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DIVISION AND EXTENSION IN WEIGHTED BERGMAN-SOBOLEV SPACES

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Abstract

Let D be a bounded strictly pseudoconvex domain of C^n with C^∞ boundary and $Y = \{z; u_1(z) = \cdots = u_l(z) = 0\}$ a holomorphic submanifold in a neighbourhood of $\bar{D}$, of codimension l and transversal to the boundary of D.

In this work we give a decomposition formula $f = u_1 f_1 + \cdots + u_l f_l$ for functions f of the Bergman-Sobolev space vanishing on $M = Y \cap D$. Also we give necessary and sufficient conditions on a set of holomorphic functions $\{f_\alpha\}_{|\alpha| \leq m}$ on M, so that there exists a holomorphic function in the Bergman-Sobolev space such that $D^\alpha f|_M = f_\alpha$ for all $|\alpha| \leq m$.

I. Introduction and main results

Let $D = \{\, z; \rho(z) < 0 \,\}$ be a bounded strictly pseudoconvex domain of C^n with C^∞-boundary. Let $Y = \{\, z; u_1(z) = \ldots = u_l(z) = 0 \,\}$ denote a holomorphic submanifold in a neighbourhood of $\bar{D}$, of codimension l and transversal to the boundary of $D \cap Y$, i.e. $\partial \rho \wedge \partial u_1 \wedge \ldots \wedge \partial u_l \neq 0$ on the intersection of the boundary of D and the submanifold Y.

For every $1 \leq p < \infty$, $\delta > 0$, and $k = 0, 1, \ldots$ we consider the weighted Sobolev space

$$L^p_{\delta,k}(D) = \{\, f \text{ measurable} \,;\, ||f||_{p,\delta,k} < \infty \,\}$$

where

$$||f||_{p,\delta,k} = \sup \left\{ \left(\int_D |\, D^\alpha \bar{D}^\beta f\,|^p \, (-\rho)^{\delta-1} \right)^{\frac{1}{p}} \,;\, |\alpha| + |\beta| \leq k \right\}$$

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and $D_z^\alpha = \frac{\partial^{|\alpha|}}{\partial z_\alpha}$, $\bar{D}_z^\beta = \frac{\partial^{|\beta|}}{\partial \bar{z}_\beta}$.

Also, we define for every p, δ, k the weighted Bergman-Sobolev space as the space of holomorphic functions $A^p_{\delta,k}(D) = L^p_{\delta,k}(D) \cap \mathcal{O}(D)$.

Replacing the derivatives D_z^α, $\bar{D}_z^\beta$ for tangent-derivatives on the submanifold Y, we define in the same way the spaces $L^p_{\delta,k}(M)$, and $A^p_{\delta,k}(M)$ in the submanifold $M = Y \cap D$.

It is well known (see for instance [**3**], [**4**]) that

$$A^p_{\delta,k}(D) \subset A^t(D), \text{ if } t = k - \frac{n+\delta}{p} > 0 \tag{1.1}$$

where $A^t(D)$ denotes the corresponding space of the holomorphic Lipschitz functions. It is also well known that

$$A^p_{\delta,k}(D) = A^{p'}_{\delta',k'}(D), \text{ if } \delta - \delta' = p(k-k'). \tag{1.2}$$

One of the main results that we will prove in this work is a result of division in the spaces $A^p_{\delta,k}(D)$.

We recall the following result of division in the holomorphic Lipschitz spaces, due to P. Bonneau, A. Cumenge and A. Zériahi ([**6**]):

If f is a holomorphic Lipschitz function of class $A^t(D)$ vanishing in the submanifold M, then there exist functions f_j, $j = 1, \dots, l$, of class $A^{t-\frac{1}{2}}(D)$ such that $f = u_1 f_1 + \dots + u_l f_l$.

We prove in this paper the following theorem:

Theorem 1.1.
If f is a function of class $A^p_{\delta,k}(D)$ vanishing on the submanifold $M = Y \cap D$ transversal to the boundary of D, then there exist functions f_j, $j = 1, \dots, l$ of class $A^p_{\delta+\frac{p}{2},k}(D)$ such that

$$f = \sum_{j=1}^{l} u_j f_j. \tag{1.3}$$

Observe that by (1.1) and (1.2) the Theorem 1.1 is in some sense a refinement of the above result of division in the holomorphic Lipschitz spaces.

In the limit case where Y is a point ζ of D, the Theorem 1.1 is the Gleason's problem. In this case (see [**11**]) it is known that

$$f(z) = \sum_{j=1}^{n} (z_j - \zeta_j) f_j(z), \quad f_j \in A^p_{\delta,k}(D).$$

The second main result that we will prove is an extension theorem of jets. This consists to give necessary and sufficient conditions on a set $\{ f_\alpha \}_{|\alpha|\le m}$ of holomorphic functions on the submanifold $M = Y \cap D$ so that there exists a $A^p_{\delta,k}(D)$-function f, such that $D^\alpha_z f\,|_M = f_\alpha$ for all $|\alpha| \le m$.

The case $m = 0$, i.e. the problem of extension and restriction of functions of class $A^p_{\delta,k}(D)$, has been studied by many authors using different methods. (See for exemple [**3**], [**4**], [**9**]). The result obtained in this case is

$$A^p_{\delta,k}(D)\Big|_M = A^p_{\delta+l,k}(M).$$

The above problem in the holomorphic Lipschitz spaces has been proved by us in [**12**].

In order to state the result of extension let us introduce the following definitions.

We consider smooth vector fields on $\bar{D}$

$$X = \sum_{i=1}^{n} a_i(z)\,\frac{\partial}{\partial z_i}.$$

For these vector fields we say that X is complex-tangential if $X\rho(z) = 0$ for every z in a neighbourhood of the boundary of D, and we define its weight $w(X)$ in the usual way:

$$w(X) = \begin{cases} \frac{1}{2} & \text{if } X \text{ is complex-tangential} \\ 1 & \text{in other case.} \end{cases}$$

If $X = X_k \dots X_1$ is a differential operator we define its weight by

$$w(X) = \sum_{i=1}^{k} w(X_i).$$

We recall that for a holomorphic function f on D the $j-th$ covariant differential of f at a point $z \in D$ is defined by:

$$\begin{aligned} d^0 f_z &= f(z) \\ d^j f_z(X_1,\dots,X_j) &= X_j\, d^{j-1} f_z(X_1,\dots,X_{j-1}) - \\ &\qquad \sum_{i=1}^{j-1} d^{j-1} f_z(X_1,\dots,\nabla_{X_j}X_i,\dots,X_{j-1}) \end{aligned}$$

and that in coordinates we can write

$$d^j f_z = \sum_{|I|=j} \frac{\partial^j f(z)}{\partial \zeta_{i_1} \dots \partial \zeta_{i_j}} \, dz_{i_1} \otimes \dots \otimes dz_{i_j}.$$

Also, fixed m, we denote by

$$J_m f_z = \left(d^0 f_z , \dots , d^m f_z \right)$$

the holomorphic jet of order m at the point $z \in D$ induced by f.

Moreover, if the function f is of class $A^p_{\delta,k}(D)$, then it is well known (see [**1**], [**3**], [**4**]) that the function $d^j f_z(X_1, \dots , X_j)\big|_M$ is of class $L^p_{\delta + l + w(X)p,k}(M)$ where X is the differential operator formed by the vector fields $X_1, \dots , X_j$.

Thus, if we define the covariant tensors of order j at a point $z \in M$ as

$$F^j_z = d^j f_z$$

then they satisfy the following conditions for every $0 \leq j \leq m$:

I-1) At every point $z \in M$, F^j_z is a j-covariant symmetric tensor.

I-2) $F^j \left(\frac{\partial}{\partial z_1}, \dots , \frac{\partial}{\partial z_1}, \dots , \frac{\partial}{\partial z_n}, \dots , \frac{\partial}{\partial z_n} \right)$ are holomorphic functions on M.

I-3) $F^j \,(X_1, \dots , X_j) = X_j \, F^{j-1} \,(X_1, \dots , X_{j-1}) -$

$$\sum_{i=1}^{j-1} F^{j-1} \,(X_1, \dots , \nabla_{X_j} X_i, \dots , X_{j-1})$$

for every tangent vector field X_j at M.

I-4) $F^j \,(X_1, \dots , X_j) \in L^p_{\delta+l+w(X)\,p,k}(M)$.

Therefore, it is natural to introduce the following definition:

Definition 1.2.

$F = (F^0, \dots , F^m)$ is an $A^p_{\delta,k}$-jet of order m on M if it satisfies the four previous conditions.

The condition I-3) just gives a relation of coherence between the tensors F^j. We point out that a $A^p_{\delta,k}$-jet on M of order 0 is a function of class $A^p_{\delta+l,k}(M)$.

The notation of $A^p_{\delta,k}$-jet is justified by the following result.

Theorem 1.3.

$F = (F^0, \dots, F^m)$ is a $A^p_{\delta,k}$-jet of order m on M if and only if there exists a function f of class $A^p_{\delta,k}(D)$ such that $J_m f = F$ on M.

We recall that in [**12**] we said that $F = (F^0, \dots, F^m)$ is an A^t-jet if it satisfies the conditions I-1, I-2, I-3 of the Definition 1.2 and the condition

$$\left| X_k \dots X_{j+1} F^j(X_1, \dots, X_j) \right| \le c\, M(t - w(X), z) \tag{1.4}$$

where the function $M(s, z)$ is defined by

$$M(s,z) = \begin{cases} 1 & \text{if } s > 0 \\ |\log|\rho(z)|| & \text{if } s = 0 \\ |\rho(z)|^s & \text{if } s < 0 \end{cases} \tag{1.5}$$

and the vector fields $X_{j+1}, \dots, X_k$ are tangential to the submanifold Y.

In the same paper [**12**] we proved that:

(1.6) *F is a A^t-jet of order m on M if and only if there exists a holomorphic Lipschitz function f of class $A^t(D)$ such that $J_m f = F$ on M.*

To prove the Theorem 1.3 we will use the Theorem 1.1, the results (1.1), (1.2) and (1.6) and a result of resolution of the $\bar\partial$-equation in the spaces $L^p_{\delta,k}(D)$.

As usually several diferent constants in the inequalities will be denoted by c.

II. Some integral formulas

In this section we give an extension operator and an explicit solution of the $\bar\partial$ -equation.

We denote by $\Phi(\zeta, z)$ the support function of Henkin and we put $a(\zeta, z) = -\rho(\zeta) + \Phi(\zeta, z)$.

Using the results of B. Berndtsson and M. Andersson [**5**], for every positive integer s we can construct kernels K^s and R^s of type

$$K^s(\zeta, z) = \left(\frac{-\rho(\zeta)}{a(\zeta, z)}\right)^{n+s} \frac{\varphi_0(\zeta, z)}{|\zeta - z|^{2n}} + \sum_{j=1}^{n-1} \frac{(-\rho(\zeta)^{n+s-j}\, \varphi_j(\zeta, z)}{a(\zeta, z)^{n+s+1}\, |\zeta - z|^{2n-2j}} \tag{2.1}$$

$$R^s(\zeta, z) = \frac{(-\rho(\zeta))^s\, \varphi_n(\zeta, z)}{a(\zeta, z)^{n+s+1}} \tag{2.2}$$

which have the following properties:

1. $d_{\zeta,z} K^s = R^s$ outside the diagonal, and R^s is holomorphic in the variable z.
2. The forms φ_j, $j = 0, \dots, n$ are of class $C^\infty(\bar{D} \times \bar{D})$.
3. $|\varphi_j(\zeta, z)| \le c|\zeta - z|$, $\qquad j = 0, \dots, n-1$.

4. Koppelman Formulas. *Let $K^s_{p,q}$ be the component of K^s of bidegree (p,q) in z, (n-p,n-q-1) in ζ, and let $R^s_{p,q}$ be the component of R^s of bidegree (p,q) in z, and (n-p,n-q) in ζ. Then, if f is a (p,q) form with coefficients in $C^1(\bar{D})$, we have*

$$\begin{aligned} f(z) &= (-1)^{p+q+1} \int_D \bar\partial f(\zeta) \wedge K^s_{p,q}(\zeta, z) + \\ &\qquad (-1)^{p+q}\, \bar\partial_z \int_D f(\zeta) \wedge K^s_{p,q-1}(\zeta, z), \qquad \text{if } q \ge 1 \\ f(z) &= (-1)^{p+1} \int_D \bar\partial f(\zeta) \wedge K^s_{p,0}(\zeta, z) - \\ &\qquad \int_D f(\zeta)\, R^s_{p,0}(\zeta, z), \qquad \text{if } q = 0. \end{aligned} \tag{2.3}$$

Now, if $Y = \{z;\, z_1 = \dots = z_l = 0\}$ and $M = Y \cap D$, then the same construction used in [5] to prove these results gives for each $s > \frac{\delta - 1}{p}$ an extension operator from the space $A^p_{\delta,k}(M)$ to the space of holomorphic functions $\mathcal{O}(D)$. This operator is defined by

$$E^s f(z) = \int_M f(\zeta)\, R^s_M(\zeta, z) \tag{2.4}$$

where

$$R^s_M(\zeta, z) = \frac{(-\rho(\zeta))^s}{a(\zeta, z)^{n-l+1+s}}\, \varphi(\zeta, z)\,, \quad \zeta \in M,\, z \in D$$

and the form φ has coefficients of class $C^\infty(\bar{M} \times \bar{D})$ and it is holomorphic in z.

Moreover, the same formula (2.3) also gives an explicit integral operator to solve the $\bar\partial$-equation for (0,q) forms $\bar\partial$-closed. This operator is given by the kernel $K^s_{0,q-1}(\zeta, z)$.

The estimates for these kernels are given by the following Lemma.

Lemma 2.1.

Let $j \leq 2n-1$ be an integer. Then with $M(s,z)$ defined as (1.5) we have

$$\int_D \frac{1}{|a|^t\,|\zeta - z|^j} \leq c \begin{cases} M\left(n+1-t-\frac{j}{2}, z\right) & \text{if } j \leq 2n-3. \\ 1 & \text{if } j = 2n-2, t < 2. \\ M(2-t,z)\,|\log|\rho(z)|| & \text{if } j = 2n-2, t \geq 2. \\ M(1-t,z) & \text{if } j = 2n-1. \end{cases}$$

Proof:

Using the usual change of coordinates and computing the respective integrals we obtain these estimates. (See for instance [**10**]). ■

Now we will state some formulas of integration by parts.

The first formula is contained in the following Lemma of [**7**]:

Lemma 2.2.

Let f be a (0,1) form $\bar\partial$-closed with coefficients of class $C^1(\bar D)$. Then

$$D_z^\alpha\, g = -\int_D D_\zeta^\alpha\, f \wedge K_{0,0}^s + \sum \int_D D_\zeta^\gamma\, f \wedge D_z^\beta\, R_{0,1,1}^{s,i}$$

where in the last terms γ and β are multiindexes with $|\gamma| + |\beta| = |\alpha| - 1$, $i = 1, \ldots, n$, and $R_{0,1,1}^{s,i}$ denotes the coefficient of $d\,z_i$ in the component of the kernel R^s of degree (1,0) in z and (n,n-1) in ζ.

Before to state the second formula we introduce the following kernels, that are a generalization of the extension kernels R_M^s.

Definition 2.3.

If $Y = \{\, z_1 = \ldots = z_l = 0\,\}$ and $M = Y \cap D$, we define the kernels

$$R_{M,\psi}^{s,r}(\zeta, z) = (-\rho(\zeta))^s\, \psi(\zeta, z), \quad \zeta \in M, z \in D$$

where the form $\psi(\zeta, z)$ has the coefficients of class $C^\infty(M \times D)$, and it satisfies

$$\left| D_z^\alpha\, D_\zeta^\beta\, \bar D_\zeta^\gamma\, \psi(\zeta, z) \right| \leq c\,|\,a(\zeta, z)\,|^{r - (|\alpha| + \frac{|\beta|}{2} + |\gamma|)}$$

for every multiindexes α, β, γ.

Let also $R_{M,\psi}^{p,r}$ denote the integral operator given by this kernel.

Observe that the extension operator $R_M^s(\zeta, z)$ is a $R_{\psi,M}^{s,-(n-l+1+s)}$ operator, because $|\,D_\zeta\, a(\zeta, z)\,| \leq c|\zeta - z| \leq c|a(\zeta, z)|^{\frac{1}{2}}$.

These operators have the following properties:

Lemma 2.4.

i) $D_z^\alpha R_{M,\psi}^{s,r} = R_{M,\psi_1}^{s,r-|\alpha|}$

ii) $\int_M \left| R_{M,\psi}^{s,r} \right| \le c M(n-l+1+s+r,z)$.

Proof:

i) is clear and ii) follows from Lemma 2.1. ■

Lemma 2.5.

If f is a function of class $C^k(\bar{M})$, then, fixed an integer q, we can find operators $R_{M,\psi_\gamma}^{s_\gamma,r_\gamma}$, $R_{M,\psi_\mu}^{s_\mu,r_\mu}$ such that

$$D_z^\alpha R_{M,\psi}^{s,r} f = \sum_{\substack{|\gamma|=k \\ s_\gamma+r_\gamma \ge s+r+k-|\alpha| \\ p_\gamma \ge s+k}} R_{M,\psi_\gamma}^{s_\gamma,r_\gamma} D_\zeta^\gamma f + \sum_{\substack{|\mu|<k \\ s_\mu+r_\mu \ge q \\ s_\mu \ge s+k}} R_{M,\psi_\mu}^{s_\mu,r_\mu} D_\zeta^\mu f$$

Remark. Roughly speaking, the Lemma 2.4 prove that the coefficient $s+r$ measures the regularity of the operator $R_{M,\psi}^{s,r}$, and therefore the operators $R_{M,\psi_\gamma}^{s_\gamma,r_\gamma}$ in Lemma 2.5 have at least the same regularity than the operator $R_{M,\psi}^{s,r}$ plus $k-|\alpha|$. On the other hand, choosing q large enought we can assume that the operators $R_{M,\psi_\mu}^{s_\mu,r_\mu}$ are as regular as required.

Proof:

Using the transversavility of the submanifold Y, we can choose a covering $\{U_i\}_{i=0}^{i_0}$ of M such that

i) $M = \cup_{i=0}^{i_0} U_i$, and $U_0 = \{ z;\, \rho(z) < -\delta \}$, $\delta > 0$.

ii) For each i, $1 \le i \le i_0$ there is $l+1 \le j_i \le n$ such that $\frac{\partial \rho(z)}{\partial z_{j_i}} \ne 0$ on U_i.

Let $\{\chi_i\}$ be a partition of the unity for this covering and we put

$$R_{M,\psi}^{s,r} = \sum_{i=0}^{i_0} R_{M,\chi_i\psi}^{s,r}.$$

We want to prove the Lemma for each one of the operators of the sum.

If $i=0$ the result is clear by the properties of ψ and the property i) of the covering.

If $i \geq 1$ by the property ii) of the covering we have

$$\int_M R^{s,r}_{M,\psi}(\zeta,z)\, f(\zeta) =$$
$$\frac{1}{s+1}\int_M (-\rho(\zeta))^{s+1}\,\frac{\partial}{\partial\zeta_{j_i}}\left(\chi_i\psi(\zeta,z)\left(\frac{\partial\rho(\zeta)}{\partial\zeta_{j_i}}\right)^{-1} f(\zeta)\right) =$$
$$\int_M R^{s+1,r}_{M,\chi_i\psi'}\,\frac{\partial f}{\partial\zeta_{j_i}} + \int_M R^{s+1,r-\frac{1}{2}}_{M,\chi_i\psi''}\, f$$

Iterating this process in the terms which have less than k derivatives on the function f, and using the Lemma 2.4 i) we obtain the result. ∎

III. Solution of the $\bar\partial$-equation in the $L^p_{\delta,k}(D)$ space

The aim of this section is to prove the following Theorems.

Theorem 3.1.
If f is a (0,q) form $\bar\partial$-closed with coefficients of class $L^p_{\delta,0}(D)$, $1 \leq p < \infty$, $\delta > 0$, then there exists a (0,q-1) form g with coefficients of class $L^p_{\delta^*,0}(D)$ for all $\delta^* \geq \delta - \frac{p}{2}$, $\delta^* > 0$ such that $\bar\partial g = f$.

Theorem 3.2.
If f is a (0,1) form $\bar\partial$-closed with coefficients of class $L^p_{\delta,k}(D)$, $1 \leq p < \infty$, $\delta > 0$, $k = 0, 1, \ldots$, then there exists a function g with coefficients of class $L^p_{\delta^*,k}(D)$ for all $\delta^* \geq \delta - \frac{p}{2}$, $\delta^* > 0$ such that $\bar\partial g = f$.

To prove these Theorems we need the following Lemma.

Lemma 3.3.
If a kernel $K(\zeta,z)$ satisfies $|K(\zeta,z)| \leq c\,\frac{(-\rho(\zeta))^s}{|a(\zeta,z)|^t\,|\zeta-z|^j}$ $s, t \geq 0$, $j = 0,\ldots,2n-1$, and f is of class $L^p_{\delta,0}(D)$, $1 \leq p < \infty$, $0 < \delta - 1 < sp$, then the function Kf is of class $L^p_{\delta^*,0}(D)$, $\delta^* \geq \delta - \lambda p$, $\delta^* > 0$, where

$$\lambda = \begin{cases} n+1+s-t-\frac{j}{2} & \text{if } j \leq 2n-2 \\ 2-\varepsilon+s-t & \text{if } j = 2n-2,\ \varepsilon > 0 \\ 1+s-t & \text{if } j = 2n-1. \end{cases}$$

Proof:

We want to see that for a δ^* fixed which satisfies the previous conditions we have

$$I = \int_D \left(\int_D |K(\zeta, z)|\, |f(\zeta)|\, d\zeta \right)^p (-\rho(z))^{\delta^*-1}\, dz \le$$
$$c \int_D |f(\zeta)|^p (-\rho(\zeta))^{\delta-1}\, d\zeta.$$

First we consider the case $p = 1$ and $j \neq 2n-2$.

In this case applying the Fubini Theorem we have

$$I \le c \int_D |f(\zeta)|\, (-\rho(\zeta))^s \int_D \frac{(-\rho(z))^{\delta^*-1}}{|a(\zeta, z)|^t\, |\zeta - z|^j}\, dz\, d\zeta$$

and using that $|a(\zeta, z)| \approx |a(z, \zeta)|$, $-\rho(z) \le c|a(\zeta, z)|$ and the Lemma 2.1 we get

$$I \le c \int_D |f(\zeta)|\, (-\rho(\zeta))^s\, M(\delta^* - 1 + \lambda - s, \zeta)\, d\zeta. \tag{3.1}$$

Now, if $\delta^* - 1 + \lambda - s \ge 0$ we have that $(-\rho(\zeta))^s\, M(\delta^* - 1 + \lambda - s, \zeta) \le c(-\rho(\zeta))^{\delta-1}$, since $s > \delta - 1$.

Moreover, if $\delta^* - 1 + \lambda - s < 0$ then $(-\rho(\zeta))^s\, M(\delta^* - 1 + \lambda - s, \zeta) \le c(-\rho(\zeta))^{\delta^*-1+\lambda} \le c(-\rho(\zeta))^{\delta-1}$ because $\delta^* \ge \delta - \lambda$.

Hence

$$I \le c \int_D |f(\zeta)|\, (-\rho(\zeta))^{\delta-1}\, d\zeta.$$

If $p = 1$ and $j = 2n-2$ we obtain in (3.1) the estimate

$$I \le c \int_D |f(\zeta)| (-\rho(\zeta))^s\, M(\delta^* + 1 - t, \zeta)\, |log|\rho(\zeta)||\, d\zeta$$

and applying the same reasoning as in the above case we prove the result.

Now we consider the case $1 < p < \infty$ and $j \le 2n-3$.

Let $p' = \frac{p}{p-1}$. Taking r such that

$$\frac{p-1}{p}\left(n + 1 - \frac{j}{2}\right) < r < \frac{p-1}{p}\left(n + 1 - \frac{j}{2} + \frac{\delta^*}{p-1}\right)$$

and applying the Hölder inequalities we get

$$I \leq c \int_D \left(\int_D |f(\zeta)|^p \frac{(-\rho(\zeta))^{sp}}{|a(\zeta,z)|^{(t-r)p}\, |\zeta - z|^j} \, d\zeta \right)$$
$$\left(\int_D \frac{1}{|a(\zeta,z)|^{rp'}\, |\zeta - z|^j} \, d\zeta \right)^{\frac{p}{p'}} (-\rho(z))^{\delta^* - 1} \, dz \leq$$
$$\leq c \int_D \int_D |f(\zeta)|^p \, (-\rho(\zeta))^{sp} \, \frac{(-\rho(z)^{(n+1-rp'-\frac{j}{2})(p-1)+\delta^*-1}}{|a(\zeta,z)|^{(t-r)p} |\zeta - z|^j} \, d\zeta \, dz.$$

By Fubini Theorem and the Lemma 4.2 we have

$$I \leq c \int_D |f(\zeta)|^p \, (-\rho(\zeta))^{sp} \, M \, \Bigg(n + 1 - (t-r)p - \frac{j}{2} +$$
$$(n + 1 - rp' - \frac{j}{2})(p-1) + \delta^* - 1, \zeta \Bigg) \, d\zeta =$$
$$c \int_D |f(\zeta)|^p \, (-\rho(\zeta))^{sp} \, M \, \left((n + 1 - t - \frac{j}{2})p + \delta^* - 1, \zeta \right) \, d\zeta \leq$$
$$c \int_D |f(\zeta)|^p \, (-\rho(\zeta))^{\delta - 1} \, d\zeta$$

and hence this case is proved.

The cases $1 < p < \infty$ and $j = 2n-2,\ 2n-1$ follow in the same way taking r such that

$$\frac{p-1}{p}(2n - j) \, < \, r \, < \, \frac{p-1}{p} \left(2n - j + \frac{\delta^*}{p-1} \right) \quad \blacksquare$$

Corollary 3.4.

If $R^{s,r}_{D,\psi}$ is the operator of the Definition 2.3 and f is of class $L^p_{\delta,k}(D)$, $\delta - 1 < sp$, then the function $R^{s,r}_{D,\psi} f$ is of class $L^p_{\delta^*,k}(D)$, for all $\delta^* \geq \delta - (n + 1 + s + r)p$, $\delta^* > 0$.

Proof:

Applying the Lemmas 2.5, 3.3 we obtain the result.

Proof of Theorem 3.1:

We take $s > 0$ such that $sp > \delta - 1$ and we define the function $g = -\int_D f \wedge K^s_{0,q-1}$, where the kernel K^s is given in (2.1).

It is clear by (2.3) that $\bar{\partial} g = f$. Now, using the estimate

$$|\, K^s \,| \leq c \left(\frac{(-\rho)^{n+s}}{|a|^{n+s} \, |\zeta - z|^{2n-1}} + \sum_{i=1}^{n-1} \frac{(-\rho)^{n+s-i}}{|a|^{n+1+s} \, |\zeta - z|^{2n-2i-1}} \right)$$

and applying the Lemma 3.3 we obtain the result. ■

Proof of Theorem 3.2:

We define g as in the previous Theorem.

By Lemmas 2.2 and 2.4 we have

$$D_z^\alpha g = -\int_D D_\zeta^\alpha f \wedge K_{0,0}^s + \sum_{|\gamma|+|\beta|<|\alpha|} \int_D D_\zeta^\gamma f \wedge R_{D,\psi_\gamma}^{s,-(n+1+s+|\beta|)}$$

where the kernels $R_{D,\psi_\gamma}^{s,-(n+1+s+|\beta|)}$ are holomorphic in z.

The same reasoning used in the proof of Theorem 3.1 shows that the term $\int_D D_\zeta f \wedge K_{0,0}^s$ is of class $L_{\delta^*,0}^p$.

Moreover the Corollary 3.4 shows that the term

$$\int_D D_\zeta^\gamma f \wedge K^{s,-(n+1+s+|\beta|)}$$

is of class $A_{\delta+|\beta|p,k-|\gamma|}^p(D) = A_{\delta,|\alpha|-|\beta|-|\gamma|}^p(D)$.

Now, using that $|\alpha| - |\beta| - |\gamma| \geq 1$ we end the proof. ■

IV. Division in the $A_{\delta,k}^p$ spaces

To prove the Theorem 1.1, we will first solve the problem locally using the following projection.

Lemma 4.1.

Let $Y = \{\, z;\, z_1 = \ldots = z_l = 0 \,\}$ be a linear submanifold transversal to the boundary of D. Then for every point w in the boundary of $M = Y \cap D$, there exists a neighbourhood V of w and a projection

$$\Pi : V \longrightarrow V \cap Y$$

of class $C^\infty(\bar{V})$, such that

i) $\Pi(z) = z + z_1 g_1 + \ldots + z_l g_l$

ii) $\rho(\Pi(z)) \leq \rho(z) - c|z'|^2, \qquad z' = (z_1, \ldots, z_l, 0 \ldots, 0),\, c > 0$

iii) $|a(\zeta, z)| \leq c\,|a(\zeta, \Pi(z)| \leq c\,\left(|a(\zeta, z)| + |z'|^2\right)$

Remark. Observe that the condition ii) implies that if $z \in V \cap D$ then $\Pi(z) \in V \cap M$.

Proof:

We write

$$\langle \zeta, z\rangle = \sum_{i=1}^{n} \zeta_i z_i \quad , \qquad z'' = z - z'$$

$$\frac{\partial \rho}{\partial \zeta} = \left(\frac{\partial \rho}{\partial \zeta_1}, \dots, \frac{\partial \rho}{\partial \zeta_n}\right) , \qquad \frac{\partial \rho}{\partial \zeta''} = \left(0, \dots, 0, \frac{\partial \rho}{\partial \zeta_{l+1}}, \dots, \frac{\partial \rho}{\partial \zeta_n}\right).$$

Let U be a neighbourhood of the boundary of M. Shrinking U and using the transversavility of Y we can assume that $\left|\frac{\partial \rho}{\partial z''}\right| \geq c > 0$ on U and therefore, for every $1 \leq j \leq l$, we can take a function $h^j : U \longrightarrow C^n$ of class $\mathcal{C}^\infty(U)$ such that

$$(4.1) \quad h^j = (0, \dots, -1_j, \dots, 0, h^j_{l+1}, \dots, h^j_n), \text{ and } \langle \frac{\partial \rho}{\partial z}, h^j \rangle = 0.$$

The next step is to see that for a certain $d > 0$ the projection

$$(4.2) \qquad \Pi(z) = z + z_1 h^1 + \dots + z_l h^l - d|z'|^2 \frac{\partial \rho(z)}{\partial z''}$$

satisfies the required conditions.

It is obvious that Π satisfies i) for every d.

Using the Taylor development and the properties (4.1) we have that

$$\rho(\Pi(z)) \leq \rho(z) - 2d|z'|^2 \left|\frac{\partial \rho}{\partial z''}\right| + c_0 |\Pi(z) - z|^2 \leq$$
$$\rho(z) - (2dc_1 - c_2)|z'|^2 + c_3 d|z'|^3$$

where c_1, c_2, $c_3 > 0$.

Now taking d such that $2dc_1 - c_2 > c > 0$ and shrinking U we obtain ii).

To prove iii) we recall that $\Phi(\zeta, z)$ is holomorphic in z and

$$\Phi(\zeta, z) = \langle P(\zeta, z), \zeta - z\rangle = \langle \frac{\partial \rho}{\partial \zeta}, \zeta - z\rangle + O(|\zeta - z|^2).$$

Using this and the properties (4.1), we have

$$(4.3) \quad a(\zeta, z) - a(\zeta, \Pi(z)) = \langle P(\zeta, z) - P(\zeta, \Pi(z)), \zeta - z\rangle +$$
$$\langle P(\zeta, \Pi(z)), z - \Pi(z)\rangle = \sum_{j=1}^{l} z_j\, \psi(\zeta, z)$$

with

$$|\psi(\zeta, z)| \leq c(|\zeta - z| + |z'|) \approx c(|\zeta - \Pi(\zeta)| + |z'|).$$

Finally, using that $|\zeta - z| \leq c|a(\zeta, z)|^{\frac{1}{2}}$ and $|\zeta - \Pi(z)|$, $|z'| \leq c|a(\zeta, \Pi(z))|^{\frac{1}{2}}$ we obtain iii). ■

Lemma 4.2.

If f is a function of class $L^p_{\delta,0}(M)$, then the function $R^{s,r}_{M,\psi} f$, $\delta-1 < sp$ is of class $L^p_{\delta^*,0}(D)$ for all $\delta^* \geq \delta - l - (n+1+s+r)p$, $\delta^* > 0$.

Proof:

Appliying the estimates of Theorem 2.4 of [**3**] and the same reasoning that in the Lemma 3.3, we obtain the result. ■

Corollary 4.3.

If f is a function of class $L^p_{\delta,k}(M)$, then the function $R^{s,r}_{M,\psi} f$, $\delta-1 < sp$ is of class $L^p_{\delta^*,k}(D)$ for all $\delta^* \geq \delta - l - (n+1+s+r)p$, $\delta^* > 0$.

Proof:

The proof is a consequence of the above Lemma and of the integration by parts formula given in the Lemma 2.5. ■

Lemma 4.4.

Let be f a (0,1) form $\bar\partial$-closed with coefficients of class $L^p_{\delta,k}(D)$, $\delta > p$ and let u be a holomorphic function on a neighbourhood of $\bar D$, such that uf has coefficients of class $L^p_{\delta-\frac{p}{2},k}(D)$. Then there exists a function g of class $L^p_{\delta-\frac{p}{2},k}(D)$ such that $\bar\partial\, g = f$ and ug is of class $L^p_{\delta-p,k}(D)$.

Proof:

We take $g = -\int_D f \wedge K^p_{00}$ as in the Theorem 3.2.

Hence, we only need to see that ug is of class $L^p_{\delta-p,k}(D)$.

By (2.3) we have $\int_D g\, R^s_{00} = 0$ and therefore we can write

$$u(z)g(z) = \int_D u(\zeta)f(\zeta) \wedge K^s_{00}(\zeta,z) + \int_D (u(z)-u(\zeta))g(\zeta)\, R^s_{00}(\zeta,z).$$

The Theorem 3.2 gives that the first term is of class $L^p_{\delta-p,k}(D)$.

Moreover $(u(z)-u(\zeta))\, R^s_{00}(\zeta,z) = R^{s,\frac{1}{2}-(n+1+s)}_{D,\psi}$ and therefore by Corollary 4.3 we obtain that the second term is of class $L^p_{\delta-p,k}(D)$. ■

To prove the result of division given in the Theorem 1.1, first we consider the linear case to obtain local solutions. Finally using these solutions, the Lemma 4.4 and a result of division in the holomorphic Lipschitz spaces ([**6**]) we will obtain the result.

Proposition 4.5.

If $Y = \{ z;\, z_1 = 0 \}$ is transversal to the boundary of D, and f is a function of class $A^p_{\delta,k}(D)$ that is zero on M, then there exists a function f_1 of class $A^p_{\delta+\frac{p}{2},k}(D)$ such that $f = z_1 f_1$.

Proof:

We consider a covering $\{U_i\}_{i=0}^{i_0}$ of D such that:

1) $U_0 = \{ z;\, \rho(z) < -\delta < 0 \}$.
2) If $1 \le i < i_1$ then $z_1 \neq 0$ on U_i.
3) If $i_1 \le i \le i_0$ then there exists a projection Π_i as the one in the Lemma 4.1.

Let $\{ \chi_i \}$ a partition of the unity for this covering.

We want to see that $\chi_i \frac{f}{z_1}$ is a function of class $L^p_{\delta+\frac{p}{2},k}(D)$.

We consider the three following cases.

1) $i = 0$.

In this case using that $U_0 \subset\subset D$ then we can take the function $\frac{f}{z_1}$ of class $C^\infty(\bar{U}_0)$ and therefore the result is true.

2) $1 \le i < i_1$

In this case (4.1) is clear.

3) $i_1 \le i \le i_0$

We will write Π instead Π_i. Thus

$$f(z) = f(z) - f(\Pi(z)) = \int_D f(\zeta)\,(\,R^s(\zeta,z) - R^s(\zeta,\Pi(z))\,)\,d\zeta =$$
$$\int_D f(\zeta) \left(\frac{(-\rho(\zeta))^s\,\varphi(\zeta,z)}{a(\zeta,z)^{n+1+s}} - \frac{(-\rho(\zeta))^s\,\varphi(\zeta,\Pi(z))}{a(\zeta,\Pi(z))^{n+1+s}} \right) d\zeta$$

where $\varphi(\zeta,z)$ is a function of class $C^\infty(\bar{D}\times\bar{D}\,)$ and holomorphic in z.

Using (4.2) $\Pi(z) - z = z_1\,h^1 - d|z_1|^2\,\frac{\partial\rho}{\partial z''}$ where h^1 is a tangential complex vector, and thus we have

i) $\varphi(\zeta,z) - \varphi(\zeta,\Pi(z)) = z_1\psi'(\zeta,z)$ with $\psi'(\zeta,z)$ of class $C^\infty(\bar{D}\times\bar{D})$
ii) $a(\zeta,z) - a(\zeta,\Pi(z)) = z_1\,\psi''(\zeta,z)$ with $\psi''(\zeta,z)$ of class $C^\infty(\bar{D}\times\bar{D})$ and $|\Psi''(\zeta,z)| = O(|\zeta-z| + |\zeta-\Pi(z)|)$. (See (4.3)).

Hence, we have

$$\chi_i(z)\frac{f(z)}{z_1} = \int_D f(\zeta)\frac{(-\rho(\zeta))^s\chi_i(z)\psi(\zeta,z)}{a(\zeta,z)^{n+1+s}} +$$
$$\sum_{j=0}^{n+s}\int_D f(\zeta)\,\frac{(-\rho(\zeta))^s\chi_i(z)\psi_1(\zeta,z)}{a(\zeta,\Pi(z))^{n+1+s-j}\,a(\zeta,z)^{j+1}}$$

where $\psi'(\zeta, z)$, $\psi_1(\zeta, z)$ are functions of class $\mathcal{C}^\infty(\bar{D} \times \bar{D})$ and $\psi_1(\zeta, z) \leq c\,(|\zeta - z| + |\zeta - \Pi(z)|)$.

With these notations we have that the above kernels are of the class $R_{D,\psi}^{s,-(n+\frac{3}{2}+s)}$ and therefore by Corollary 3.4 we obtain that $\chi_i(z)\frac{f(z)}{z_1}$ is a function of class $L^p_{\delta+\frac{p}{2},k}(D)$.

Thus finally $f_1 = \frac{f}{z_1}$ is of class $A^p_{\delta+\frac{p}{2},k}(D)$. ■

Definition 4.6.

We say that the holomorphic submanifold $Y = \{\, z;\, u_1(z) = \ldots = u_l(z) = 0 \,\}$ is totally transversal to the boundary of D if for every $1 \leq j_1 < \ldots < j_s \leq l$, $Y_J = \{\, z;\, u_{j_1}(z) = \ldots = u_{j_s}(z) = 0 \,\}$ is a holomorphic submanifold of codimension s and transversal to the boundary of D.

Proposition 4.7.

If $Y = \{\, z : z_1 = \ldots = z_l = 0 \,\}$ is a holomorphic submanifold totally transversal to the boundary of D and f is a function of class $A^p_{\delta,k}(D)$ such that is zero on M, then there exist functions f_j, $j = 1, \ldots, l$, of class $A^p_{\delta+\frac{p}{2},k}(D)$ such that

$$f = \sum_{j=1}^{l} z_j\, f_j$$

Moreover, for all $j = 1, \ldots, l$ the functions $z_j\, f_j$ are of class $A^p_{\delta,k}(D)$.

Proof:

We will construct the functions f_j inductively.

Say $Y_m = \{\, z : z_{m+1} = \ldots = z_l = 0 \,\}$, $Y_l = C^n$ and $M_m = Y_m \cap D$.

Using the hypothesis of total transversavility we have that for each m, M_m is a strictly pseudoconvex domain with boundary of class $\mathcal{C}^\infty$ and that Y_{m-1} is transversal to the boundary of M_m.

By (1.1) we say that $f|_{M_1}$ is a function of class $A^p_{\delta+l-1,k}(M_1)$ that is zero on M_0 and hence by Proposition 4.5 there exists a function h_1 of class $A^p_{\delta+l-1+\frac{p}{2}}(M_1)$ such that $f = z_1\, h_1$ on M_1.

We define $f_1(z) = \int_{M_1} R^s_{M_1}(\zeta, z)\, h_1(\zeta)\, d\zeta$ where $R^s_{M_1}$ is the extension operador (2.3).

By Lemma 4.2 we have that f_1 is of class $A^p_{\delta+\frac{p}{2},k}(D)$.

Also putting

$$z_1 f_1(z) = \int_{M_1} (z_1 - \zeta_1) f_1(\zeta)\, R^s_{M_1}(\zeta, z)\, d\zeta + \int_{M_1} f(\zeta)\, R^s_{M_1}(\zeta, z)\, d\zeta$$

and using that $|z_1 - \zeta_1| \le c\,|a(\zeta,z)|^{\frac{1}{2}}$ and the Corollary 4.3 we have that $z_1 f_1$ is of class $A^p_{\delta,k}(D)$.

If we consider the function $f - z_1 f_1$ and we repeat the above method on M_2 we will find f_2, and by iteration we will obtain the remaining f_j. ∎

We introduce the following covering of D which is a variation of the one of A.Cumenge [**9**].

Lemma 4.8.

For $0 < \varepsilon_1 < \ldots < \varepsilon_{r_0}$ there exist points $\{z_i\}_{i=1,\ldots,i_0}$ of D and strictly pseudoconvex domains with C^∞ boundary $\{D_i^r\}_{i=1,\ldots,i_0}^{r=1,\ldots,r_0}$, such that:

i) $B(w_i,\varepsilon_{r-1}) \cap D \subset D_i^r \subset B(w_i,\varepsilon_r) \cap D$ *if* $1 \le r \le r_0$.

ii) $\cup_{i=1}^{i_0} D_i^1 = D$.

iii) *If* $i_1 < i \le i_0$ *there is* $1 \le i_j \le l$ *such that* $u_{i_j} \ne 0$ *in* D_i^r.

iv) *If* $1 \le i \le i_1$ *then*

  a) $D_i^r \cap Y \ne \emptyset$.

  b) *For every* $D_i^{r_0}$ *there exists a holomorphic system of coordinates such that the* l *first are* $u_1,\ldots,u_l$.

v) Y *is totally transversal to* D_i^r *for all* $1 \le i \le i_1$, $1 \le r \le r_0$.

vi) *If* $r < r'$ *and* $D_{i_1}^r \cap \ldots \cap D_{i_s}^r \ne \emptyset$ *then there exists a strictly pseudoconvex domain* D_I^r *with* C^∞ *boundary, such that*

  a) $D_{i_1}^r \cap \ldots D_{i_s}^r \subset D_I^r \subset D_{i_1}^{r'} \cap \ldots \cap D_{i_s}^{r'}$.

  b) *If* $D_I^r \cap Y \ne \emptyset$ *then* Y *is totally transversal to the boundary of* D_I^r.

Proof of Theorem 1.1:

We take the covering of D of the Lemma 4.8. We fix an r and we write D_i instead D_i^r.

By Proposition 4.7 in each D_i we have:

$$f(z) = \sum_{j=1}^{l} u_j(z)\, f_j^i(z)$$

$$f_j^i(z) \in A^p_{\delta+\frac{p}{2},k}(D_i)\,, \qquad u_j\, f_j^i \in A^p_{\delta,k}(D_i)$$

We define $g_j(z) = \sum_i \chi_i(z)\, f_j^i(z)$ where $\{\chi_i\}$ is a partition of the unity with respect to the covering $\{D_i\}$.

It is clear that $\sum_{j=1}^{l} u_j\, g_j = f$.

For each j we denote by w_j the solution of the equation $\bar\partial w_j = \bar\partial g_j$ given by the Lemma 4.3 and we put

$$f = \sum_{j=1}^{l} u_j\,(g_j - w_j) + \sum_{j=1}^{l} u_j\,w_j.$$

By Lemma 4.4 and (1.2) we have

$$h_j = g_j - w_j \in A^p_{\delta+\frac{p}{2},k}(D)$$
$$h = \sum_{j=1}^{l} u_j\,w_j \in A^p_{\delta+\frac{p}{2},k+1}(D).$$

Hence, we have proved that for every function $f \in A^p_{\delta,k}(D)$ that is zero on M, there exist functions $h_j \in A^p_{\delta+\frac{p}{2},k}(D)$ and $h \in A^p_{\delta+\frac{p}{2},k+1}(D)$ such that

i) $f = \sum_{j=1}^{l} u_j\,h_j + h$
ii) h is zero on M.

Iterating this method with the function h we obtain

i) $f = \sum_{j=1}^{l} u_j\,h_j^r + h^r$
ii) $h^r \in A^p_{\delta+\frac{rp}{2},k+r}(D)$ and is zero on M
iii) $h_j^r \in A^p_{\delta+\frac{p}{2},k}(D) \quad j = 1,\dots,l.$

Taking r such that $t = k - \frac{n+\delta}{p} + \frac{r}{2} > k + \frac{1}{2}$ and applying (1.1) we have that h^r is a holomorphic Lipschitz function of class $A^t(D)$ that is zero on M. Therefore by a result of [6] we have

$$h^r = \sum_{j=1}^{l} u_j\,h_j^{r+1}, \qquad h_j^{r+1} \in A^{t-\frac{1}{2}}(D) \subset C^k(\bar D) \cap \mathcal{O}(D).$$

Finally, if we define $f_j = h_j^r + h_j^{r+1}$ we end the proof. ■

V. Extension of $A^p_{\delta,k}$-jets

First we prove the extesion result in the linear case.

Theorem 5.1.

If the linear submanifold $Y = \{z \in C^n;\, z_1 = \dots = z_l = 0\}$ is transversal to the boundary of D and F is an $A^p_{\delta,k}$-jet of order m on M

then there exists a function f of class $A^p_{\delta,k}(D)$ such that $J_m f = F$ on M.

Proof:

First we consider the case $Y = \{ z;\, z_1 = 0 \}$.

We take $s > \frac{\delta}{p}$ and for $j = 0, \dots, m$ we define by induction the functions

$$
(5.1)\qquad
\begin{aligned}
g_0 &= E^s F^0 \\
&\dots\dots\dots\dots \\
g_j &= g_{j-1} + \frac{z_1^j}{j!} E^s \left(F^j - d^j g_{j-1} \right) \left(\frac{\partial}{\partial \zeta_1}, \dots, \frac{\partial}{\partial \zeta_1} \right)
\end{aligned}
$$

where the operator E^s is the extension operator (2.4) given by the kernel R^s_M.

It is clear that the function $f = g_m$ satisfies $J_m f = F$ on M.

To prove the Theorem we will show by induction on the index j in (5.1) that the functions g_j are of class $A^p_{\delta,k}(D)$.

If $j = 0$, using that $R^s_M = R^{s,-(n+1+s)}_{M,\psi}$ and applying the Corolary 4.3, we obtain the result.

Now we assume that $g_{j-1} \in A^p_{\delta,k}(D)$. As follows from (5.1), to prove that $g_j \in A^p_{\delta,k}(D)$ is sufficient to see that

$$
h_j = z_1^j \int_M R^s_M \left(F^j - d^j g_{j-1} \right) \left(\frac{\partial}{\partial \zeta_1}, \dots, \frac{\partial}{\partial \zeta_1} \right)
$$

is of class $A^p_{\delta,k}(D)$.

Consider the normal complex field

$$
N = \frac{1}{|\partial \rho|^2} \sum_{i=1}^n \frac{\partial \rho}{\partial \bar\zeta_i} \frac{\partial}{\partial \zeta_i}
$$

defined in a neighbourhood of the boundary of D, and the decomposition of the vector field

$$
Z = \sum_{i=1}^n (z_i - \zeta_i) \frac{\partial}{\partial \zeta_i} = \sum_{i=1}^n (z_i - \zeta_i) \left(\frac{\partial}{\partial \zeta_i} - \chi \frac{\partial \rho}{\partial \zeta_i} N \right) + \chi Z \rho N
$$

where χ is a function with compact support and that is 1 in a neighbourhood of the boundary of D.

We denote by T_i the complex tangent vector field $T_i = \frac{\partial}{\partial \zeta_i} - \frac{\partial \rho}{\partial \zeta_i} N$.

With these notations and by the properties I-1, I-2 and I-3 of the Definition 1.1, we can write

$$h_j = \int_M R^p_M \left(F^j - d^j g_{j-1} \right) (z - \zeta, \dots, z - \zeta) =$$

$$\sum_{|\beta|=j} \int_M R^p_M (z_1 - \zeta_1)^{\beta_1} \dots (z_n - \zeta_n)^{\beta_n} (Z \rho)^{\beta_{n+1}} g_\beta$$

where $g_\beta = (F^j - d^j g_{j-1})(T_1, \overset{(\beta_1)}{\dots}, T_1, \dots, N, \overset{(\beta_{n+1})}{\dots}, N)$.

Observe that by the hypothesis of induction and the property I-4, we have that the function g_β is of class $L^p_{\delta + \frac{\beta_1 + \dots + \beta_n}{2} + \beta_{n+1}, k}(M)$.

Moreover, using that $|\zeta - z|^2, |Z\rho| \leq c|a(\zeta, z)|$ we can write

$$h_j = \sum_{|\beta|=j} R^{s, r_\beta - (n+1-l+s)}_{M, \psi_\beta} g_\beta, \qquad r_\beta = \frac{\beta_1 + \dots + \beta_n}{2} + \beta_{n+1}$$

and applying the Corollary 4.3 we end the proof in this case.

The proof in the case $Y = \{ z; z_1 = \dots = z_l = 0 \}$ is similar to the case $Y = \{ z; z_1 = 0 \}$. In the same way, in this case the function f is defined by $f = g_m$, where

$$g_0 = E^p F^0$$

....

$$g_j = g_{j-1} + E^p \left(\left(F^j - d^j g_{j-1} \right) (z - \zeta, \dots, z - \zeta) \right). \blacksquare$$

Before proving the Theorem 1.3 we introduce the following definition.

Definition 5.2.

For every $\varepsilon \geq 0$ small enough, we define

$$D_\varepsilon = \{ \zeta; \rho(\zeta) - \varepsilon |u(\zeta)|^2 < 0 \}$$

where $|u|^2 = |u_1|^2 + \dots + |u_l|^2$.

It is clear that these domains are strictly pseudoconvex domains with C^∞ boundary, $D_\varepsilon \cap Y = M$ and Y is transversal to D_ε.

Lemma 5.2.

If $f \in L^p_{\delta,k}(D_{\varepsilon'})$, $\delta > \frac{p}{2}$, then $u_j f \in L^p_{\delta - \frac{p}{2}, k}(D_\varepsilon)$ for every $j = 1, \dots, l$, and $0 \leq \varepsilon < \varepsilon'$.

Proof:

The result is a consequence of the fact that

$$|u_j| \le \frac{1}{(\varepsilon' - \varepsilon)^{\frac{1}{2}}} (-\rho + \varepsilon' |u|^2)^{\frac{1}{2}}, \qquad \text{on} \quad D_\varepsilon$$

for all $\delta^* \geq \delta - p$, $\delta^* > 0$. ■

Proof of Theorem 1.3:

We take a covering $\{D_i^r\} = \{D_i^r\}_{i=1,\dots,N}^{0 \le r \le r_0}$ of D as the one in the Lemma 4.8 and we also consider the domains $\{D_{i,\varepsilon}^r\}$, $\varepsilon \geq 0$.

We also take $0 < r < r'' < r'$, $0 < \varepsilon < \varepsilon'$.

By Proposition 5.1 we have that for every $D_{i,\varepsilon'}^{r'}$ such that $D_i \cap Y \neq \emptyset$, there exists a function $f_i \in A_{\delta,k}^p(D_{i,\varepsilon'}^{r'})$ such that $J_m f_i = F$ on $Y \cap D_i$.

Using (1.2) we can assume that $\delta > p$.

For the remaining D_i we define $f_i = 0$.

We consider the function $g = \sum_i \chi_i f_i$ where χ_i is a partition of the unity with respect to the $\{D_{i,\varepsilon'}^{r'}\}$.

This function g is of class $L_{\delta,k}^p(D)$ and verifies $J_m g = F$.

Let $w \in L_{\delta-\frac{p}{2},k}^p(D)$ be the solution of the $\bar\partial w = \bar\partial g$ given by Lemma 4.4.

Note that $h = g - w \in A_{\delta,k}^p(D)$ and that $F = J_m h + J_m w$.

The next step is to see that $J_m w$ is an $A_{\delta+\frac{p}{2},k+1}^p$-jet.

We say $f_{ij} = f_i - f_j$ in $D_{ij}^{r''} \subset D_i^{r'} \cap D_j^{r'}$.

Using the Theorema 1.1 we can write

$$f_{ij} = \sum_{|\gamma|=m+1} u^\gamma g_{ij}^\gamma, \qquad g_{ij}^\gamma \in A_{\delta+\frac{(m+1)p}{2},k}^p(D_{ij,\varepsilon'}^{r''}).$$

We define in D_i^r the function $g_i^\gamma = \sum_s \chi_s g_{is}^\gamma$.

This function satisfies

$$\sum_{|\gamma|=m+1} u^\gamma g_i^\gamma = f_i - \sum_s \chi_s f_s = f_i - g.$$

By Lemma 4.4 we can take w_i^γ such that

$$\bar\partial w_i^\gamma = \bar\partial g_i^\gamma \qquad , \qquad w_i^\gamma \in L_{\delta+\frac{mp}{2},k}^p(D_{i,\varepsilon'}^{r''}).$$

Moreover, using the Lemma 4.4, the Lemma 5.2 and (1.2) we have that

$$h_i' = w - \sum_{|\gamma|=m+1} u^\gamma w_i^\gamma \in A^p_{\delta+\frac{p}{2},k+1}(D^r_{i,\epsilon})$$

and also $J_m h_i' = J_m g$ on $Y \cap D_i^r$.

Hence, we have that $J_m g$ is a $A^p_{\delta+\frac{p}{2},k+1}$-jet of order m on M.

By iteration of this method we obtain

$$F = J_m h^s + J_m g^s$$

with

$$h^s \in A^p_{\delta,k}(D) \qquad \text{and} \qquad J_m g^s \ \text{ is in } \ A^p_{\delta+\frac{sp}{2},k+s}\text{-jet.}$$

Now if we take s such that $t = k + s - \frac{n+\delta}{p} - \frac{s}{2} > k + \frac{1}{2}$, then (1.1), (1.4) and (1.6) shows that $J_m g^s$ is a A^t-jet of order m. Finally applying the extension result of A^t-jets (1.6) we can take a function h of class $A^t(D)$ such that $J_m h = J_m g^s$ on M and defining $f = h^s + h$ we end the proof. ■

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