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A NOTE ON SUPERSOLUBLE MAXIMAL SUBGROUPS AND THETA-PAIRS

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Abstract

A θ-pair for a maximal subgroup M of a group G is a pair (A, B) of subgroups such that B is a maximal G-invariant subgroup of A with B but not A contained in M. θ-pairs are considered here in some groups having supersoluble maximal subgroups.

1. Introduction

Let M be a maximal subgroup of the group G. An ordered pair (A, B) of subgroups of G is called a θ-pair for M if B is a G-invariant subgroup of A such that (i) $B \leq M$ but $A \not\leq M$ and (ii) A/B contains properly no nontrivial normal subgroup of G/B.

The set of all θ-pairs for M is denoted by $\theta(M)$ (see [**3**]). A partial order is defined on $\theta(M)$ by means of $(A, B) \leq (C, D)$ if and only if $A \leq C$. In this case $B \leq D$ also. It is then clear what is meant by saying that (A, B) is a maximal θ-pair for M. If (A, B) is in $\theta(M)$ and $A \lhd G$ then A/B is a chief factor of G.

This brief note is concerned with θ-pairs in relation to the property of supersolubility. Our principal result is Theorem 1, which bears some relation to Theorem 1 of [**1**]. It will be seen that Theorem 1 is an easy consequence of Theorem 2. The concepts and results found here can be found in [**4**].

Let $\mathrm{Fit}(G)$ denote the Fitting subgroup of the group G. The main results presented here are as follows.

Theorem 1. *Let G be a group with a supersoluble maximal subgroup M and suppose that* $\mathrm{Fit}(G) \cap M$ *is a maximal subgroup of* $\mathrm{Fit}(G)$. *Then G is supersoluble.*

Theorem 2. *Let G be a group and M a supersoluble maximal subgroup of G not containing* $\mathrm{Fit}(G)$. *If $\theta(M)$ has a maximal pair (A, B) such that A/B is cyclic and A is subnormal in G then G is supersoluble.*

An argument similar to that employed in proving Theorem 2 allows us to establish the following result (the proof of which is omitted).

Theorem 3. *Let G be a group and M a supersoluble maximal subgroup not containing* $\mathrm{Fit}(G)$. *If A/B is cyclic for each maximal pair (A, B) in $\theta(M)$ then G is supersoluble.*

2. Proofs

We require two preliminary lemmas.

Lemma 1. *Let G be a group and M a maximal subgroup of finite index in G. Let (A, B) be a maximal θ-pair for M. Then, given any G-invariant subgroup N of finite index in M, there exists a maximal θ-pair $(C/N, D/N)$ for M/N such that C/D is isomorphic to a normal section of A/B. Further, if A is subnormal in G then C may be chosen subnormal in G.*

Proof: If $N \leq B$ then $(A/N, B/N)$ is a maximal member of $\theta(M/N)$ and there is nothing to prove. Suppose that N is not contained in B. Then N is not contained in A, otherwise $A = BN \leq M$, a contradiction. Let K be the normal core of $AN \cap M$ in G. Then $BN \leq K$. Since $A < AN$ and (A, B) is maximal, (AN, K) is not in $\theta(M)$. Let H/K be a minimal G-invariant subgroup of (the finite group) AN/K. Then (H, K) belongs to $\theta(M)$ and is contained in some maximal member (C, D) of $\theta(M)$. Now $C = HD$ is normal in G and $C/D = HD/D \cong H/H \cap D$, an image of H/K, which is, in turn, normal in the image AN/K of A/B. Finally, $(C/N, D/N)$ is a maximal member of $\theta(M/N)$. Note that $C = A$ in the case where $N \leq B$, while if $N \not\leq B$ then $C \lhd G$. ∎

Note that some of the ideas in the proof of Lemma 2.1 of [**3**] are used to establish Lemma 1.

Lemma 2. *Let G be a group and M a polycyclic maximal subgroup of G not containing* $\mathrm{Fit}(G)$. *Then G is polycyclic.*

Proof: Let N be a nilpotent normal subgroup of G not contained in M. Then $G = MN$ and so G is soluble. We may assume M is core-free in G. Then G is a soluble primitive group and it is known that G has a unique non-trivial abelian normal subgroup A which satisfies $G = MA$, $A \cap M = 1$ and $A = C_G(A)$. Thus A is a simple ZM-module and, by a result of Roseblande [**5**, p. 308], A is finite. Therefore, G is polyciclic. ∎

Proof of Theorem 2: By Lemma 2, G is polycyclic and so, by a theorem of Baer [**6**, 11.11], it suffices to prove that every finite image G/N of G is supersoluble. Clearly we may assume that $N \leq M$ and hence, by Lemma 1, that G is finite. Suppose that G is not supersoluble and let T be a nontrivial normal subgroup of G. By Lemma 1 and an obvious induction, G/T is supersoluble. Thus G has a unique minimal normal subgroup W and G/W is supersoluble. If $\phi(G) \neq 1$ then $W \leq \phi(G)$ and G is supersoluble, by a result of Huppert [**4**, 9.4.5]. Thus $\phi(G) = 1$ and $\mathrm{Fit}(G) = W$, by a result of Gaschütz [**4**, 5.2.15]. Since $W \nleq M$ we see that $M_G = 1$ and hence $B = 1$ and A is cyclic and subnormal in G. Thus $A \leq W$. Certainly $(W, 1)$ belongs to $\theta(M)$ and so, by maximality, $A = W$. Thus G is supersoluble and we have the required contradiction. ■

Proof of Theorem 1: By Lemma 2, G is polycyclic and so, by a result of Hirsch [**4**, 5.4.19], $\phi(G) \leq \mathrm{Fit}(G)$ and $\mathrm{Fit}(G)/\phi(G) = \mathrm{Fit}(G/\phi(G))$. Hence, by a result of Lennox [**2**], we may assume that $\phi(G) = 1$. Since G is polyciclic, $F = \mathrm{Fit}(G)$ is nilpotent ([**4**, p. 129]) and consequently every maximal subgroups of F is normal and of prime index in F. Therefore, F is abelian and hence $F \cap M$ is normal in G and of prime index in F. It follows that $(F, F \cap M) \in \theta(M)$. Let (A, B) be a maximal member of $\theta(M)$ containing $(F, F \cap M)$. Then either $FB = A$ or $B \leq F = A$. In either case, A/B is cyclic and A is normal in G. By Theorem 2, G is supersoluble. ■

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