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SUPERSOLUTIONS AND STABILIZATION OF THE SOLUTIONS OF THE EQUATION $\frac{\partial u}{\partial t} - \mathrm{div}(|\nabla u|^{p-2}\,\nabla u) = h(x,u)$, PART II

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Abstract

In this paper we consider a nonlinear parabolic equation of the following type:

$$(\mathcal{P}) \qquad \frac{\partial u}{\partial t} - \mathrm{div}(|\nabla u|^{p-2}\,\nabla u) = h(x,u)$$

with Dirichlet boundary conditions and initial data in the case when $1 < p < 2$.

We construct supersolutions of $(\mathcal{P})$, and by use of them, we prove that, for $t_n \to +\infty$, the solution of $(\mathcal{P})$ converges to some solution of the elliptic equation associated with $(\mathcal{P})$.

0. Introduction

This is the second part of a work concerning the existence and asymptotic behaviour of bounded non negative solutions of the following problem:

$$(0.1) \qquad \mathcal{P}(\Omega)\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u - h(x,u) = 0 \text{ in } \Omega \times \mathbb{R}_+ \\ u(x,t) = 0 \text{ in } \partial\Omega \times \mathbb{R}_+ \\ u(x,0) = u_0(x) \text{ in } \Omega \end{cases}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\,\nabla u)$, $1 < p < +\infty$ and Ω is a regular open subset of $\mathbb{R}^N$, $N \geq 1$.

These problems arise from nonnewtonian fluid mechanics for $1 < p < +\infty$ ([**3**]), and from glaciology for $p = \frac{N+1}{N}$ ([**2**]). In the first part ([**5**]), we were concerned with the case $p > 2$ and have proved that if $\mathcal{P}(\Omega)$ admits a uniform supersolution with spatially bounded support which is independent on T, then the orbits are compacts and any w in the ω-limit set:

$$\omega(u_0) = \left\{ w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \,\middle|\, \exists t_n \longrightarrow +\infty : u(.,t_n) \longrightarrow w \text{ in } W_0^{1,p}(\Omega) \right\}$$

is a solution of the elliptic problem associated with $\mathcal{P}(\Omega)$. Here, we study the case $1 < p < 2$ and obtain similar results in the case when

$$\frac{2N}{N+2} < p < 2.$$

In addition, we give in this paper the justification of the formal derivation of the regularized equation associated with (0.1), by means of finite dimensional problems. That was not done in [**5**]. We also show the following regularizing effect:

$$|\nabla u|^{\frac{p-2}{2}} \frac{\partial}{\partial t} \nabla u \in L^2\left(t_0, +\infty; L^2(\Omega)\right).$$

Existence and regularity results can be found in Tsutsumi [**18**], Nakao [**13**], Diaz and Herrero [**3**]. Stabilization results are obtained by Otani [**15**] for the one dimensional case and by Langlais and Phillips [**8**] for a problem closely related to $\mathcal{P}(\Omega)$ and including the case $p = 2$. But they do not prove the compacity of the orbits. All our results for $p > 2$ were however extended to the case of a system by Elouardi and de Thelin [**6**] when Ω is bounded.

As in [**5**] our technique is based upon a comparaison principle and the construction of supersolutions. Some proofs already made in [**5**] are omited here. So we refer the reader [**5**] for completeness.

In the first section of this paper we give some preliminaries and state the main results. The proofs are given in section 2 and section 3 is devoted to the justification of the formal derivation.

1. Preliminaries and main results

1.1. Preliminaries. In all this paper Ω stands for a regular open subset of $\mathbb{R}^N$ and may be unbounded. Let h be an application from $\mathbb{R}^{N+1}$ to $\mathbb{R}$ such that:

$$h \in C(\overline{\Omega} \times \mathbb{R}) \text{ and } h(x,0) \geq 0 \text{ for any } x \in \Omega \tag{1.1}$$

and, for any $M > 0$, there exists $K_M > 0$ such that:

$$h(x,u) - h(x,v) \leq K_M(u-v) \quad \forall x \in \Omega,\ \forall u,v : 0 \leq v \leq u \leq M. \tag{1.2}$$

Note that (1.2) is satisfied if for some $\lambda > 0$, $h - \lambda I$ is nonincreasing.

First we recall some notations and definitions used in [**5**]:

For $T > 0$,

$$Q_T = \Omega \times [0,T],\ S_T = \partial\Omega \times [0,T]$$

and for $R > 0$

$$\Omega_R = \overline{\Omega} \cap B(0,R).$$

$$F(\nabla u) = |\nabla u|^{p-2}\nabla u \text{ with } : 1 < p < 2$$
$$\Delta_p u = \operatorname{div}(F(\nabla u)).$$

Let u be given in $L^\infty(0,T;W^{1,p}(\Omega))$; we say that $u \geq 0$ in S_T [resp. $u = 0$ in S_T] iff

$$(-u)_+ \,[\text{resp. } u] \in L^\infty(0,T;W_0^{1,p} \cap L^\infty(\Omega)).$$

Let u_0 be given such that:

$$u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \tag{1.3}$$

we say that u is a solution of $\mathcal{P}(\Omega)$ in Q_T [resp. $\hat{u}$ is a supersolution of $\mathcal{P}(\Omega)$ in Q_T] iff:

$$u\,[\text{resp. } \hat{u}] \in L^\infty(0,T;W^{1,p}(\Omega) \cap L^\infty(\Omega)) \tag{1.4}$$

$$\frac{\partial u}{\partial t}\left[\text{resp. } \frac{\partial \hat{u}}{\partial t}\right] \in L^2(Q_T) \tag{1.5}$$

$$Au \equiv \frac{\partial u}{\partial t} - \Delta_p u - h(x,u) = 0 \text{ in } Q_T\,[\text{resp. } A\hat{u} \geq 0 \text{ in } Q_T] \tag{1.6}$$

(in the distribution sense)

$$u = 0\,[\text{resp. } \hat{u} \geq 0] \text{ in } S_T \tag{1.7}$$

$$u(.,0) = u_0\,[\text{resp. } \hat{u}(.,0) \geq u_0] \text{ in } \Omega. \tag{1.8}$$

We say that $u \in L^\infty(Q_T)$ [resp. $\hat{u} \in L^\infty(\Omega)$] has a spatially bounded support in Q_T [resp. has a bounded support in Ω] iff there exists $R > 0$ such that:

$$\operatorname{Supp} u \subset \overline{\Omega}_R \times [0,T]\,(\text{resp. } \operatorname{Supp} u \subset \overline{\Omega}_R).$$

Let $\hat{u}$ be a supersolution of $\mathcal{P}(\Omega)$ in Q_T for any $T > 0$. We say that $\hat{u}$ is a uniform supersolution with spatially bounded support iff there exists $R_2 > 0$ and $M > 0$ both independent on T such that:

$$\operatorname{Supp} \hat{u} \subset \overline{\Omega}_{R_2} \times \mathbb{R}_+$$
$$\|\hat{u}\|_{L^\infty(Q_T)} = M(T) \leq M.$$

Supersolutions are very useful in our problem owing to the following comparison principle.

Theorem 0. [5] *Suppose that f satisfies (1.1) and (1.2), that u_0 and $\hat{u}_0$ satisfy (1.3) and that u and $\hat{u}$ satisfy (1.4) and (1.5). If*

$$\begin{aligned} u(.,0) =& u_0 \leq \hat{u}_0 = \hat{u}(.,0) \text{ in } \Omega \\ & u \leq \hat{u} \text{ in } S_T \\ & Au \leq A\hat{u} \text{ in } \Omega \text{ (in the distribution sense).} \end{aligned}$$

Then $u \leq \hat{u}$ in Q_T.

1.2. Main results. First we give some sufficient conditions for existence of supersolutions of $\mathcal{P}(\Omega)$.

Theorem 1. *Let $1 < p < 2$ be given.*

If $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ has a bounded support and if there exists $\lambda > 0$, $\mu \geq 0$, $\sigma > 0$, $R_0 > 0$ and $\gamma_0, \gamma \in]0, p-1[$ such that:

(i) For any $x \in \overline{\Omega}_{R_0}$ and any $u \in \mathbb{R}_+ : f(x,u) \leq \mu + \lambda u^{\gamma_0}$.

(ii) For any $x \in \Omega$, $|x| > R_0$ and for any $u \in \mathbb{R}_+$, $f(x,u) \leq -\sigma u^\gamma$.

Then $\mathcal{P}(\Omega)$ has a nonnegative uniform supersolution with spatially bounded support.

Theorem 2. (Existence) *Let $1 < p < 2, T > 0$ and $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $u_0 \geq 0$ be given. Suppose that h satisfies (1.1) and (1.2) and that $\mathcal{P}(\Omega)$ admits a nonnegative supersolution $\hat{u}$ with spatially bounded support in Q_T. Then $\mathcal{P}(\Omega)$ has a unique solution u in Q_T satisfying:*

$$0 \leq u \leq \hat{u} \text{ in } Q_T.$$

Remark 1. Theorem 2 extends some of Nakao's results [**12**] when Ω is unbounded and generalizes Diaz-Herrero's results [**3**] in the case when h may be nonmonotone.

When $h(x,u) = |u|^{\gamma-1}u$, by use of Theorem 1, we can find again Tsutsumi's results [**18**].

Corollary. (Semi-group property) *If the hypothesis of Theorem 2 are satisfied, $\mathcal{P}(\Omega)$ generates a continuous semi-group on $L^2(\Omega)$.*

Theorem 3. (Regularizing effects) *Let $1 < p < 2$, $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be given.*

Suppose that h satisfies (1.1) and (1.2) and that $\mathcal{P}(\Omega)$ has a nonnegative uniform supersolution $\hat{u}$ with spatially bounded support. Then for any $t_0 \in]0,1[$, the solution u of $\mathcal{P}(\Omega)$ satisfies the following regularity estimates:

$$\frac{\partial u}{\partial t} \in L^2\left(t_0, +\infty; L^2(\Omega)\right) \cap L^\infty\left(t_0, +\infty; L^2(\Omega)\right) \tag{1.9}$$

$$|\nabla u|^{\frac{p-2}{2}} \frac{\partial}{\partial t} \nabla u \in L^2\left(t_0, +\infty; L^2(\Omega)\right) \tag{1.10}$$

and for any p such that

$$\frac{2N}{N+2} < p < 2 \tag{1.11}$$

there exists some $\sigma : 0 < \sigma < 1$ such that

$$u \in L^\infty\left(t_0, +\infty; B_\infty^{1+\sigma,p}(\Omega)\right) \tag{1.12}$$

where $B_\infty^{1+\sigma,p}(\Omega)$ is a Besov space [**16**] *defined by the real interpolation method.*

Let u be the solution of $\mathcal{P}(\Omega)$, we define the ω-limit set by:

$$\omega(u_0) = \left\{ w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) | \exists t_n \to \infty : u(., t_n) \to w \text{ in } W_0^{1,p}(\Omega) \right\}.$$

Let $\mathcal{E}$ be the set of nonnegative solutions w of the elliptic problem:

$$\begin{cases} -\Delta_p w = h(x, w) \text{ in } \Omega \\ w = 0 \text{ in } \partial\Omega. \end{cases}$$

Our main result is the following:

Theorem 4. (Stabilization) *Let $\frac{2N}{N+2} < p < 2$, $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $u_0 \geq 0$ be given. Suppose that h satisfies (1.1) and (1.2) and that $\mathcal{P}(\Omega)$ has a nonnegative uniform supersolutions $\hat{u}$ with spatially bounded support. Then $\omega(u_0) \neq \phi$ and $\omega(u_0) \subset \mathcal{E}$.*

Remark 2. In some cases [4], [**10**], [**11**] $\mathcal{E}$ contains at least one nontrivial element w; if in addition we can construct some subsolution $\underline{u} \not\equiv 0$, $\underline{u} \geq 0$ of $\mathcal{P}(\Omega)$ (see [**5**, corollary of Theorem 4 for sufficient conditions]), then $\omega(u_0) = \{w\}$ and $\lim u(., t) = w$.

1.3. Examples. Theorems 2, 3 and 4 apply to the following examples:

1) Ω is a nonnecessarily bounded set and

$$h(x, u) = g(x)(1 + u^2)^{\frac{\gamma}{2}}$$

where $0 < \gamma < p - 1$ and $g \in \mathcal{C}(\overline{\Omega})$ satisfies:

$$g(x) \leq -\sigma < 0 \text{ for any } x \in \Omega, \ |x| > R_0 > 0.$$

(Apply Theorem 1).

2) Ω is a bounded set and

$$h(x,u) = g(x)\,|u|^{\gamma-1}u$$

where $\gamma \geq 1$, $(\gamma+1)(N-p) < Np$, $g \in C(\overline{\Omega})$, $\|g\|_{L^\infty} = \sigma$ and $u_0 \leq w$, $w \in W_0^{1,p}(\Omega)$ being a nontrivial solution of the equation [17]

$$-\Delta_p w = \sigma|w|^{\gamma-1}w \text{ in } \Omega.$$

3) Ω is a bounded set and $h \in C(\overline{\Omega} \times \mathbb{R})$ is any function such that $h(x,0) = 0$, $u \to h(x,u)$ is a non increasing function and $h(x,u) \leq 0$ for $u \geq M > 0$.

4) Ω is a bounded set, $0 \leq u_0 \leq 1$, $a \in C(\overline{\Omega})$ satisfies $0 \leq a(x) \leq 1$ and

$$h(x,u) = u(1-u)(u-a(x)).$$

2. Proofs of the main results

2.1. Sketch of the proof of Theorem 1: Let $M_0 = \|u_0\|_{L^\infty(\Omega)}$, let R_0' be such that supp $u_0 \subset \overline{\Omega}_{R_0'}$ and $R = \max(R_0, R_0')$. Define $\hat{u}$ by $\hat{u}(x,t) = \varphi(r)$ where $r = |x|$ and:

$$\varphi(r) = \begin{cases} ar^{p^*} + b \text{ for } 0 \leq r \leq R \\ \alpha r + \beta \text{ for } R < r \leq R_1 \\ K(R_2 - r)^m \text{ for } R_1 < r \leq R_2 \\ 0 \text{ for } r > R_2 \end{cases}$$

with $m = \frac{p}{p-1-\gamma} > 1$.

As in [5] straithforward considerations enable us to choose the constants a, b, α, β, K, R_1, R_2 so that $\hat{u}$ be a uniform supersolutions of $P(\Omega)$ in Q_T is for any $T > 0$. Whence Theorem 1 is proved. ■

2.2. Proof of Theorem 2: Let $T > 0$ be given and consider $R > 0$ such that

$$\text{Supp } \hat{u} \subset \overline{\Omega}_R \times [0,T].$$

Let ω and ω' be bounded regular open sets such that:

$$\Omega \cap \overline{\Omega}_R \subset \omega' \subset \Omega \cap \overline{\omega}' \subset \omega \subset \Omega.$$

Note $q_T = \omega \times [0,T]$ and $S_T = \partial\omega \times [0,T]$.

It is well known (see for instance [8]) that there exists a sequence $h_\varepsilon \in C^1(\overline{\Omega} \times \mathbb{R}_+)$ such that:

$$\begin{cases} h_\varepsilon \searrow h \text{ uniformly as } \varepsilon \to 0, \text{ and for any } \varepsilon > 0 \\ \frac{\partial h_\varepsilon}{\partial u}(x,u) \leq K_{M(T)},\ h_\varepsilon(x,0) \geq 0 \\ h_\varepsilon(x,u) = 0 \text{ if } u \geq 3m(T) \end{cases}$$

on the other hand, let $(u_{0^\varepsilon}) \subset \mathcal{D}(\omega)$, $0 \le u_{0^\varepsilon} \le M(T)$, be such that $u_{0^\varepsilon} \to u_0$ in $W_0^{1,2}(\omega)$.

From [7, pp. 457–459], for each $\varepsilon > 0$, there is a unique classical solution $u_\varepsilon \in \mathcal{C}(\overline{q}_T) \cap \mathcal{C}^{2,1}(q_T)$ of:

$$P_\varepsilon(\omega)\begin{cases} A_\varepsilon u_\varepsilon \equiv \frac{\partial u_\varepsilon}{\partial t} - \Delta_p^\varepsilon u_\varepsilon - h_\varepsilon(x, u_\varepsilon) = 0 \text{ in } Q_T & (2.1)\\ u_\varepsilon(x,t) = 0 \text{ in } s_T & (2.2)\\ u_\varepsilon(x,0) = u_{0^\varepsilon}(x) \text{ in } \omega & (2.3)\end{cases}$$

where $\Delta_p^\varepsilon u_\varepsilon = \operatorname{div} F_\varepsilon(\nabla u_\varepsilon)$, $F_\varepsilon(\nabla u_\varepsilon) = (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon$.

Remark 3. Hereafter $C(M(T))$ stands for any constant which depends only on $M(T)$. In the case when $\mathcal{P}(\Omega)$ has a nonnegative uniform supersolution, $C(M(T))$ does not depend on T.

We have the following:

Lemma 1. *There exists $C(M(T))$ such that for any $\varepsilon \in]0,1[$.*

$$\|u_\varepsilon\|_{L^\infty(q_T)} \le C(M(T)) \tag{2.4}$$

$$\|u_\varepsilon\|_{L^\infty(0,T;W_0^{1,p}(\omega))} \le C(M(T)) \tag{2.5}$$

$$\left\|\frac{\partial u_\varepsilon}{\partial t}\right\|_{L^2(q_T)} \le C(M(T)). \tag{2.6}$$

Proof: 0 and $3M(T)$ are respectively subsolutions and supersolutions of $\mathcal{P}_\varepsilon(\omega)$; hence by Theorem 0, we have:

$$0 \le u_\varepsilon \le 3M(T) \text{ in } q_T \text{ whence (2.4).}$$

By the properties of h_ε we have that $h_\varepsilon(.,u_\varepsilon)$ is bounded in q_T. This implies that H_ε defined by $H_\varepsilon(x,u) = \displaystyle\int_0^u h_\varepsilon(x,v)\,dv$ satisfies

$$|H_\varepsilon(.,u_\varepsilon)| \le C(M(T))$$

whence:

$$\int_{q_\tau} h_\varepsilon(x,u_\varepsilon)\frac{\partial u_\varepsilon}{\partial t}dx = \int_\omega [H_\varepsilon(.,u_\varepsilon(.,\tau)) - H_\varepsilon(.,0))]dx \le C(M(T)),$$
$$\forall \tau : 0 < \tau < T.$$

Multiplying (2.1) by $\frac{\partial u_\varepsilon}{\partial t}$ and integrating on q_τ we get:

$$\int_{q_\tau}\left(\frac{\partial u_\varepsilon}{\partial t}\right)^2 dx\,dt + \frac{1}{p}\int_\omega \left(|\nabla u_\varepsilon(.,\tau)|^2 + \varepsilon\right)^{p/2} dx$$
$$\le \frac{1}{p}\int_\omega \left(|\nabla u_\varepsilon(.,0)|^2 + \varepsilon\right)^{p/2} dx + C(M(T)).$$

By Hölder inequality, $u_\varepsilon(.,0)$ converging to $u(.,0)$ we get

$$\int_\omega \left(|\nabla u_\varepsilon(.,0)|^2 + \varepsilon\right)^{p/2} dx \le C(M(T))$$

whence (2.5) and (2.6) hold. ■

Lemma 2. *$\mathcal{P}(\omega)$ has a unique solution u satisfying:*

$$0 \leq u \leq \hat{u} \text{ in } q_T.$$

Moreover u_ε converges strongly to u in $L^p(0,T;W^{1,p}(\omega))$.

Proof: By (2.4), (2.5), (2.6), there is a subsequence denoted again by u_ε which converges to u in weak $*L^\infty(0,T;W_0^{1,p}(\omega) \cap L^\infty(\omega))$ and in weak $L^p(0,T;W_0^{1,p}(\omega))$ such that $\frac{\partial u_\varepsilon}{\partial t}$ converges to $\frac{\partial u}{\partial t}$ in weak $L^2(q_T)$ and $\Delta_p^\varepsilon u_\varepsilon$ converges to χ in $L^{p^*}(0,T;W^{-1,p^*}(\omega))$.

Moreover, multiplying (2.1) by u_ε, we have:

$$\begin{aligned}(2.7)\quad E_\varepsilon \equiv \int_{q_T} \left(|\nabla u_\varepsilon|^2+\varepsilon\right)^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2 dx\, dt \\ = \int_{q_T} u_\varepsilon h_\varepsilon(.,u_\varepsilon) dx\, dt + \frac{1}{2}\int_\omega u_\varepsilon^2(.,0)dx - \frac{1}{2}\int_\omega u_\varepsilon^2(.,T)dx\end{aligned}$$

$u_\varepsilon h_\varepsilon$ being bounded, the same argument that [**9**, p. 160] shows that $u_\varepsilon(.,T)$ converges to $u(.,T)$ in weak $L^2(\omega)$ and therefore:

$$(2.8)\qquad \limsup_{\varepsilon\to 0} -\int_\omega u_\varepsilon^2(.,T) \leq -\int_\omega u^2(.,T).$$

Moreover, by lemma 1, u_ε is bounded in the space:

$$W = \left\{ v \in L^p(0,T;W_0^{1,p}(\omega));\, \frac{\partial v}{\partial t} \in L^p(q_T)\right\}$$

and by [**9**, p. 58], u_ε converges to u in strong $L^p(q_T)$. By (2.7), (2.8) and the use of the dominated convergence theorem we obtain:

$$\limsup_{\varepsilon\to 0} E_\varepsilon \leq \int_{q_T} uh(.,u)dx\,dt + \frac{1}{2}\int_\omega u_0^2 dx - \frac{1}{2}\int_\omega u^2(.,T)dx = \int_0^T \langle -\chi, u\rangle.$$

By standard monotonicity argument [**9**, p. 160], $\chi = \Delta_p u$; so u is a solution of $\mathcal{P}(\omega)$ satisfying $0 \leq u \leq \hat{u}$ and we have:

$$(2.9)\qquad \limsup_{\varepsilon\to 0} E_\varepsilon \leq \int_{q_T} |\nabla u|^p dx\, dt.$$

Now, for any $m > 0$, we define:

$$q_{T,m} = \left\{(x,t) \in q_T : |\nabla u_\varepsilon(x,t)|^2 \geq \frac{\varepsilon}{m}\right\}$$

we get:

$$\int_{q_{T,m}} |\nabla u_\varepsilon|^p dx\, dt \leq (1+m)^{\frac{2-p}{2}} \int_{q_{T,m}} \left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2 dx\, dt$$

whence:

$$\int_{q_T} |\nabla u_\varepsilon|^p dx\, dt \leq \left(\frac{\varepsilon}{m}\right)^p \text{ meas } (q_T) + (1+m)^{\frac{2-p}{2}} E_\varepsilon$$

with (2.9), we therefore obtain for any $m > 0$:

$$\limsup_{\varepsilon \to 0} \int_{q_T} |\nabla u_\varepsilon|^p dx\, dt \leq (1+m)^{\frac{2-p}{2}} \int_{q_T} |\nabla u|^p dx\, dt$$

whence

$$\limsup_{\varepsilon \to 0} \|\nabla u_\varepsilon\|_x^p \leq \|\nabla u\|_x^p \tag{2.10}$$

where $X = L^p(0,T; W_0^{1,p}(\omega))$ is an uniformly convex space. So (2.10) and weak convergence of u_ε to u imply strong convergence of u_ε to u in X. ■

End of proof of Theorem 2: The supersolution $\hat{u}$ vanishes in $(\omega \backslash \overline{\Omega}_R) \times [0,T]$ and, by lemma 2, u has the same property; so we can extend u by 0 out of ω and we get a unique solution of $\mathcal{P}(\Omega)$ notes also by u and satisfying:

$$0 \leq u \leq \hat{u} \text{ in } q_T. \quad ■$$

Proof of Theorem 3: Straightforward calculations give:

$$\frac{\partial}{\partial t} F_\varepsilon(\nabla u_\varepsilon) = \left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p-2}{2}} \frac{\partial}{\partial t} \nabla u_\varepsilon + (p-2)\left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p-4}{2}} \left(\nabla u_\varepsilon \cdot \frac{\partial}{\partial t} \nabla u_\varepsilon\right) \nabla u_\varepsilon$$

whence:

$$\frac{\partial}{\partial t} F_\varepsilon(\nabla u_\varepsilon) \cdot \frac{\partial}{\partial t} \nabla u_\varepsilon = \left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p-2}{2}} \left|\frac{\partial}{\partial t} \nabla u_\varepsilon\right|^2 + (p-2)\left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p-4}{2}} \left(\nabla u_\varepsilon \cdot \frac{\partial}{\partial t} \nabla u_\varepsilon\right)^2 \geq (p-1)\left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p-2}{2}} \left|\frac{\partial}{\partial t} \nabla u_\varepsilon\right|^2. \tag{2.11}$$

On the other hand, by formal derivation of (2.1) we get:

$$\frac{\partial^2 u_\varepsilon}{\partial t^2} - \text{div} \frac{\partial}{\partial t} F_\varepsilon(\nabla u_\varepsilon) = \frac{\partial}{\partial t} h_\varepsilon(x, u_\varepsilon). \tag{2.12}$$

Multiplying (2.12) by $\frac{\partial u_\varepsilon}{\partial t}$ and integrating, we get with (2.11):

$$\text{(2.13)} \quad \frac{1}{2}\frac{\partial}{\partial t}\left\|\frac{\partial u_\varepsilon}{\partial t}(.,t)\right\|^2_{L^2(\omega)} + (p-1)\int_\omega \left(|\nabla u_\varepsilon|^2+\varepsilon\right)^{\frac{p-2}{2}}\left|\frac{\partial}{\partial t}\nabla u_\varepsilon\right|^2 dx \leq K\int_\omega \left(\frac{\partial u_\varepsilon}{\partial t}\right)^2 dx$$

Furthermore by (2.6), there exists $t_\epsilon \in]0,t_0[$ such that:

$$\left\|\frac{\partial u_\varepsilon}{\partial t}(.,t_\varepsilon)\right\|^2_{L^2(\omega)} = \frac{1}{t_0}\int_0^{t_0}\left\|\frac{\partial u_\varepsilon}{\partial t}(.,t)\right\|^2_{L^2(\omega)} dt \leq C \leq +\infty.$$

Integrating (2.13) on $[t_\varepsilon, T]$ we get with (2.6) and remark 3:

$$\text{(2.14)} \quad \frac{1}{2}\left\|\frac{\partial u_\varepsilon}{\partial t}(.,T)\right\|^2_{L^2(\omega)} + (p-1)\int_{t_0}^T\int_\omega \left(|\nabla u_\varepsilon|^2+\varepsilon\right)^{\frac{p-2}{2}}\left|\frac{\partial}{\partial t}\nabla u_\varepsilon\right|^2 dx\,dt \leq K\int_{t_\varepsilon}^T\int_\omega \left(\frac{\partial u_\varepsilon}{\partial t}\right)^2 dx\,dt + \frac{1}{2}\left\|\frac{\partial u_\varepsilon}{\partial t}(.,t_\varepsilon)\right\|^2_{L^2(\omega)} \leq C < +\infty.$$

From lemma 2 we deduce that

$$\text{(2.15)} \quad \nabla u_\varepsilon \longrightarrow \nabla u \text{ a.e. on } q_T.$$

By (2.14) we obtain for any $T > 0$

$$\text{(2.16)} \quad \left\|\frac{\partial u_\varepsilon}{\partial t}(.,T)\right\|_{L^2(\omega)} \leq C < +\infty$$

and

$$\text{(2.17)} \quad \left\||\nabla u_\varepsilon|^{\frac{p-2}{2}}\frac{\partial}{\partial t}\nabla u_\varepsilon\right\|_{L^2([t_0,T]\times\omega)} \leq C < +\infty$$

From (2.16) we get:

$$\left\|\frac{\partial u}{\partial t}(.,T)\right\|_{L^2(\omega)} \leq C < +\infty \text{ for any } T \geq t_0.$$

Thus by (2.6) and remark 3 we get:

$$\text{(2.18)} \quad \frac{\partial u}{\partial t} \in L^\infty(t_0,+\infty;L^2(\omega)) \cap L^2(t_0,+\infty;L^2(\omega)).$$

Furthermore by (2.15) and (2.17) we have:

$$\left\| |\nabla u|^{\frac{p-2}{2}} \frac{\partial}{\partial t} \nabla u \right\|_{L^2([t_0,T]\times\omega)} \leq C < +\infty \text{ for any } T > 0.$$

Therefore

$$|\nabla u|^{\frac{p-2}{2}} \frac{\partial}{\partial t} \nabla u \in L^2(t_0, +\infty; L^2(\omega)). \tag{2.19}$$

Thus (1.9) and (1.10) hold respectively by (2.18) and (2.19), because u vanishes on $(\Omega\backslash\overline{\omega}) \times \mathbb{R}_+$.

On the other hand, by (1.11) there is some σ', $0 < \sigma' < 1$, such that

$$L^2(\Omega) \hookrightarrow W^{-\sigma',p^*}(\Omega).$$

Simon's regularity results [17] concerning the equation:

$$-\Delta_p u = h(x,u) - \frac{\partial u}{\partial t} \in L^\infty(t_0, +\infty; B_\infty^{-\sigma',p^*}(\Omega))$$

then give for any t:

$$\|u(.,t)\|_{B_\infty^{1+(1-\sigma')(1-p)^2,p}(\Omega)} \leq C \left\| h(.,u) - \frac{\partial u}{\partial t}(.,t) \right\|_{B_\infty^{-\sigma',p^*}} + C'$$

where C and C' do not depend on t; whence (1.12) holds. ■

Remark 4. The compactness of the embedding

$$B_\infty^{1+(1-\sigma')(1-p)^2,p}(\Omega) \subset W^{1,p}(\Omega)$$

ensures the compactness of the orbit

$$\omega(u_0) = \{w \in w_0^{1,p}(\Omega) \cap L^\infty(\Omega)/\exists t_n \to \infty : u(.,t_n) \to w \text{ in } W_0^{1,p}(\Omega)\}.$$

Proof of Theorem 4 and its corollary:

a) $\omega(u_0) \neq \phi$ because $\operatorname{supp} u \subset \omega \times \mathbb{R}_+$ and $B_\infty^{1+\sigma,p}(\omega)$ is compactly imbedded in $W^{1,p}(\omega)$.

b) Let $\omega = \lim_{n+\infty} u(.,t_n) \in \omega(u_0)$, we get $w \in \mathcal{E}$.

The proof of this, as well as the proof of corollary is the same as in [5] and is omited. ■

3. Justification of the formal proof in section 2

Let (w_j) be a basis of $W_0^{1,p}(\Omega)$ consisting of $C_0^\infty(\Omega)$-functions. For $\varepsilon > 0$ given, we seek a sequence of functions u_m such that

$$u_m = \sum_{j=1}^{m} g_{jm}(t) w_j \text{ and } u_m \longrightarrow u_\varepsilon \text{ in } W_0^{1,p}(\Omega).$$

The $g_{jm}(t)$ being solutions of the following system of ordinary differential equations:

$$(3.1) \qquad (S)\begin{cases} (u'_m(t), w_j) + a_\varepsilon(u_m(t), w_j) = (h_\varepsilon(., u_m(t)), w_j),\ 1 \leq j \leq m \\ u_m(0) = u_{0m}. \end{cases}$$

where: $(.,.)$ is the canonical inner product in $L^2(\Omega)$

$$u_{0m} = \sum_{j=1}^{m} \alpha_{jm} w_j \longrightarrow u_{0\varepsilon} \text{ in } W_0^{1,p}(\Omega)$$

and $a_\varepsilon(u, v) = \displaystyle\int_\Omega F_\varepsilon(\nabla u).\nabla v\, dx$ for any $u, v \in W_0^{1,p}(\Omega)$.

We shall use the following notations:

$*$ For $q \in \mathbb{N}$, $\xi = (\xi_1, \dots, \xi_q)$ and $\eta = (\eta_1, \dots, \eta_q)$ in $\mathbb{R}^q$

$$\xi.\eta = \sum_{j=1}^{q} \xi_j\, \eta_j$$

$$\text{and } |\xi| = \left(\sum_{j=1}^{q} |\xi_j|^2\right)^{1/2}.$$

$*$ For any matrix $\mathcal{U} = (a_{ij})$ in $\mathcal{M}(m, N)$

$$\|\mathcal{U}\| = \left(\sum_{i=1}^{m}\sum_{j=1}^{N} (a_{ij})^2\right)^{1/2}.$$

$*$ $G_m(t) = (g_{1m}(t), \dots, g_{mm}(t))$ for any $t \in [0, T]$.

$*$ For any $x \in \Omega$:

$w(x) = (w_1(x), \dots, w_m(x))$ and $W(x)$ is the matrix $: W(x) = \left(\dfrac{\partial w_j}{\partial x_i}\right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq N}}$

* B is the Gram matrix of the system $(w_1, \dots, w_m)$.
* For any $\xi \in \mathbb{R}^m$

$$\begin{aligned}
&\varphi_j(\xi) = \int_\Omega F_\varepsilon(W(x)\xi).\nabla w_j(x)\,dx, \qquad 1 \le j \le m \\
&\text{and } \varphi = (\varphi_1, \dots, \varphi_m), \\
&\Psi_j(\xi) = \int_\Omega h_\varepsilon(x, w(x).\xi) w_j(x)\,dx \\
&\text{and } \Psi = (\Psi_1, \dots, \Psi_m).
\end{aligned}$$

With these relations we have

$$\begin{aligned}
u_m(x,t) &= w(x).G_m(t) \text{ and} \\
\nabla u_m(x,t) &= W(x).G_m(t).
\end{aligned}$$

Now, we go back to (S). Since B is inversible, we can write (S) in the form:

$$(S')\begin{cases} \frac{dG_m}{dt} = \phi(G_m(t)) \\ G_m(0) = \alpha_m \end{cases}$$

where $\phi(\xi) = B^{-1}[\Psi(\xi) - \varphi(\xi)]$ for any $\xi \in \mathbb{R}^n$ and $\alpha_m = (\alpha_{1m}, \dots, \alpha_{mm})$.

We shall prove that (S') admits a unique solution G_m in $C^2(0, T; \mathbb{R}^m)$. We begin by the following:

Lemma 3. *Suppose that the hypothesis (1.1), (1.2) and (1.3) are satisfied and that Ω bounded.*

Then (S') admits a unique solution on $]0, T[$.

Proof: Let $F_j(x,\xi) = (|W(x)\xi|^2 + \varepsilon)^{\frac{p-2}{2}} W(x)\xi.\nabla w_j(x)$ and $\hat{h}_\varepsilon(x,\xi) = h_\varepsilon(x, w(x).\xi)$ for any $x \in \Omega$ and $\xi \in \mathbb{R}^m$.

F_j and $\hat{h}_\varepsilon$ are locally lipschitz with respect to ξ. Thus ϕ satisfies the same property. This ensures the existence of G_m on an interval $]0, t_m[$. The estimates that follows enable us to have in fact $t_m = T$. Multiply (3.1) by $g_{jm}(t)$; after adding from $j = 1$ to $j = m$, we get:

$$(3.3) \quad \frac{1}{2}\frac{d}{dt}\left(\|u_m(t)\|^2_{L^2(\Omega)}\right) + \int_\Omega \left(|\nabla u_m|^2 + \varepsilon\right)^{\frac{p-2}{2}} |\nabla u_m(t)|^2\,dx = \int_\Omega h_\varepsilon(., u_m).u_m\,dx.$$

Since $h_\varepsilon(x,u) \le K u_\varepsilon + C_0$, where $C_0 = \sup_{x\in\Omega} h_\varepsilon(x, 0)$, the left hand side of (3.3) is bounded by

$$\left(K + \frac{1}{2}\right) \|u_m(t)\|_{L^2(\Omega)} + \frac{1}{2} C_0 \text{ meas } (\Omega).$$

Whence, by Gronwall's lemma, we get:

$$\|u_m\|_{L^\infty(0,T;L^2(\Omega))} \leq C(M(T)). \tag{3.4}$$

On the other hand, multiplying (3.1) by g'_{jm} and adding from $j = 1$ to $j = m$, we get:

$$\int_0^T \|u'_m(t)\|^2_{L^2(\Omega)}\,dt + \frac{1}{p}\int_\Omega \left(|\nabla u_m(T)|^2 + \varepsilon\right)^{p/2} dx \leq \tag{3.5}$$
$$\frac{1}{p}\int_\Omega \left(|\nabla u_m(0)|^2 + \varepsilon\right)^{p/2} dx + \int_\Omega [H_\varepsilon(x, u_m(T)) - H_\varepsilon(x, u_m(0))]dx \leq C(M_1)$$

where $H_\varepsilon(x,u) = \displaystyle\int_0^u h_\varepsilon(x,v)\,dv$.

Whence we obtain the estimate:

$$\|u'_m\|_{L^2(0,T;L^2(\Omega))} \leq C(M). \tag{3.6}$$

From (3.5) and (3.6) we deduce:

$$u_m \in C(0,T;\mathbb{R}^m).$$

Therefore, by classical theory of ordinary differential equations, see for example [1], we get $t_m = T$. ■

Now we have the main result of this section:

Theorem 5.
$$G_m \in \mathbb{C}^2(0,T;\mathbb{R}^m).$$

Proof: By classical theory of ordinary differential equations it suffices to show that $\phi \in C^1(\mathbb{R}^m, \mathbb{R}^m)$.

For any $x \in \Omega$, we have:

$$F_j(x,.) \in C^1(\mathbb{R}^m, \mathbb{R})$$
$$\text{and } \frac{\partial F_j}{\partial \xi_h}(x,\xi) = (|W(x)\xi|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla w_h.\nabla w_j + (p-2).$$

$$(|W(x)\xi|^2 + \varepsilon)^{\frac{p-4}{2}} \left(\sum_k \xi_k \nabla w_k.\nabla w_h\right)\left(\sum_k \xi_k \nabla w_k.\nabla w_j\right). \tag{3.7}$$

It's straithforward that $|\nabla w_j| \leq \|W(x)\|$ for any $j : 1 \leq j \leq m$; therefore using Cauchy-Schwarz inequality we get:

(3.8)
$$\left|\frac{\partial F_j}{\partial \xi_h}\right| \leq (|W(x)\xi|^2+\varepsilon)^{\frac{p-2}{2}}\|W(x)\|^2+(2-p)(|W(x)\xi|^2+\varepsilon)^{\frac{p-4}{2}}|W(x)\xi|^2\|W(x)\|^2$$
$$\leq \varepsilon^{\frac{p-2}{2}}\|W(x)\|^2 + (2-p)\varepsilon^{\frac{p-4}{2}}|\xi|^2\|W(x)\|^4.$$

From (3.8) and Lebesgue's theorem, we obtain:

$$\frac{\partial F_j}{\partial \xi_h}(.,\xi) \in L^1(\Omega) \text{ for any } h : 1 \leq h \leq m \text{ and any } \xi \in \mathbb{R}.$$

By the same way we get that $\frac{\partial \varphi_j}{\partial \xi_h}$ exists and is continuous on $\mathbb{R}^m$ whence:

$$\varphi \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^m) \tag{3.9}$$

On the other hand let $h_\varepsilon^j(x,\xi) = h_\varepsilon(x, w(x).\xi)w_j(x)$, we have:

$$\Psi_j(\xi) = \int_\Omega h_\varepsilon^i(x,\xi)\,dx$$
$$\text{and } h_\varepsilon^j(x,.) \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}) \text{ for any } x \in \Omega.$$

Furthermore:

$$\left|\frac{\partial h_\varepsilon^j}{\partial \xi_h}(x,\xi)\right| = \left|\frac{\partial h_\varepsilon}{\partial u}(x, w(x).\xi).w_h(x)w_j(x)\right| \leq K|w_h(x)||w_j(x)|.$$

Thus: $\frac{\partial h_\varepsilon^j}{\partial \xi_h}(.,\xi) \in L^1(\Omega)$ for any h, $j : 1 \leq h$, $j \leq m$ and any $\xi \in \mathbb{R}^m$.

Once again, by Lebesgue's continuity and derivability theorems, we obtain:

$$\Psi \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^m). \tag{3.10}$$

By (3.9) and (3.10), we get $\phi \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^m)$. The proof of theorem 5 is now complete. ■

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