original proof).

Theorem 9. *Let m be the Fourier transform of a compactly supported positive measure μ in $\mathbf{R}^n$ such that*

$$|m\,(\xi)| \leq C|\xi|^{-a} \text{ for some } a > \frac{1}{2}. \tag{18}$$

Then, the maximal operator

$$T^* f(x) = \sup_{t>0} (m\,(t\cdot)\hat{f})^\vee(x)$$

is bounded in $L^p(\mathbf{R}^n)$, $p > \dfrac{2a+1}{2a}$.

($\vee$ stands for the inverse Fourier transform.)

Let us sketch the proof: take a cutting function $\psi \in C^\infty(\mathbf{R}^n)$ supported in $\frac{1}{2} < |\xi| < 2$ and also $\varphi \in C^\infty(\mathbf{R}^n)$ supported in $|\xi| < 1$, such that

$$\varphi(\xi) + \sum_{j=0}^{\infty} \psi(2^{-j}\xi) \equiv 1$$

and consider $m_j = m\,\psi(2^{-j}\cdot)$. Since the maximal function associated to $m\,\varphi$ is bounded by the Hardy-Littlewood maximal function, it is enough to prove that

$$T_j^* f(x) = \sup_{t>0} (m_j(t\cdot)\hat{f})^\vee(x)$$

satisfies

$$\|T_j^* f\|_p \leq C 2^{-\epsilon j} \|f\|_p$$

for the desired range of p's. This is achieved in a way similar to the one in the preceding paragraph, starting with an L^2–inequality. In fact, due to the size hypothesis on m one gets

$$\|T_j^* f\|_2 \leq C 2^{-j(a-\frac{1}{2})} \|f\|_2.$$

For $p = 1$, looking at T_j^* as a vector valued singular integral, one can compute the Hörmander constant of the kernel to obtain

$$\|T_j^* f\|_1 \leq C j 2^j \|f\|_{H^1}.$$

Interpolating and summing in j gives the result for $p \leq 2$. But for $p = \infty$ the theorem holds trivially and the proof is ended.

When μ is the Lebesgue measure over the unit sphere, (18) is satisfied with $a = \dfrac{n-1}{2}$ and Stein's result is obtained. For other hypersurfaces a theorem of Greenleaf in [**18**] is obtained.

Since this method is based on a good L^2–estimate which is false in $n = 2$, it is not applicable to get Bourgain's result [**1**].

If m is not the Fourier transform of a measure as before, the L^∞ estimate can be false. In [**25**] there is also a theorem concerning this case.

Theorem 10. *Let s be an integer $> \frac{n}{2}$, $m \in C^{s+1}(\mathbf{R}^n)$ such that*

$$|D^\alpha m(\xi)| \leq C|\xi|^{-a} \qquad \forall |\alpha| \leq s+1 \text{ with } a > \frac{1}{2}.$$

Then, T^* is bounded in $L^p(\mathbf{R}^n)$ for

$$\frac{2n}{n+2a-1} < p < \frac{2n-2}{n-2a}$$

(1 on the left if $a \geq \frac{n+1}{2}$, ∞ on the right if $a \geq \frac{n}{2}$.)

For $p \leq 2$ the proof follows the same way as in Theorem 9 but now

$$\|T_j^* f\|_1 \leq C_\beta 2^{j(\frac{1}{2}+\beta-a)} \|f\|_{H^1} \qquad \forall \beta > \frac{n}{2}.$$

When $p > 2$, the theory of vector valued singular integrals is again applied to get an L^∞–BMO estimate:

$$\|T_j^* f\|_{BMO} \leq C_\beta 2^{j(\beta-a)} \|f\|_\infty \qquad \forall \beta > \frac{n}{2}.$$

This constant is smaller than the $H^1 - L^1$ constant so that the range in Theorem 10 is not symmetric.

Notice that for $p < 2$, Theorem 9 is better than theorem 10 if $a < \frac{n-1}{2}$ and conversely if $a > \frac{n-1}{2}$. Since this second theorem applies also in the first case one must combine both results to obtain the optimal situation when $\mathcal{M}$ comes as the Fourier transform of a measure.

3. Operators related to the method of rotations and the Radon transform [10]

a) Given an one-dimensional operator S bounded in $L^p(\mathbf{R})$, we can define a collection of n-dimensional analogues: for any $u \in S^{n-1}$,

$$S_u f(x) = S(f(x + \cdot u))(0).$$

All these operators are uniformly bounded in $L^p(\mathbf{R}^n)$. We look for inequalities of the type

$$\left(\int_{\mathbf{R}^n} \left(\int_{S^{n-1}} |S_u f(x)|^q du\right)^{p/q} dx\right)^{1/p} \leq C\|f\|_p. \tag{19}$$

The left-hand side is called the $L^p(L^q)$ mixed norm of the family $\{S_u f\}$. Inequality (19) is trivial for $p = q$ (hence for $p > q$) because the order of integration can be reversed.

When S is the Hilbert transform, Hardy-Littlewood maximal function or maximal Hilbert transform, inequalities like (19) are used in the method of rotations for singular integral operators with variable kernel ([2]):

$$Tf(x) = p.v. \int \frac{\Omega(x, y')}{|y|^n} f(x - y)\, dy \tag{20}$$

where

$$\sup_x \|\Omega(x, \cdot)\|_{L^r(S^{n-1})} < +\infty \text{ for some } r > 1.$$

Taking f = characteristic function of the unit ball, one proves that (19) only can be true when $\dfrac{1}{q} > \dfrac{n-p}{(n-1)p}$ for the three operators listed above. In [10] the following is proved

Theorem 11.

(i) *When* $S = M$, (19) *holds for* $\dfrac{1}{q} > \dfrac{n-p}{(n-1)p}$ *whenever* $p \leq \max(2, \dfrac{n+1}{2})$.

(ii) *When S is an one-dimensional operator bounded in $L^r(w)$ for every weight $w \in A_r(\mathbf{R})$, then* (19) *holds in the same range as (i).*

In both cases interpolation with the trivial case $p = q$ gives a result for $p > \max(2, \frac{n+1}{2})$ which is sharp if $n = 2$ but leaves an undecided region when $n \geq 3$.

The proof of (a) for $p = 2$ uses an estimate from L^2 to $L^2(L^2_\beta)$ where L^2_β is a Sobolev space. By the embedding theorems, for any $q < \frac{2(n-1)}{n-2}$, there exists $\beta < \frac{1}{2}$ such that $L^2_\beta(S^{n-1}) \subset L^q(S^{n-1})$. The estimate is obtained via the Fourier transform. For $p = \frac{n+1}{2}$, the X-ray transform and its L^p mapping properties are used. See [10] for details. (b) is then obtained from (a).

Applying theorem 11 to the method of rotations in [2] we get:

Corollary 12. *Let T be a singular integral operator like (20) and T^* the maximal operator of the truncated integrals. Then, T and T^* are bounded in $L^p(\mathbf{R}^n)$ if $1 < p \leq \max(2, \frac{n+1}{2})$ and $r > \frac{n-1}{n} p'$. If $n = 2$ also for $2 \leq p < \infty$ for any $r > 1$ and if $n \geq 3$, for $\frac{n+1}{2} \leq p < \infty$ if $r > \frac{2p}{2p-1}$.*

This result is sharp in $n = 2$ but there is probably a better result if $n \geq 3$.

A maximal operator related to the Bochner-Riesz multipliers is given by

$$M_\delta f(x) = \sup_R \frac{1}{|R|} \int_R |f(x-y)|\, dy$$

where for a fixed $\delta > 0$, the supremum is taken over all the parallelopipeds containing the origin and having one side of length r and $(n-1)$ sides of length δr, $\forall r > 0$. The conjecture is

$$\|M_\delta f\|_n \leq C(\log \delta)^a \|f\|_n \tag{21}$$

which was proved by Córdoba [12] for $n = 2$, the only case where it is known.

If (21) was true, interpolating with the trivial estimate with constant $C\delta^{1-n}$ for p close to 1 would give

$$\|M_\delta f\|_p \leq C(\log \delta)^a \delta^{1-n/p} \|f\|_p, \quad 1 < p \leq n \tag{22}$$

As a consequence of theorem 10 (a) we have:

Corollary 13. (22) *holds if* $1 < p \leq \max(2, \frac{n+1}{2})$.

For $n = 2$ this gives a new proof of Córdoba's result.

b) A maximal operator associated to the Radon transform is

$$Rf(x,u) = \sup_{r>0} \frac{1}{r^{n-1}} \int_{\substack{|y|\le r \\ <y,u>=0}} |f(x-y)| d\lambda(y), \quad x \in \mathbf{R}^n, \; u \in S^{n-1}$$

where λ is the Lebesgue measure on the hyperplane $< y,u >= 0$. Remember that for $u \in S^{n-1}$ and $t \in \mathbf{R}$, the Radon transform of f in (u,t) is obtained by integrating f over the hyperplane $< x,u >= t$ (see [23]). Again one can consider mixed norm estimates

$$(23) \qquad \left(\int_{\mathbf{R}^n} \left(\int_{S^{n-1}} |Rf(x,u)|^q du \right)^{p/q} dx \right)^{1/p} \le C\|f\|_p$$

and the theorem proved in [10] is:

Theorem 14. *Inequality* (23) *holds whenever*

$$1 < p \le \frac{n+1}{n} \qquad \textit{and} \qquad \frac{1}{q} > \frac{n}{p} - (n-1)$$

or

$$\frac{n+1}{n} \le p \le 2 \qquad \textit{and} \qquad \frac{1}{q} > (\frac{2}{p} - 1)\frac{1}{n-1}$$

or

$$p \ge 2 \qquad \textit{and} \qquad q < \infty.$$

It is enough to prove the theorem for $p = 2$, $q < \infty$ and $q = n+1$, $p > \dfrac{n+1}{n}$ and interpolate with the trivial case $p = q$. The L^2–theory is handled with the Fourier transform and the mapping properties of the Radon transform are used in the remainder.

c) Let us include two more results from [10]:

Theorem 15. *Let Γ be a rectifiable curve in $\mathbf{R}^n$ which crosses at most M times ($M > 0$ given) every hyperplane in $\mathbf{R}^n$ and $u_1, ..., u_N$, N points over Γ. If $H_{u_1}, ..., H_{u_N}$ are the Hilbert transforms in these directions, then*

$$\| \left(\frac{1}{N} \sum_{j=1}^{N} |H_{u_j} f|^q \right)^{1/q} \|_2 \le Cq\|f\|_2$$

and

$$\| \sup_{1\le j\le N} |H_{u_j} f| \|_2 \le C \log N \|f\|_2.$$

Notice that S^1 has the finite crossing property of the theorem so that any set of N points in $\mathbf{R}^2$ satisfies those inequalities.

As a consequence we have

Corollary 16. *Let $u_1, \ldots, u_N$ as in the preceding theorem, if $\frac{4}{3} \leq p \leq 4$,*

$$\| \left(\sum_{j=1}^{N} |H_{u_j} f_j|^2 \right)^{1/2} \|_p \leq C(\log N)^{4|\frac{1}{p}-\frac{1}{2}|} \| \left(\sum_{j=1}^{N} |f_j|^2 \right)^{1/2} \|_p$$

with C independent of N.

4. Singular integrals with rough kernel: weak (1,1) estimates [11]

The singular integral (8) and the related maximal operator

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|<r} |\Omega(y')f(x-y)|\, dy \tag{24}$$

are easily seen to be bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, by using the method of rotations. But this method does not apply to obtain a weak (1,1) estimate because the weak L^1–space is not a normed space. The question remained open for a long time until M. Christ gave in [7] the first proof of the weak (1,1) estimate for M_Ω in the two-dimensional case. Subsequently, in a joint paper with José L. Rubio de Francia [11], they were able to extend the result to all dimensions for M_Ω and to prove it for $n = 2$ for the singular integral (they claim that in this case the proof can be extended to $n \leq 5$). S. Hofmann proved independently the result for the singular integral in [19].

Theorem 17. *The maximal operator M_Ω given by (24) is of weak type (1,1) when $n \geq 2$ and $\Omega \in L \log L(S^{n-1})$. The singular integral operator T given by (8) is of weak type (1,1) when $n = 2$, $\Omega \in L \log L(S^1)$ and $\int \Omega = 0$.*

The proof follows the usual Calderón-Zygmund argument with just one modification: after decomposing $f = g + b$ where b is a sum of functions b_i supported in disjoint dyadic cubes Q_i and with mean value zero, one takes away an exceptional set E formed by dilations of the cubes Q_i and usually proves that $\|Tb\|_{L^1(\mathbf{R}^n \setminus E)} \leq C\|b\|_1$. Instead of this inequality, what is used in the above papers is

$$\|Tb\|^2_{L^2(\mathbf{R}^n \setminus E)} \leq C\lambda \|b\|_1$$

where $\lambda > 0$ is the height at which the Calderón-Zygmund decomposition has been made. The idea of using this inequality goes back to a paper of C. Fefferman.

In practice, one takes $K_j(x) = 2^{-jn}\eta(2^{-j}x)\Omega(x)$ where η is a radial C^∞ function, nonnegative, supported in $\frac{1}{2} \leq |x| \leq 4$ and identically one on $1 \leq$

$|x| \le 2$. Then $M_\Omega f \le C \sup_j f * K_j$ if $f \ge 0$ and it is enough to prove the weak (1,1) estimate for $\sup_j f * K_j$.

If b is the "bad function" in the Calderón-Zygmund decomposition of f and the exceptional set E is constructed by taking the union of the cubes with same centers as Q_i and five times their sides, for a fixed j, $b_i * K_j(x)$ is different from zero in some point $x \notin E$ only if the side of the cube Q_i where b_i is supported is less than 2^j. For each $s \in \mathbf{Z}$, denote by B_s the sum of the b_i for which the sidelength of Q_i is 2^s; then, the key estimate is the following: assuming $\Omega \in L^\infty(S^{n-1})$ and $s > 0$,

$$\| \sup_j |K_j * B_{j-s}| \|^2_{L^2(\mathbf{R}^n)} \le C 2^{-\varepsilon s} \|\Omega\|^2_\infty \lambda \|b\|_1$$

for some $\varepsilon > 0$. By dilation invariance it is enough to prove it for $j = 0$. If $\tilde{K}_0(x) = \overline{K_0(-x)}$ we have

$$\|K_0 * B_{-s}\|_2^2 = < \tilde{K}_0 * K_0 * B_{-s}, B_{-s} >$$

and it is enough to prove

$$\|\tilde{K}_0 * K_0 * B_{-s}\|_\infty \le C \|\Omega\|^2_\infty 2^{-\varepsilon s} \lambda.$$

The convolution $\tilde{K}_0 * K_0$ has better properties than K_0 alone and this makes possible the above estimate to hold. When $n = 2$, $\tilde{K}_0 * K_0$ is Hölder continuous outside the origin and this is enough (see [7]) but for $n \ge 3$ this Hölder property does not hold and one has to go into harder geometric considerations (see [11]).

For the singular integral there is an additional complication coming from the fact that the key estimate must be now proved for a sum instead of a supremum

$$\| \sum_j K_j * B_{j-s} \|^2_{L^2(\mathbf{R}^n)} \le C 2^{-\varepsilon s} \lambda \|b\|_1 \|\Omega\|^2_\infty.$$

The square of the sum presents cross terms which are hard to handle. This is the reason why the proof works only in low dimensions.

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THE WORK OF JOSÉ LUIS RUBIO DE FRANCIA IV

José Luis and I first met at the famous - and hugely enjoyable 1983 El Escorial conference of which he and Ireneo Peral were the chief organisers, but we did not really discuss mathematics together until the spring and summer of 1985. There is an old question - formally posed by Stein in the proceedings of the 1978 Williamstown conference [**St**] - concerning the disc multiplier and the Bochner-Riesz means.

For $\lambda \geq 0$, let

$$(T_R^\lambda f)\hat{}(\xi) = (1 - |\xi|^2/R^2)_+^\lambda \hat{f}(\xi),$$

and for $\delta > 0$ (small) let

$$(S^\delta f)\hat{}(\xi) = m_\delta(\xi)\hat{f}(\xi)$$

where m_δ is a smooth radial bump function associated to the annulus $\{1 - \delta \leq |\xi| \leq 1\}$. It is known, [ClS], [Hö], [F], [**Có1**] that in $\mathbf{R}^2$, for $\lambda > 0$, T_1^λ is bounded on L^4, and that the L^4 operator norm of m_δ is $O(\log(1/\delta)^{1/4})$. Thus, according to the "Boundedness Principle" of José Luis, (see the lecture of García-Cuerva in this volume) there must be L^2-weighted inequalities reflecting these facts. In particular, there is an inequality of the form

$$\int_{\mathbf{R}^2} |S_\delta f|^2 \omega \leq \int_{\mathbf{R}^2} |f|^2 \tilde{\omega}$$

with $\omega \to \tilde{\omega}$ bounded on $L^2_+(\mathbf{R}^2)$ with constant $O(\log(1/\delta)^{1/2})$. In analogy with what happens in the one-dimensional case, where the Hardy-Littlewood maximal function controls things, one would hope that it would be possible to choose the operator $M_\delta : \omega \to \tilde{\omega}$ to be the Kakeya maximal function of C. Fefferman and A. Córdoba [F], [**Có1**]. This, roughly speaking, was what Stein proposed.

As we know, José Luis had a great interest in both the disc multiplier operators and, of course, weighted inequalities. So this was a problem he found very attractive - and about which he was very optimistic - so much so that he bet me L5 during the Arcata conference that it would be resolved in the affirmative by the end of 1988. (He later remarked that I had the better end of the bet - if he wanted L5 he'd have to work for it, whereas I could just sit back and hope that no - one else was working very hard!).

What I want to describe here is some of the progress made by José Luis in the last few years in the direction of this and related problems. One such related problem is that of almost-everywhere convergence for Bochner-Riesz means. That is, do we have, as $R \to \infty$,

$$T_R^\lambda f(x) \to f(x) \qquad a.e.\ \forall f \in L^p(\mathbf{R}^n), \qquad 2 \leq p \leq 2n/(n-1), \qquad \lambda > 0?*$$

Of course the one dimensional problem is classical, and it was known (see **[C1]**) that the answer was affirmative when $n = 2$ for $2 \leq p \leq 4$. In higher dimensions, even the *boundedness* of T_1^λ on $L^p(\mathbf{R}^n)$, $2 \leq p < 2n/(n-1)$ remains unsolved. Imagine, then, my reaction, while visiting Madrid in March 1987, when José Luis asked me what I would say were he to tell me that "almost-everywhere convergence holds in the optimal range of p in all dimensions"!. While my mouth was still open he changed the question - what if he were to tell me that "almost-everywhere convergence holds in the optimal range of p in all *even* dimensions"!. Laughing, I replied that I would say that he was making a joke with me. But he wasn't making a joke, and it's the circle of ideas surrounding this result that I want to relate in this article.

José Luis' last work is entitled "Transference Principles for Radial Multipliers", and the result he told me about was precisely that, a transference result, starting out with the (known) result in 2 dimensions, and transferring up 2 dimensions at a time. Of course one is not going to transfer a.e. convergence properties directly, but some quantitative estimates which imply them. The standard thing to consider is the maximal operator

$$T_*^\lambda f(x) = \sup_R |T_R^\lambda f(x)|,$$

but immediately one realises that the usual method - involving boundedness of the maximal operator on L^p - is not appropriate here because boundedness of T_1^λ is not known. Thus (and this was a typical José Luis idea) one should show that T_*^λ is well-behaved on a *larger* space than $L^p(\mathbf{R}^n)$ - and what better place to start than with the mixed-norm spaces $L^p(L^2)$ - old friends of José Luis? In this case the index p refers to the radial variable and the 2 refers to the angular variable - so that $L^p(L^2)$ is the space of functions for which the norm

$$\|f\|_{p,2} = \left\{ \int_0^\infty \Big(\int_{S^{n-1}} |f(r\theta)|^2 d\theta \Big)^{p/2} r^{n-1} dr \right\}^{1/p}$$

is finite. (The reader should beware that $\| \ \|_{p,2}$ here is what José Luis called $\| \ \|_{2,p}$). Notice that by Hölder's inequality, when $p \geq 2$ we have $L^p \subseteq L^p(L^2)$.

The José Luis philosophy (the "Boundedness Principle") tells us that to understand bounded linear operators, we need to understand the weighted L^2

*One can also pose the problem for $2n/(n+1) \leq p < 2$, but this version has a somewhat different flavour.

inequalities they satisfy. For the case of $L^p(L^2)$, $(p \geq 2)$, these are precisely the L^2 inequalities with *radial weights;* that is, of the form

$$\int |Tf(r\theta)|^2 d\theta \omega(r) r^{n-1} dr \leq \int |f(r\theta)|^2 d\theta \tilde{\omega}(r) r^{n-1} dr$$

with

$$\|\tilde{\omega}\|_{L_+^{(p/2)'}(r^{n-1}dr)} \leq C\|\omega\|_{L_+^{(p/2)'}(r^{n-1}dr)}.$$

From this one can immediately see that if T is both bounded on $L^p(\mathbf{R}^n)$ and rotationally invariant - for example, given by a radial Fourier multiplier - then it is bounded on $L^p(L^2)$. For the boundedness of T on L^p and the boundedness principle give a weighted inequality of the form

$$\int |Tf|^2 \omega \leq \int |f|^2 \tilde{\omega}$$

with $\omega \to \tilde{\omega}$ bounded on $L^{(p/2)'}(\mathbf{R}^n)$, and if we now choose $\omega \in L^{(p/2)'}(\mathbf{R}^n)$ radial and use rotation invariance of T, we see that $\tilde{\omega}(\rho\cdot)$ works just as well as $\tilde{\omega}$, for any $\rho \in SO(n)$. Hence so does its average $\tilde{\tilde{\omega}} = \int_{SO(n)} \tilde{\omega}(\rho x) d\rho$ which is now a radial weight satisfying $\|\tilde{\tilde{\omega}}\|_{L^{(p/2)'}(r^{n-1}dr)} \leq C\|\omega\|_{L^{(p/2)'}(r^{n-1}dr)}$.

This latter fact (that, for rotationally invariant linear operators, boundedness on L^p implies boundedness on $L^p(L^2)$) is essentially a particular case of the result of Herz and Rivière in [HR]. However, it is nice to notice that one can use the boundedness principle instead of the usual argument using the Marcinkiewicz-Zygmund theorem to prove it. (Similar reasoning applies to show that translation-invariant linear operators which are bounded on $L^p(T^{n_1+n_2})$ or $L^p(\mathbf{R}^{n_1+n_2})$ are also bounded on $L^p_{dx_1}(L^2_{dx_2})$.)

THE FIRST TRANSFERENCE PRINCIPLE

Let $\mathcal{M} = [0,\infty) \to \mathbf{C}$ be a bounded function, and define T simultaneously on $\mathbf{R}^k$ (for each $k \in \mathbf{N}$) by

$$(Tf)\hat{}(\xi) = \mathcal{M}(|\xi|)\hat{f}(\xi), \qquad \xi \in \mathbf{R}^k.$$

The First Transference Principle. *Let $n \geq 2$, and let T be as above. Suppose for given measurable $\omega, \tilde{\omega} : [0,\infty) \to [0,\infty)$ and for all continuous f with compact support we have*

$$\text{(1)} \qquad \int_{\mathbf{R}^n} |Tf(x)|^2 \omega(|x|) dx \leq \int_{\mathbf{R}^n} |f(x)|^2 \tilde{\omega}(|x|) dx.$$

Then

$$\int_{\mathbf{R}^{n+2m}} |Tf(x)|^2 \omega(|x|) dx \leq \int_{\mathbf{R}^{n+2m}} |f(x)|^2 \tilde{\omega}(|x|) dx$$

for each $m \in \mathbf{N}$, and all suitable f.

The proof is a model of elegance and simplicity. Let T_k be the action of T on radial functions in $\mathbf{R}^k$, that is (with $\mathcal{F}_k$ denoting the action of the k-dimensional Fourier transform on radial functions in $\mathbf{R}^k$,)

$$T_k g(r) = \mathcal{F}_k^{-1}\{\mathcal{F}_k g(\cdot)\mathcal{M}(\cdot)\}(r).$$

Lemma. *If $n \geq 2$, then (1) holds $\Leftrightarrow$*

$$(2) \quad \begin{cases} \displaystyle\int_0^\infty |T_{n+2k}g(r)|^2\omega(r)r^{n+2k-1}dr \leq \int_0^\infty |g(r)|^2\tilde{\omega}(r)r^{n+2k-1}dr \\ \forall k \in \mathbf{N},\ \forall g \in C_c([0,\infty])). \end{cases}$$

Proof: A function $f \in C_c(\mathbf{R}^n)$ has a spherical harmonic development

$$\sum_{k=0}^\infty f_k(|x|)P_k(x)$$

with P_k a homogeneous harmonic polynomial of degree k satisfiying

$$\int_{S^{n-1}} |P_k(\theta)|^2 d\theta = 1.$$

Hence

$$(3) \quad \begin{cases} \displaystyle\int_0^\infty |f(x)|^2\tilde{\omega}(|x|)dx &= \displaystyle\int_0^\infty \int_{S^{n-1}} |f(r\theta)|^2 d\theta\tilde{\omega}(r)r^{n-1}dr \\ &= \displaystyle\sum_{k\geq 0}\int_0^\infty |f_k(r)|^2\tilde{\omega}(r)r^{2k+n-1}dr. \end{cases}$$

By the Hecke-Bochner formula,

$$Tf(x) = \sum_{k=0}^\infty (-1)^{k+1}P_k(-x)T_{n+2k}f_k(|x|)$$

and so

$$(4) \quad \int |Tf(x)|^2\omega(|x|)dx = \sum_{k\geq 0}\int |T_{n+2k}f_k(r)|^2\omega(r)r^{2k+n-1}dr.$$

Combining (3) and (4) and "cancelling out the $\sum$ signs" completes the proof. ■

The First Transference Principle is now immediate: if the set of inequalities (2) holds for a given $n \geq 2$, it also holds for $n + 2m$, $m \in \mathbf{N}$.

The hypothesis that n be at least 2 is essential: although $\mathcal{X}_{[-1,1]}$ is a bounded multiplier on $L^2(\mathbf{R},\omega)$ for even A_1 weights, $\mathcal{X}_{|\xi|\leq 1}$ is *not* a multiplier on $L^2(\mathbf{R}^n,\omega)$ for radial A_1 weights. In particular, Andersen's conjecture **[A]** is false; this had also been observed by Mockenhaupt **[M]**. On the other hand, as the proof shows, the theorem works just as well in the Hilbert-space valued setting.

Corollary 1. *If n is even, $\lambda > 0$ and $2 \le p \le \frac{2n}{n-1}$, then T_*^λ is bounded on $L^p(L^2)(\mathbf{R}^n)$.*

Proof: There is a standard majorisation

$$T_*^\lambda f(x) \le C\{Mf(x) + G^\lambda f(x)\}$$

where M is the (harmless) Hardy-Littlewood maximal function and G^λ, $\lambda > 0$ is a square function given by a radial multiplier. It is known (see **[C1]**, **[C2]**) that G^λ is bounded on $L^4(\mathbf{R}^2)$ and hence that there is an inequality

$$\int_{\mathbf{R}^2} |G^\lambda f(x)|^2 \omega(|x|)dx \le \int_{\mathbf{R}^2} |f(x)|^2 \tilde{\omega}(|x|)dx$$

where

$$\int_0^\infty |\tilde{\omega}(r)|^2 r dr \le C_\lambda \int_0^\infty |\omega(r)|^2 r dr.$$

By the First Transference Principle, the same inequality holds in all higher even dimensions, i.e.

$$\int_0^\infty \int_{S^{n-1}} |G^\lambda f(r\theta)|^2 d\theta \omega(r) r^{n-1} dr \le \int_0^\infty \int_{S^{n-1}} |f(r\theta)|^2 d\theta \tilde{\omega}(r) r^{n-1} dr.$$

Given ω and $\tilde{\omega}$, define u by $u(r) = r^{-(n-2)/2}\omega(r)$ and $\tilde{u}(r) = r^{-(n-2)/2}\tilde{\omega}(r)$ so that

$$\int u^2(r) r^{n-1} dr = \int \omega^2(r) r dr$$

and

$$\int \tilde{u}^2(r) r^{n-1} dr = \int \tilde{\omega}^2(r) r dr.$$

Thus, if n is even

$$\int_0^\infty \int_{S^{n-1}} |G^\lambda f(r\theta)|^2 d\theta u(r) r^{(n-1)+2(n-2)/4} dr$$

$$\le \int_0^\infty \int_{S^{n-1}} |f(r\theta)|^2 d\theta \tilde{u}(r) r^{(n-1)+2(n-2)/4} dr,$$

from which it follows that $|x|^{(n-2)/4} G^\lambda(|\cdot|^{-(n-2)/4} f(\cdot))(x)$ is bounded on $L^4(L^2)(\mathbf{R}^n)$. Moreover $|x|^{-1/2} G^\lambda(|\cdot|^{1/2} f(\cdot))(x)$ is bounded on $L^2(L^2)(\mathbf{R}^n) = L^2(\mathbf{R}^n)$ for all n. (This merely says that $\int |G^\lambda f(x)|^2 dx/_{|x|} \le C \int |f|^2 dx/_{|x|}$ - for which fact see below). Therefore, by interpolation, $G^\lambda f(x)$ is bounded on $L^{2n/(n-1)}(L^2)(\mathbf{R}^n)$ provided that n is even. ■

* * *

With his First Transference Principle, José Luis solved the almost-everywhere convergence problem for Bochner-Riesz means in even dimensions by means of a judicious use of weighted inequalities for radial weights. But what about odd dimensions? Once one realises that L^p boundedness of the maximal operator is not the right way to proceed, many possibilities arise. In particular, one only needs to show that $T_*^\lambda f(x)$ is finite almost everywhere for all f in L^p - in particular boundedness of T_*^λ from L^p to $L^2(\omega)$ for any positive weight ω would be sufficient. Thus one is led to study L^2 weighted inequalities for particular radial weights - for example for radial *power* weights - as well as for the general radial weights considered above. Luis Vega had already been studying L^2 power weighted inequalities for the Schrödinger operator in his thesis, and many of his ideas came in useful in the Bochner-Riesz context too. He, José Luis and I showed in **[CR de FV]** that G^λ and T_*^λ are indeed finite almost everywhere for $f \in L^p(\mathbf{R}^n)$, $2 \le p \le \dfrac{2n}{n-1}$, $\lambda > 0$, all n, by establishing the following estimate:

Theorem 2. *Let $\lambda > 0$ and $0 \le \alpha < 1 + 2\lambda \le n$. Then*

$$\int_{\mathbf{R}^n} |G^\lambda f(x)|^2 \frac{dx}{|x|^\alpha} \le C_{\alpha,\lambda,n} \int_{\mathbf{R}^n} |f(x)|^2 \frac{dx}{|x|^\alpha}. \tag{5}$$

(The assertion about L^p now follows, because if $2 \le p \le \dfrac{2n}{n-1}$, then $L^p \subseteq L^2 + L^2(dx/_{|x|^\alpha})$ for some $0 \le \alpha < 1$; notice that this theorem also fills in the gap left in the proof of Corollary 1).

José Luis observed that the "essential" part of this result - when $0 \le \alpha < 1$ - can also be obtained in $\mathbf{R}^n$ once it is known in $\mathbf{R}^1$. This led him to his Second Transference Principle.

THE SECOND TRANSFERENCE PRINCIPLE

Suppose $\mathcal{M} : [0,\infty) \to \mathbf{C}$ satisfies $|\mathcal{M}(r)| \le 1$, and that T is defined simultaneously on each $\mathbf{R}^k$ by

$$(Tf)\hat{}(\xi) = \mathcal{M}(|\xi|)\hat{f}(\xi).$$

The Second Transference Principle. *Let T be as above. Suppose that for some $-1 < \alpha < 1$ and for all continuous f of compact support we have*

$$\int_{\mathbf{R}} |Tf(x)|^2 |x|^\alpha dx \le \int_{\mathbf{R}} |f(x)|^2 |x|^\alpha dx.$$

Then for each $n \in \mathbf{N}$ there is a $C = C(\alpha, n)$ such that

$$\int_{\mathbf{R}^n} |Tf(x)|^2 |x|^\alpha dx \le C \int_{\mathbf{R}^n} |f(x)|^2 |x|^\alpha dx.$$

As the proof will show, the theorem is also valid in the Hilbert-space valued setting (with possibly different Hilbert spaces for the domain and range). By duality is suffices to consider the case $\alpha > 0$.

Lemma. *Let $\|\xi\| = \max |\xi_i|$, and with T as above let*

$$(Sf)\check{}(\xi) = \mathcal{M}(\|\xi\|)\hat{f}(\xi).$$

Under the hypothesis of the Second Transference Principle, we have

$$\int_{\mathbf{R}^n} |Sf(x)|^2 |x|^\alpha dx \le C \int_{\mathbf{R}^n} |f(x)|^2 |x|^\alpha dx.$$

Proof: The idea is to split $\mathbf{R}^n$ into cones, on each of which S looks like its one-dimensional version. Let $\Gamma_i = \{\xi \in \mathbf{R}^n \mid \|\xi\| = |\xi_i|\}$. Let $(P_i f)\check{}(\xi) = \mathcal{X}_{\Gamma_i}(\xi)\hat{f}(\xi) = \hat{f}_i(\xi)$. Then $(Sf_i)\check{}(\xi) = \mathcal{M}(|\xi_i|)\hat{f}_i(\xi)$. Since

$$\int |P_i g|^2 |x|^\alpha dx \le C \int |g|^2 |x|^\alpha dx,$$

it suffices to show that

$$\int |Sf_i|^2 |x|^\alpha dx \le C \int |f_i|^2 |x|^\alpha dx.$$

Let $(Rf)\check{}(\xi) = \mathcal{M}(|\xi_1|)\hat{f}(\xi)$. By symmetry in the coordinate variables, we will be done if we can show

$$\int |Rf|^2 |x|^\alpha dx \le C \int |f|^2 |x|^\alpha dx.$$

But, by hypothesis, for each fixed $\bar{x} = (x_2, \ldots, x_n)$, we have

$$\int |Rf(x_1, \bar{x})|^2 dx_1 \le \int |f(x_1, \bar{x})|^2 dx_1$$

and

$$\int |Rf(x_1, \bar{x})|^2 |x_1|^\alpha dx_1 \le \int |f(x_1, \bar{x})|^2 |x_1|^\alpha dx_1.$$

Multiply the first of these inequalities by $|\bar{x}|^\alpha$ and add the result to the second; now integrate with respect to $\bar{x}$. Using the fact that for $\alpha > 0$, $|x_1|^\alpha + |\bar{x}|^\alpha \approx |x|^\alpha$ completes the proof. ∎

Proof of the Second Transference Principle: By Plancherel's Theorem, what we are trying to show is that $\mathcal{M}(|\xi|)$ is a pointwise multiplier for the space $L^2_{\alpha/2}(\mathbf{R}^n) = \{g \mid \int_{\mathbf{R}^n} |\mathcal{D}^{\alpha/2} g(x)|^2 dx < \infty\}$, where

$$\mathcal{D}^\beta g(x) = \left(\int_{\mathbf{R}^n} \frac{|g(x+y) - g(x)|^2}{|y|^{2\beta}} \frac{dy}{|y|^n} \right)^{1/2}.$$

It is quite apparent that $L^2_\beta(\mathbf{R}^n)$ is invariant under bilipschitzian changes of variables provided $\beta < 1$: just think about changing variables in the formula for $\mathcal{D}^\beta$. Hence the space of pointwise multipliers of $L^2_\beta(\mathbf{R}^n)$ is also invariant under a bilipschitzian change of variables, if $\beta < 1$. Finally, since by the lemma $\mathcal{M}(\|\cdot\|)$ is a multiplier of $L^2_{\alpha/2}$, so is $\mathcal{M}(\|\Lambda\xi\|) = \mathcal{M}(|\xi|)$ where Λ is the bilipschitzian function on $\mathbf{R}^n$ defined by $\Lambda\xi = \xi|\xi|/_{\|\xi\|}$. ∎

The same example as before shows that the Second Transference Principle does not extend to general A_1 weights. José Luis noted that the proof of this principle is really just a simple adaptation of old ideas of Hirschman, [Hi]; nevertheless, it was José Luis' own notion that such a general statement would hold.

Let us now see an application of the Second Principle to José Luis' "arbitrary intervals" Littlewood-Paley operator.

Theorem 3. *Let $\{I_j\}$ be a sequence of disjoint intervals in $[0,\infty)$, and let $\Delta_j = \{\xi \in \mathbf{R}^n \mid |\xi| \in I_j\}$. Let $(\Delta_j f)\check{}(\xi) = \mathcal{X}_{\Delta_j}(\xi)\hat{f}(\xi)$. Then for $0 \leq \alpha < 1$, we have*

$$\int_{\mathbf{R}^n} \sum_j |\Delta_j f(x)|^2 \frac{dx}{|x|^\alpha} \leq C_{\alpha,n} \int_{\mathbf{R}^n} |f|^2 \frac{dx}{|x|^\alpha}.$$

(The result concerning the square function G^λ alluded to above is essentially a consequence of this theorem).

Proof: By the Second Transference Principle, it is enough to establish the inequality when $n = 1$. By duality, what we need is equivalent to

$$\int |\sum_j \Delta_j g_j(x)|^2 |x|^\alpha dx \leq C_\alpha \sum_j \int |g_j|^2 |x|^\alpha dx,$$

and by Plancherel's theorem this is equivalent to

$$\|\mathcal{D}^{\alpha/2}\{\sum_j \mathcal{X}_{\Delta_j} h_j\}\|_2^2 \leq C_\alpha \sum_j \|\mathcal{D}^{\alpha/2} h_j\|_2^2,$$

with $\mathcal{D}^\beta$ defined as above. In fact,

$$
\begin{aligned}
(\mathcal{D}^{\alpha/2}\{\sum_j \mathcal{X}_{\Delta_j} h_j\})^2(\xi) &= \int \frac{|\sum_j (\mathcal{X}_{\Delta_j} h_j)(\xi+\eta) - (\mathcal{X}_{\Delta_j} h_j)(\eta)|^2}{|\eta|^\alpha} \frac{d\eta}{|\eta|} \\
&\leq \sum_j (\mathcal{D}^{\alpha/2}(\mathcal{X}_{\Delta_j} h_j))^2(\xi)(\sup_{\xi,\eta} \#\{j | \xi+\eta \in \Delta_j \text{ or } \eta \in \Delta_j\}) \\
&\leq \sum_j (\mathcal{D}^{\alpha/2}(\mathcal{X}_{\Delta_j} h_j))^2(\xi)
\end{aligned}
$$

since the I_j are mutually disjoint. Hence we want to show that

$$\sum_j \|\mathcal{D}^{\alpha/2} \mathcal{X}_{\Delta_j} h_j\|_2^2 \leq C_\alpha \sum_j \|\mathcal{D}^{\alpha/2} h_j\|_2^2.$$

We cancel out the $\sum$ signs, and appeal to the standard result (Hirschman) that if $I \subseteq \mathbf{R}$ is an interval and $(S_I f)\hat{}(\xi) = \mathcal{X}_I(\xi)\hat{f}(\xi)$, then

$$\int_{\mathbf{R}} |S_I f(x)|^2 \frac{dx}{|x|^\alpha} \leq C_\alpha \int_{\mathbf{R}} |f(x)|^2 \frac{dx}{|x|^\alpha} \tag{6}$$

for $0 \leq \alpha < 1$, with C_α independent of I. ■

(The proof given here is different from the one José Luis gave in [**R de F2**]; in fact it is more reminiscent of the ideas contained in [**CR de FV**]).

Any radial multiplier operator whose one-dimensional version is controlled by the arbitrary intervals square function will now automatically be bounded on $L^2(\mathbf{R}^n, |x|^\alpha dx)$ for $-1 < \alpha < 1$. This applies in particular to multipliers such that

$$\sup_{k\in\mathbf{Z}} \sup_{2^k = t_0 < \ldots < t_j = 2^{k+1}} \left(\sum_{i=1}^{j} |m(t_i) - m(t_{i-1})|^2 \right)^{1/2} < \infty$$

- see [**CoR de FS**].

* * *

The reader may notice a certain historical anomaly in the above (chronological) account. In order to prove Corollary 1, José Luis needed Theorem 2 to finish off the argument. What had happened, of course, was that José Luis was thinking in terms of the scalar-valued operator T_1^λ where instead of (5) one would only need (6). "Vectorialising" (6) leads to (5) and to the result for T_*^λ. What had escaped José Luis' notice was that Theorem 2 *already* implied almost everywhere convergence in *all* dimensions! Had the realised this from the outstart we might never have had the beautiful Transference Principles to behold today.

* * *

The influence of José Luis will be with us for a long time to come. His understanding of vector-valued inequalities, weights and Littlewood-Paley theory was profound, his ability to apply this understanding to concrete operators such as the disc multiplier and singular integrals along curves astounding. Even within the relatively narrow field covered by this lecture, he has inspired and continues to inspire much work. We now wish to give a few examples of recent developments:

1) Weighted inequalities for the disc multiplier. Recently, G. Mockenhaupt proved the following theorem:

Theorem [M]. *Let ω be an even non-negative measurable function on* $\mathbf{R}$ *which satisfies*

$$\left|\frac{1-Rt}{1+Rt}\right|^{1/2}\omega(t)\in A_2(\mathbf{R})$$

uniformly in $R \geq 0$. Then

$$\int |T_R^0 f(x)|^2\omega(|x|)dx \leq C_\omega \int |f(x)|^2\omega(|x|)dx.$$

Mockenhaupt also indicates that there are explicit weighted inequalities for the disc multiplier which lead to its boundedness on $L^p(L^2)$, $\frac{2n}{n+1} < p < \frac{2n}{n-1}$, (a result rediscovered in [**Có2**]). A restricted weak type endpoint result and a positive resolution of the aforementioned question of Stein for radial weights have recently been obtained by Carbery, Romera and Soria.

2) Localisation for the disc multipliers. While the conjecture that $T_R^0 f(x) \to f(x)$ a.e. as $R \to \infty$ for $f \in L^p(\mathbf{R}^n)$ $2 \leq p < 2n/_{(n-1)}$ still seems very far away from being solved, nevertheless a localisation principle has been established: if f as above is zero on an open set Ω in $\mathbf{R}^n$, then $T_R^0 f(x) \to 0$ a.e. on Ω. This result, in [**CS**], uses very heavily the ideas of [**CR de FV**] and [**R de F2**].

3) Results for the maximal disc multiplier acting on radial functions. In this connection, we refer only to the work of Prestini [**P**], Kanjin [**K**], Romera and Soria [**RS**], and Crespi [**Cr**].

I would also like to recall two problems, explicity posed by José Luis, which are closely related to the material of this lecture.

Problem 1. Is the First Transference Principle valid for transference through all dimensions, not just even ones? José Luis said: "I believe that transference of a weighted L^2 estimate (with a radial weight) from a given dimension ≥ 2 to any higher dimension must hold; this would be a very satisfactory improvement ..., but the method of proof should be quite different". In particular, a positive solution to this problem would also solve:

Problem 1(a). Establish whether the maximal Bochner Riesz operator is bounded in $L^{2n/(n-1)}(L^2)$ for $\lambda > 0$ in all dimensions.

The second problem José Luis mentioned a number of times. (See [**R de F1**] and [**R de F2**]).

Problem 2. Let $G(f)^2(x) = \sum |\Delta_j f|^2(x)$ be the arbitrary intervals square function on $\mathbf{R}$. Is it true that

$$\int_{\mathbf{R}} G(f)^2(x)\omega(x)dx \leq C \int_{\mathbf{R}} |f(x)|^2\omega(x)dx$$

for all A_1 weights ω? It holds in the equally spaced and lacunary cases - "extreme cases"; the theorem above with weights $|x|^\alpha$, $-1 < \alpha \leq 0$ was regarded by José Luis as fairly strong evidence that it is true for general A_1 weights.

To conclude, I would like to mention one further corollary to the "arbitrary intervals" Theorem 3.

Corollary 4. *If* $0 < R_1 < ... < R_j \to \infty$, *then for all* $f \in L^p(\mathbf{R}^n)$, $2 \leq p < \frac{2n}{n-1}$, *we have*

$$\lim_{j\to\infty} |T^0_{R_j} f(x) - T^0_{R_{j-1}} f(x)| \to 0$$

almost everywhere.

The proof is a triviality: $\left(\sum |T^0_{R_j} f - T^0_{R_{j-1}} f|^2\right)^{1/2} = \left(\sum |\Delta_j f|^2\right)^{1/2}$ and $L^p \subseteq L^2 + L^2(dx/_{|x|^\alpha})$, $(0 \leq \alpha < 1)$. Nevertheless, this result, for me, captures the spirit of José Luis, his ability to use his remarkably powerful imagination with great elegance to bring deceptively simple abstract ideas to bear on genuine problems in concrete analysis. His sense of direction was unerring: all the tools and techniques which he helped to develop, and in some cases pioneer - vector-valued inequalities, weights, mixed norms, the Boundedness Principle, Littlewood-Paley theory - play some role here. With a certain sad and ironic symmetry, this last result about almost everywhere convergence for Fourier integrals brought him back to the very same problems he began his mathematical career with such a short time ago.

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