

GALOIS $\mathbf{H}$ -OBJECTS WITH A NORMAL BASIS IN CLOSED CATEGORIES. A COHOMOLOGICAL INTERPRETATION

J. N. ALONSO ÁLVAREZ AND J. M. FERNÁNDEZ VILABOA

Abstract

In this paper, for a cocommutative Hopf algebra $\mathbf{H}$ in a symmetric closed category $\mathcal{C}$ with basic object K , we get an isomorphism between the group of isomorphism classes of Galois $\mathbf{H}$ -objects with a normal basis and the second cohomology group $H^2(\mathbf{H}, K)$ of $\mathbf{H}$ with coefficients in K . Using this result, we obtain a direct sum decomposition for the Brauer group of $\mathbf{H}$ -module Azumaya monoids with inner action:

$$\mathbf{B}M_{\text{inn}}(\mathcal{C}, \mathbf{H}) \cong \mathbf{B}(\mathcal{C}) \oplus H^2(\mathbf{H}, K)$$

In particular, if $\mathcal{C}$ is the symmetric closed category of K -modules with K a field, $H^2(H, K)$ is the second cohomology group introduced by Sweedler in [21]. Moreover, if H is a finitely generated projective, commutative and cocommutative Hopf algebra over a commutative ring with unit K , then the above decomposition theorem is the one obtained by Beattie [5] for the Brauer group of H -module algebras.

Preliminary

A monoidal category $(\mathcal{C}, \otimes, K)$ consists of a category $\mathcal{C}$ with a bifunctor $-\otimes- : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and a basic object K , and with natural isomorphisms:

$$\begin{aligned}a_{ABC} : A \otimes (B \otimes C) &\cong (A \otimes B) \otimes C \\l_A : K \otimes A &\cong A \\r_A : A \otimes K &\cong A\end{aligned}$$

such that

$$\begin{aligned}(A \otimes a_{BCD}) \circ a_{A(B \otimes C)D} \circ (a_{ABC} \otimes D) &= a_{AB(C \otimes D)} \circ a_{(A \otimes B)CD} \\(A \otimes l_B) \circ a_{AKB} &= r_A \otimes B\end{aligned}$$

If there is a natural isomorphism $\tau_B^A : A \otimes B \cong B \otimes A$ such that $\tau_A^B \circ \tau_B^A = A \otimes B$, $\tau_{B \otimes C}^A = (B \otimes \tau_C^A) \circ (\tau_B^A \otimes C)$, then $\mathcal{C}$ is called a symmetric monoidal category.

A closed category is a symmetric monoidal category in which each functor $- \otimes A : \mathcal{C} \rightarrow \mathcal{C}$ has a specified right adjoint $[A, -] : \mathcal{C} \rightarrow \mathcal{C}$ ([12], [18]).

Examples:

- 1) The category of sets and mappings.
- 2) The category of R -modules over a commutative ring R .
- 3) The category of chain complexes of R -modules and morphisms of degree 0, with R a commutative ring.
- 4) The category of sheaves of θ -modules over a topological space, with θ a sheaf of commutative rings.
- 5) The category of coherent sheaves of modules over a scheme.
- 6) The category of all R -graded modules with morphisms of degree 0 (R is a commutative graded ring).
- 7) $(R, \sigma)\text{-Mod}$, with R a commutative ring and σ an idempotent kernel functor in $R\text{-Mod}$.

In what follows, $\mathcal{C}$ denotes a symmetric closed category with equalizers, co-equalizers and projective basic object K . We denote by α_M and β_M the unit and the co-unit, respectively, of the $\mathcal{C}$ -adjunction $M \otimes - \dashv [M, -] : \mathcal{C} \rightarrow \mathcal{C}$ which exists for each object M of $\mathcal{C}$.

1. An object M of $\mathcal{C}$ is called profinite in $\mathcal{C}$ if the morphism $[M, \beta_M(K) \otimes M] \circ \alpha_M(\hat{M} \otimes M) : \hat{M} \otimes M \rightarrow [M, M] = E(M)$ is an isomorphism, where $\hat{M} = [M, K]$. If, moreover, the factorization of $\beta_M(K) : M \otimes \hat{M} \rightarrow K$ through the co-equalizer of the morphisms $\beta_M(M) \otimes \hat{M}$ and $M \otimes ([M, \beta_M(K) \circ (\beta_M(M) \otimes \hat{M})] \circ \alpha_M(E(M) \otimes \hat{M})) : M \otimes E(M) \otimes \hat{M} \rightarrow M \otimes \hat{M}$ is an isomorphism, we say that M is a progenerator in $\mathcal{C}$.

2. A monoid in $\mathcal{C}$ is a triple $\mathbf{A} = (A, \eta_A, \mu_A)$ where A is an object in $\mathcal{C}$ and $\mu_A : A \otimes A \rightarrow A$, $\eta_A : K \rightarrow A$ are morphisms in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (\mu_A \otimes A) = \mu_A \circ (A \otimes \mu_A)$. If $\mu_A \circ \tau_A^A = \mu_A$, then we will say that $\mathbf{A}$ is a commutative monoid. Given two monoids $\mathbf{A} = (A, \eta_A, \mu_A)$ and $\mathbf{B} = (B, \eta_B, \mu_B)$ in $\mathcal{C}$, $f : A \rightarrow B$ is a monoid morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$.

A comonoid (cocommutative), $\mathbf{D} = (D, \varepsilon_D, \delta_D)$ is an object D in $\mathcal{C}$ together with two morphisms $\varepsilon_D : D \rightarrow K$, $\delta_D : D \rightarrow D \otimes D$, such that $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$ and $(\varepsilon_D \otimes D) \circ \delta_D = 1_D = (D \otimes \varepsilon_D) \circ \delta_D$ ($\tau_D^D \circ \delta_D = \delta_D$). If $\mathbf{D} = (D, \varepsilon_D, \delta_D)$ and $\mathbf{E} = (E, \varepsilon_E, \delta_E)$ are comonoids, $f : D \rightarrow E$ is a comonoid morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$ and $\varepsilon_E \circ f = \varepsilon_D$.

3. For a monoid $\mathbf{A} = (A, \eta_A, \mu_A)$ and a comonoid $\mathbf{D} = (D, \varepsilon_D, \delta_D)$ in $\mathcal{C}$, we denote by $\text{Reg}(D, A)$ the group of invertible elements in $\mathcal{C}(D, A)$ (morphisms in $\mathcal{C}$ from D to A) with the operation "convolution" given by: $f * g = \mu_A \circ (f \otimes g) \circ \delta_D$. The unit element is $\varepsilon_D \otimes \eta_A$.

Observe that $\text{Reg}(D, A)$ is an abelian group when $\mathbf{D}$ is cocommutative and $\mathbf{A}$ is commutative.

4. **Definition.** Let $\Pi = (C, \eta_C, \mu_C)$ be a monoid and $\mathbf{C} = (C, \varepsilon_C, \delta_C)$ a comonoid in $\mathcal{C}$ and let $\lambda : C \rightarrow C$ be a morphism. Then $\mathbf{H} = (\mathbf{C} = (C, \varepsilon_C, \delta_C), \Pi = (C, \eta_C, \mu_C), \tau^C, \lambda)$ is a Hopf algebra in $\mathcal{C}$ with respect to the comonoid $\mathbf{C}$ if ε_C and δ_C are monoid morphisms (equivalently, η_C and μ_C are comonoid morphisms) and λ is the inverse of $1_C : C \rightarrow C$ in $\text{Reg}(C, C)$.

We say that $\mathbf{H}$ is a finite Hopf algebra if C is profinite in $\mathcal{C}$.

5. **Definition.** $(\mathbf{A}, \varphi_A) = (A, \eta_A, \mu_A; \varphi_A)$ is a left $\mathbf{H}$ -module monoid if:

- i) $\mathbf{A} = (A, \eta_A, \mu_A)$ is a monoid in $\mathcal{C}$.
- ii) (A, φ_A) is a left $\mathbf{H}$ -module ($\varphi_A \circ (C \otimes \varphi_A) = \varphi_A \circ (\mu_C \otimes A)$, $\varphi_A \circ (\eta_C \otimes A) = A$).
- iii) η_A, μ_A are morphisms of left $\mathbf{H}$ -modules ($\varphi_A \circ (C \otimes \eta_A) = \eta_A \otimes \varepsilon_C$ and $\varphi_A \circ (C \otimes \mu_A) = \mu_A \circ \varphi_{A \otimes A}$, where $\varphi_{A \otimes A} = (\varphi_A \otimes \varphi_A) \circ (C \otimes \tau_A^C \otimes A) \circ (\delta_C \otimes A \otimes A)$).

We say that the action φ_A of $\mathbf{H}$ in $\mathbf{A}$ is inner if there exists a morphism f in $\text{Reg}(C, A)$ such that $\varphi_A = \mu_A \circ (A \otimes (\mu_A \circ \tau_A^A)) \circ (f \otimes f^{-1} \otimes A) \circ (\delta_C \otimes A) : C \otimes A \rightarrow A$, where f^{-1} is the convolution inverse of f .

6. **Definition.** If $\mathbf{H}$ is a cocommutative Hopf algebra and $(\mathbf{A}, \varphi_A)$ is a commutative $\mathbf{H}$ -module monoid, then, we say that a morphism σ in $\text{Reg}(C \otimes C, A)$ is a 2-cocycle if $\partial_1(\sigma) * \partial_3(\sigma) = \partial_2(\sigma) * \partial_4(\sigma)$, where $\partial_1(\sigma) = \varphi_A \circ (C \otimes \sigma)$, $\partial_2(\sigma) = \sigma \circ (\mu_C \otimes C)$, $\partial_3(\sigma) = \sigma \circ (C \otimes \mu_C)$ and $\partial_4(\sigma) = \sigma \otimes \varepsilon_C$.

Two 2-cocycles σ and γ are said to be cohomologous, written $\sigma \sim \gamma$, if there exists a morphism $v \in \text{Reg}(C, A)$ such that $\sigma * \partial_2(v) = \partial_1(v) * \partial_3(v) * \gamma$, where $\partial_1(v) = \varphi_A \circ (C \otimes v)$, $\partial_2(v) = v \circ \mu_C$ and $\partial_3(v) = v \otimes \varepsilon_C$.

Trivially, " $\sim$ " is an equivalence relation.

The set of equivalence classes shall be called the second cohomology group of the cocommutative Hopf algebra $\mathbf{H}$ with the coefficients in the left $\mathbf{H}$ -module monoid $(\mathbf{A}, \varphi_A)$, and will be denoted by $H^2(\mathbf{H}, \mathbf{A})$.

If σ is a 2-cocycle in $\text{Reg}(C \otimes C, A)$, then the morphism $\bar{\sigma} = \sigma * \partial_1(\pi) * \partial_2(\pi^{-1}) * \partial_3(\pi)$ is a 2-cocycle in $\text{Reg}(C \otimes C, A)$ cohomologous with σ such that $\bar{\sigma} \circ (\eta_C \otimes C) = \varepsilon_C \otimes \eta_A = \bar{\sigma} \circ (C \otimes \eta_C)$, where $\pi = \sigma^{-1} \circ (C \otimes \eta_C)$ is a morphism in $\text{Reg}(C, A)$ with inverse $\pi^{-1} = \sigma \circ (C \otimes \eta_C)$. Moreover, if

γ is cohomologous with σ , then there exists a morphism $\bar{\vartheta} \in \text{Reg}(C, A)$ such that $\bar{\vartheta} \circ \eta_C = \eta_A$ and $\bar{\sigma} * \partial_2(\bar{\vartheta}) = \partial_1(\bar{\vartheta}) * \partial_3(\bar{\vartheta}) * \bar{\gamma}$.

Remark. Let $\mathcal{C}$ the category of K -modules over a field. In this case, $H^2(\mathbf{H}, \mathbf{A})$ is the second cohomology group of the Sweedler's complex $\{\text{Reg}(\otimes^q C, A); \Delta_q\}_{q \geq 0}$

$$\begin{aligned} \text{Reg}(K, A) \xrightarrow{\Delta_0} \text{Reg}(C, A) \xrightarrow{\Delta_1} \dots \xrightarrow{\Delta_{q-1}} \text{Reg}(\otimes^q C, A) \\ \xrightarrow{\Delta_q} \text{Reg}(\otimes^{q+1} C, A) \xrightarrow{\Delta_{q+1}} \dots \end{aligned}$$

where $\Delta_q := \partial_1 * \partial_2^{-1} * \dots * \partial_{q+2}^{(-1)^{q+1}}$ and for each $f \in \text{Reg}(\otimes^q C, A)$,

$$\begin{aligned} \partial_1(f) &= \varphi_A \circ (C \otimes f) \\ &\vdots \\ \partial_i(f) &= f \circ (C \otimes \overset{i-2}{\tau} \otimes C \otimes \mu_C \otimes C \otimes \overset{q-i+1}{\tau} \otimes C) \\ &\vdots \\ \partial_{q+2}(f) &= f \otimes \varepsilon_C \end{aligned}$$

([21]).

7. Definition. $(\mathbf{B}, \rho_B) = (B, \eta_B, \mu_B; \rho_B)$ is a right $\mathbf{H}$ -comodule monoid if:

- i) $\mathbf{B} = (B, \eta_B, \mu_B)$ is a monoid in $\mathcal{C}$
- ii) (B, ρ_B) is a right $\mathbf{H}$ -comodule $((\rho_B \otimes C) \circ \rho_B = (B \otimes \delta_C) \circ \rho_B; (B \otimes \varepsilon_C) \circ \rho_B = B)$.
- iii) $\rho_B : B \rightarrow B \otimes C$ is a monoid morphism from (B, η_B, μ_B) to the product monoid $\mathbf{B} \amalg = (B \otimes C, \eta_B \otimes \eta_C, (\mu_B \otimes \mu_C) \circ (B \otimes \tau_B^C \otimes C))$ (that is, $\rho_B \circ \eta_B = \eta_B \otimes \eta_C$ and $\rho_B \circ \mu_B = (\mu_B \otimes \mu_C) \circ (B \otimes \tau_B^C \otimes C) \circ (\rho_B \otimes \rho_B)$).

From now on we assume that $\mathbf{H}$ is a finite cocommutative and commutative Hopf algebra.

8. Definition. A right $\mathbf{H}$ -comodule monoid $(\mathbf{B}, \rho_B)$ is said to be a Galois $\mathbf{H}$ -object if and only if:

- i) The morphism $\gamma_B := (\mu_B \otimes C) \circ (B \otimes \rho_B) : B \otimes B \rightarrow B \otimes C$ is an isomorphism.
- ii) B is a progenerator in $\mathcal{C}$.

For example, in the case of $(R, \sigma)\text{-Mod}$, a commutative $\mathbf{H}$ -comodule monoid is a couple $(\mathbf{B}, \rho_B)$, where B is a commutative (R, σ) -algebra and $\rho_B : B \rightarrow B \amalg H := Q_\sigma(B \otimes_R H)$ is a morphism of algebras and it defines a right $\mathbf{H}$ -comodule structure over B .

(B, ρ_B) is a Galois $\mathbf{H}$ -object if and only if B is a (R, σ) -progenerator and the mapping $\rho_B^{\hat{H}} : B \#_{\sigma} \hat{H} \rightarrow \text{Hom}(B, B)$ arising from the left $B \#_{\sigma} \hat{H}$ -module structure on B is an isomorphism ([15, (1.3.17)]).

If a Galois $\mathbf{H}$ -object is isomorphic to $\mathbf{H}$ as an $\mathbf{H}$ -comodule then we say that it has a normal basis.

If B_1 and B_2 are Galois $\mathbf{H}$ -objects, $f : B_1 \rightarrow B_2$ is a morphism of Galois $\mathbf{H}$ -objects if it is a morphism of $\mathbf{H}$ -comodules ($\rho_{B_2} \circ f = (f \otimes C) \circ \rho_{B_1}$) and of monoids.

If (A, ρ_A) and (B, ρ_B) are $\mathbf{H}$ -comodule monoids, then $A \circ B$, defined by the following equalizer diagram

$$A \circ B \xrightarrow{i_{AB}} A \otimes B \xrightleftharpoons[\partial_{AB}^2]{\partial_{AB}^1} A \otimes B \otimes C$$

where

$$\begin{aligned} \partial_{AB}^1 &= (A \otimes \tau_B^C) \circ (\rho_A \otimes B), \text{ and} \\ \partial_{AB}^2 &= A \otimes \rho_B \end{aligned}$$

is an $\mathbf{H}$ -comodule monoid to be denoted by $(A \circ B, \rho_{AB})$.

If moreover (A, ρ_A) and (B, ρ_B) are Galois $\mathbf{H}$ -objects, then $(A \circ B, \rho_{AB})$ is also a Galois $\mathbf{H}$ -object, where ρ_{AB} is the factorization of the morphism $\partial_{AB}^1 \circ i_{AB}$ (or $\partial_{AB}^2 \circ i_{AB}$) through the equalizer $i_{AB} \otimes C$.

The set of isomorphism classes of Galois $\mathbf{H}$ -objects (with a normal basis), with the operation induced by the one given above, is an abelian group to be denoted by $\text{Gal}_C(\mathbf{H})(\mathbf{N}_C(\mathbf{H}))$. The unit element is the class of (Π, δ_C) and the opposite of $[(B, \rho_B)]$ is $[(B^{\text{op}}, (B \otimes \lambda) \circ \rho_B)]$ where $B^{\text{op}} = (B, \eta_B, \mu_B \circ \tau_B^B)$.

Remark. In the case of a finitely generated projective, commutative and cocommutative Hopf algebra $\mathbf{H}$ over a commutative ring R , $\text{Gal}_C(\mathbf{H})$ is the group of Galois $\mathbf{H}$ -objects in the sense of S. Chase and M. Sweedler in [9].

9. Proposition. *If $[(B, \rho_B)] \in \mathbf{N}_C(\mathbf{H})$, then, there is a 2-cocycle σ in $\text{Reg}(C \otimes C, K)$ satisfying $\sigma \circ (\eta_C \otimes C) = \varepsilon_C = \sigma \circ (C \otimes \eta_C)$.*

Proof.

Let (B, ρ_B) a Galois $\mathbf{H}$ -object with a normal basis. Then we have an isomorphism $\gamma_B : B \otimes B \rightarrow B \otimes C$ and an $\mathbf{H}$ -comodule isomorphism $r : C \rightarrow B$. Therefore the morphism of $\mathbf{H}$ -comodules $f = (\varepsilon_C \otimes B) \circ (r^{-1} \otimes r) \circ (\eta_B \otimes C) : C \rightarrow B$ is in $\text{Reg}(C, B)$ with inverse

$$f^{-1} = \mu_B \circ (B \otimes \varepsilon_C \otimes B) \circ (B \otimes r^{-1} \otimes r) \circ (\gamma_B^{-1} \otimes \eta_C) \circ (\eta_B \otimes C)$$

and satisfying $f \circ \eta_C = \eta_B$.

Indeed:

$$\begin{aligned}
 f * f^{-1} &= \mu_B \circ (\varepsilon_C \otimes B \otimes \varepsilon_C \otimes B) \circ (r^{-1} \otimes B \otimes r^{-1} \otimes r) \circ \\
 &\quad \circ (\eta_B \otimes \gamma_B^{-1} \otimes \eta_C) \circ (r \otimes C) \circ \delta_C = \\
 &= \mu_B \circ (\varepsilon_C \otimes B \otimes \varepsilon_C \otimes B) \circ (r^{-1} \otimes B \otimes r^{-1} \otimes r) \circ \\
 &\quad \circ (B \otimes [\gamma_B^{-1} \circ \gamma_B] \otimes C) \circ (\eta_B \otimes \eta_B \otimes B \otimes \eta_C) \circ r = \\
 &= (\varepsilon_C \otimes B \otimes \varepsilon_C) \circ (r^{-1} \otimes r \otimes C) \circ ([\rho_B \circ \eta_B] \otimes C) = \\
 &= \varepsilon_C \otimes \eta_B \\
 f^{-1} * f &= \mu_B \circ (B \otimes [\varepsilon_C \circ r^{-1}] \otimes B) \circ (\gamma_B^{-1} \otimes r) \circ (\eta_B \otimes \delta_C) = \\
 &= \mu_B \circ (B \otimes [\varepsilon_C \circ r^{-1}] \otimes r) \circ (B \otimes \rho_B) \circ \gamma_B^{-1} \circ (\eta_B \otimes C) = \\
 &= \mu_B \circ (B \otimes \varepsilon_C \otimes r) \circ (B \otimes \delta_C) \circ (B \otimes r^{-1}) \circ \gamma_B^{-1} \circ (\eta_B \otimes C) = \\
 &= (B \otimes \varepsilon_C) \circ \gamma_B \circ \gamma_B^{-1} \circ (\eta_B \otimes C) = \\
 &= \varepsilon_C \otimes \eta_B
 \end{aligned}$$

because r is an $\mathbf{H}$ -comodule isomorphism and the equalities:

$$\begin{aligned}
 (\mu_B \otimes B) \circ (B \otimes \gamma_B^{-1}) &= \gamma_B^{-1} \circ (\mu_B \otimes C) \\
 (\varepsilon_C \otimes B) \circ (r^{-1} \otimes r) \circ (\eta_B \otimes \eta_C) &= (\varepsilon_C \otimes B) \circ (r^{-1} \otimes r) \circ \rho_B \circ \eta_B = \eta_B \\
 (\gamma_B^{-1} \otimes C) \circ (B \otimes \delta_C) &= (B \otimes \rho_B) \circ \gamma_B^{-1}
 \end{aligned}$$

Trivially, f is a morphism of $\mathbf{H}$ -comodules and $f \circ \eta_C = \eta_B$.

The morphism $\sigma_f = (\mu_B \circ (f \otimes f)) * (f^{-1} \otimes \mu_C) : C \otimes C \rightarrow B$ factors through the equalizer

$$K \xrightarrow[\eta_C]{\eta_B} B \xrightarrow[\rho_B]{\rho_B} B \otimes C$$

because $\mathbf{H}$ is a Hopf algebra, (B, ρ_B) an $\mathbf{H}$ -comodule monoid, f a morphism of $\mathbf{H}$ -comodules and the equality $\rho_B \circ f^{-1} = (f^{-1} \otimes \lambda) \circ \tau_C^C \circ \delta_C$ ([14, (2.3)]).

Moreover, the factorization, $\bar{\sigma}_f$, of σ_f is in $\text{Reg}(C \otimes C, K)$ with inverse the factorization of the morphism $\sigma_f^{-1} = (f \circ \mu_C) * (\mu_B \circ \tau_B^B \circ (f^{-1} \otimes f^{-1})) : C \otimes C \rightarrow B$ through the equalizer η_B .

The morphism $\bar{\sigma}_f : C \otimes C \rightarrow K$ is a 2-cocycle. Indeed:

$$\begin{aligned}
 \eta_B \circ (\partial_1(\bar{\sigma}_f) * \partial_3(\bar{\sigma}_f)) &= \\
 &= (f * f^{-1}) \circ (\bar{\sigma}_f \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) \circ \\
 &\circ (C \otimes \bar{\sigma}_f \otimes \mu_C) \circ (C \otimes C \otimes \tau_C^C \otimes C) \circ (C \otimes \delta_C \otimes \delta_C) = \\
 &= \mu_B \circ (B \otimes f^{-1}) \circ \rho_B \circ f \circ (\bar{\sigma}_f \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) \circ \\
 &\circ (C \otimes \bar{\sigma}_f \otimes \mu_C) \circ (C \otimes C \otimes \tau_C^C \otimes C) \circ (C \otimes \delta_C \otimes \delta_C) = \\
 &= \mu_B \circ (B \otimes f^{-1}) \circ \rho_B \circ \mu_B \circ (f \otimes f) \circ (C \otimes \bar{\sigma}_f \otimes \mu_C) \circ \\
 &\circ (C \otimes C \otimes \tau_C^C \otimes C) \circ (C \otimes \delta_C \otimes \delta_C) = \\
 &= \mu_B \circ (B \otimes f^{-1}) \circ \rho_B \circ \mu_B \circ (\mu_B \otimes C) \circ (f \otimes f \otimes f) = \\
 &= \mu_B \circ (B \otimes f^{-1}) \circ \rho_B \circ f \circ (\bar{\sigma}_f \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) \circ \\
 &\circ (\bar{\sigma}_f \otimes \mu_C \otimes C) \circ (C \otimes \tau_C^C \otimes C \otimes C) \circ (\delta_C \otimes \delta_C \otimes C) = \\
 &= (f * f^{-1}) \circ (\bar{\sigma}_f \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) \circ \\
 &\circ (\bar{\sigma}_f \otimes \mu_C \otimes C) \circ (C \otimes \tau_C^C \otimes C \otimes C) \circ (\delta_C \otimes \delta_C \otimes C) = \\
 &= \eta_B \circ (\partial_2(\bar{\sigma}_f) * \partial_4(\bar{\sigma}_f))
 \end{aligned}$$

because f is a morphism of $\mathbf{H}$ -comodules, $\mathbf{H}$ a cocommutative Hopf algebra and the equality

$$f \circ (\bar{\sigma}_f \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \mu_B \circ (f \otimes f)$$

and then, since η_B is a monomorphism, $\bar{\sigma}_f$ is a 2-cocycle.

Trivially, $\bar{\sigma}_f \circ (\eta_C \otimes C) = \varepsilon_C = \bar{\sigma}_f \circ (C \otimes \eta_C)$. ■

Remark. If $[(B_1, \rho_{B_1})] = [(B_2, \rho_{B_2})] \in N_C(\mathbf{H})$ then there is an isomorphism of $\mathbf{H}$ -comodule monoids $h : B_1 \rightarrow B_2$. Clearly, $\bar{\sigma}_{f_1} = \bar{\sigma}_{h \circ f_1}$. (Notice that $h \circ f_1 \in \text{Reg}(C, B_2)$ with inverse $h \circ f_1^{-1}$). Moreover, $\bar{\sigma}_{f_1} \sim \bar{\sigma}_{f_2}$.

Indeed:

The morphism $e = (h \circ f_1) * f_2^{-1} : C \rightarrow B_2$ factors through the equalizer η_{B_2} :

$$\begin{aligned}
 \rho_{B_2} \circ e &= (\mu_{B_2} \otimes \mu_C) \circ ((h \circ f_1) \otimes \tau_{B_2}^C \otimes \lambda) \circ (\delta_C \otimes f_2^{-1} \otimes C) \circ \\
 &\circ (C \otimes (\tau_C^C \circ \delta_C)) \circ \delta_C = (B_2 \otimes \eta_C) \circ e
 \end{aligned}$$

and then, there exists a morphism $\bar{e} : C \rightarrow K$ such that $\eta_{B_2} \circ \bar{e} = e$. Clearly, $\bar{e} \circ \eta_C = K$. Moreover, $\bar{e}$ is in $\text{Reg}(C, K)$ with inverse $\bar{e}^{-1}$, the factorization of $e^{-1} = f_2 * (h \circ f_1^{-1})$ through the equalizer η_{B_2} .

We also have that:

$$\begin{aligned}
 \eta_{B_2} \circ (\overline{\sigma_{f_1}} * \partial_2(\bar{e})) &= \\
 &= \mu_{B_2} \circ (\eta_{B_2} \otimes \eta_{B_2}) \circ (\overline{\sigma_{h \circ f}} * \partial_2(\bar{e})) = \\
 &= \mu_{B_2} \circ (\mu_{B_2} \otimes \mu_{B_2}) \circ [(h \circ f_1) \otimes (h \circ f_1) \otimes ((h \circ f_1^{-1}) * (h \circ f_1)) \otimes \\
 &\quad \otimes f_2^{-1}] \circ (C \otimes C \otimes \delta_C) \circ (C \otimes C \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 &= \mu_{B_2} \circ (\mu_{B_2} \otimes B_2) \circ (\mu_{B_2} \otimes (\mu_{B_2} \circ \tau_{B_2}^{B_2}) \otimes \mu_{B_2}) \circ \\
 &\quad \circ ((h \circ f_1) \otimes (h \circ f_1) \otimes (f_2^{-1} * f_2) \otimes f_2^{-1} \otimes f_2 \otimes f_2^{-1}) \circ \\
 &\quad \circ (C \otimes C \otimes C \otimes \delta_C \otimes \mu_C) \circ (C \otimes C \otimes C \otimes \tau_C^C \otimes C) \circ \\
 &\quad \circ (C \otimes C \otimes \delta_C \otimes \delta_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 &= \mu_{B_2} \circ (\mu_{B_2} \otimes B_2) \circ ((\eta_{B_2} \circ \bar{e}) \otimes (\eta_{B_2} \circ \bar{e}) \otimes (\eta_{B_2} \circ \overline{\sigma_{f_2}})) \circ \\
 &\quad \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 &= \eta_{B_2} \circ (\partial_1(\bar{e}) * \partial_3(\bar{e}) * \overline{\sigma_{f_2}})
 \end{aligned}$$

and then, since η_{B_2} is a monomorphism, $\overline{\sigma_{f_1}} \sim \overline{\sigma_{f_2}}$.

10. Proposition. *If σ is a 2-cocycle in $\text{Reg}(C \otimes C, K)$ such that $\sigma \circ (\eta_C \otimes C) = \varepsilon_C = \sigma \circ (C \otimes \eta_C)$, then $(C_\sigma = (C, \eta_C, \mu_{C_\sigma} = (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C)); \delta_C)$ is a Galois $\mathbf{H}$ -object with a normal basis.*

Proof:

Trivially, (C_σ, δ_C) is an $\mathbf{H}$ -comodule monoid.

The morphism $\gamma_{C_\sigma} = (\mu_{C_\sigma} \otimes C) \circ (C \otimes \delta_C) : C \otimes C \rightarrow C \otimes C$ is an isomorphism with inverse

$$\begin{aligned}
 \gamma_{C_\sigma}^{-1} &= (\mu_C \otimes C) \circ (C \otimes \sigma^{-1} \otimes \lambda \otimes C) \circ (C \otimes \mu_C \otimes C \otimes \delta_C) \circ (\delta_C \otimes \lambda \otimes C \otimes C) \circ \\
 &\quad \circ (C \otimes \delta_C \otimes C) \circ (C \otimes \delta_C)
 \end{aligned}$$

Indeed:

$$\begin{aligned}
 \gamma_{C_\sigma}^{-1} \circ \gamma_{C_\sigma} &= \\
 &= (\mu_C \otimes C) \circ (\mu_C \otimes \sigma^{-1} \otimes \lambda \otimes C) \circ (C \otimes C \otimes \mu_C \otimes C \otimes \delta_C) \circ \\
 &\quad \circ (C \otimes C \otimes \mu_C \otimes \lambda \otimes C \otimes C) \circ (C \otimes \tau_C^C \otimes C \otimes \delta_C \otimes C) \circ \\
 &\quad \circ (\sigma \otimes \delta_C \otimes \delta_C \otimes \delta_C) \circ (C \otimes \tau_C^C \otimes C \otimes C) \circ (\delta_C \otimes \delta_C \otimes C) \circ (C \otimes \delta_C) = \\
 &= (\mu_C \otimes C) \circ (\mu_C \otimes \sigma^{-1} \otimes \lambda \otimes C) \circ (C \otimes C \otimes \mu_C \otimes \delta_C \otimes C) \circ \\
 &\quad \circ (C \otimes \tau_C^C \otimes [\mu_C \circ (C \otimes \lambda) \circ \delta_C] \otimes \delta_C) \circ (C \otimes C \otimes \delta_C \otimes C) \circ \\
 &\quad \circ (\sigma \otimes \delta_C \otimes \delta_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) =
 \end{aligned}$$

$$\begin{aligned}
 &= (\mu_C \otimes C) \circ (\mu_C \otimes \sigma^{-1} \otimes \lambda \otimes C) \circ (C \otimes \tau_C^C \otimes [\tau_C^C \circ \delta_C] \otimes C) \circ \\
 &\circ (\sigma \otimes \delta_C \otimes C \otimes \delta_C) \circ (C \otimes \tau_C^C \otimes \delta_C) \circ (\delta_C \otimes \delta_C) = \\
 &= (\mu_C \otimes C) \circ (C \otimes [\mu_C \circ (C \otimes \lambda) \circ \delta_C] \otimes C) \circ \\
 &\circ (\sigma \otimes C \otimes C \otimes \sigma^{-1} \otimes C) \circ (C \otimes C \otimes C \otimes \tau_C^C \otimes C \otimes C) \circ \\
 &\circ (C \otimes C \otimes \delta_C \otimes \delta_C \otimes C) \circ (C \otimes \tau_C^C \otimes \delta_C) \circ (\delta_C \otimes \delta_C) = \\
 &= (\sigma \otimes C \otimes \sigma^{-1} \otimes C) \circ (C \otimes C \otimes [\tau_C^C \circ \delta_C] \otimes \delta_C) \circ \\
 &\circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 &= ([\sigma * \sigma^{-1}] \otimes C \otimes C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 &= C \otimes C \\
 \gamma_{C_\sigma} \circ \gamma_{C_\sigma}^{-1} &= \\
 &= (\sigma \otimes \mu_C \otimes C) \circ (\mu_C \otimes \tau_C^C \otimes C \otimes C) \circ (C \otimes C \otimes \mu_C \otimes \delta_C \otimes C) \circ \\
 &\circ (C \otimes \tau_C^C \otimes C \otimes \delta_C) \circ (\delta_C \otimes \sigma^{-1} \otimes \delta_C \otimes C) \circ (C \otimes \mu_C \otimes C \otimes \lambda \otimes C) \circ \\
 &\circ (C \otimes C \otimes \lambda \otimes C \otimes \delta_C) \circ (\delta_C \otimes \delta_C \otimes C) \circ (C \otimes \delta_C) = \\
 &= (\sigma \otimes \mu_C \otimes C) \circ (C \otimes \tau_C^C \otimes C \otimes C) \circ (\mu_C \otimes \mu_C \otimes [\tau_C^C \circ \delta_C] \otimes C) \circ \\
 &\circ (C \otimes \tau_C^C \otimes C \otimes \delta_C) \circ (\delta_C \otimes \sigma^{-1} \otimes [(\lambda \otimes \lambda) \circ \tau_C^C \circ \delta_C] \otimes C) \circ \\
 &\circ (C \otimes \mu_C \otimes C \otimes C \otimes C) \circ (\delta_C \otimes \lambda \otimes C \otimes \delta_C) \circ \\
 &\circ (C \otimes \delta_C \otimes C) \circ (C \otimes \delta_C) = \\
 &= (\sigma \otimes C \otimes C) \circ (C \otimes \tau_C^C \otimes C) \circ (\mu_C \otimes \mu_C \otimes \delta_C) \circ \\
 &\circ (C \otimes \tau_C^C \otimes [\mu_C \circ (\lambda \otimes C) \circ \delta_C] \otimes C) \circ (\delta_C \otimes \sigma^{-1} \otimes \lambda \otimes \delta_C) \circ \\
 &\circ (C \otimes \mu_C \otimes C \otimes \delta_C) \circ (\delta_C \otimes \lambda \otimes C \otimes C) \circ (C \otimes \delta_C \otimes C) \circ (C \otimes \delta_C) = \\
 &= (\sigma \otimes C \otimes C) \circ (\mu_C \otimes \tau_C^C \otimes C) \circ (C \otimes \tau_C^C \otimes \delta_C) \circ \\
 &\circ ([\tau_C^C \circ \delta_C] \otimes \sigma^{-1} \otimes \lambda \otimes C) \circ (C \otimes \mu_C \otimes C \otimes C \otimes C) \circ \\
 &\circ (C \otimes C \otimes \lambda \otimes C \otimes C \otimes C) \circ (\delta_C \otimes \delta_C \otimes C \otimes C) \circ \\
 &\circ (C \otimes [\tau_C^C \circ \delta_C] \otimes C) \circ (C \otimes \delta_C) = \\
 &= (C \otimes \sigma \otimes C) \circ (C \otimes C \otimes \sigma^{-1} \otimes \delta_C) \circ (C \otimes [\delta_C \circ \mu_C] \otimes C \otimes C) \circ \\
 &\circ (C \otimes C \otimes \lambda \otimes C \otimes C) \circ (\delta_C \otimes \delta_C \otimes C) \circ (C \otimes \delta_C) = \\
 &= (C \otimes \sigma \otimes \sigma^{-1} \otimes C) \circ (C \otimes C \otimes \tau_C^C \otimes C \otimes C) \circ \\
 &\circ (C \otimes \delta_C \otimes \delta_C \otimes C) \circ (C \otimes \mu_C \otimes \delta_C) \circ (\delta_C \otimes \lambda \otimes C) \circ (C \otimes \delta_C) = \\
 &= (C \otimes [\sigma * \sigma^{-1}] \otimes C) \circ (C \otimes \mu_C \otimes \delta_C) \circ (\delta_C \otimes \lambda \otimes C) \circ (C \otimes \delta_C) = \\
 &= C \otimes C
 \end{aligned}$$

and thus (C_σ, δ_C) is a Galois **H**-object with a normal basis. ■

Remark. If σ and γ are two 2-cocycles such in Proposition 10 and cohomologous, then $(\bar{e} \otimes C) \circ \delta_C : (C_\sigma, \delta_C) \rightarrow (C_\gamma, \delta_C)$ is an isomorphism

of $\mathbf{H}$ -comodule monoids where $\bar{e}$ is the morphism which exists because $\sigma \sim \gamma$, and thus, $[(C_\sigma, \delta_C)] = [(C_\gamma, \delta_C)]$ in $N_C(\mathbf{H})$.

11. Theorem. *There is an isomorphism of abelian groups $F : N_C(\mathbf{H}) \rightarrow H^2(\mathbf{H}, K)$ defined by $F([(B, \rho_B)]) = [\bar{\sigma}_f]$ with inverse $G([\sigma]) = [(C_\sigma, \delta_C)]$.*

Proof:

The morphisms F and G are well defined by Propositions 9 and 10. Moreover, the morphism of $\mathbf{H}$ -comodules $f : (C_{\bar{\sigma}_f} = (C, \eta_C, \mu_{C_{\bar{\sigma}_f}}); \delta_C) \rightarrow (B = (B, \eta_B, \mu_B); \rho_B)$ is a morphism of monoids and therefore it is an isomorphism ([13, (4.3.9)]) and $[(B, \rho_B)] = [(C_{\bar{\sigma}_f}, \delta_C)]$ in $N_C(\mathbf{H})$.

If $[\sigma] \in H^2(\mathbf{H}, K)$ then the 2-cocycle γ defined from C_σ (Proposition 9) equals to σ :

$$\begin{aligned}
 \eta_C \circ \gamma &= \mu_{C_\sigma} \circ (\mu_{C_\sigma} \circ [(C \otimes \varepsilon_C) \circ \gamma_{C_\sigma}^{-1} \circ (\eta_C \otimes C)]) \circ (C \otimes C \otimes \mu_C) \circ \\
 &\quad \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 &= (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) \circ (\sigma \otimes \mu_C \otimes \sigma^{-1} \otimes C) \circ \\
 &\quad \circ (C \otimes \tau_C^C \otimes C \otimes \lambda \otimes C \otimes \lambda) \circ (\delta_C \otimes \delta_C \otimes \delta_C \otimes C) \circ \\
 &\quad \circ (C \otimes C \otimes [\delta_C \circ \mu_C]) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 &= (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \sigma^{-1} \otimes \delta_C) \circ \\
 &\quad \circ (C \otimes \lambda \otimes C \otimes \lambda) \circ (C \otimes \delta_C \otimes C) \circ (C \otimes \delta_C) \circ \delta_C \circ \\
 &\quad \circ (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 &= (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \sigma^{-1} \otimes C \otimes C) \circ \\
 &\quad \circ (C \otimes \lambda \otimes C \otimes [(\lambda \otimes \lambda) \circ \tau_C^C \circ \delta_C]) \circ (C \otimes \delta_C \otimes C) \circ \\
 &\quad \circ (C \otimes [\tau_C^C \circ \delta_C]) \circ \delta_C \circ (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 &= (\sigma \otimes C \otimes \sigma^{-1}) \circ (C \otimes \tau_C^C \otimes \lambda \otimes C) \circ \\
 &\quad \circ (C \otimes [\mu_C \circ (C \otimes \lambda) \circ \delta_C] \otimes \lambda \otimes \delta_C) \circ (C \otimes C \otimes \delta_C) \circ (C \otimes \delta_C) \circ \\
 &\quad \circ \delta_C \circ (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 &= (\sigma \otimes \sigma^{-1} \otimes \eta_C) \circ (C \otimes \lambda \otimes \lambda \otimes C) \circ (\delta_C \otimes \delta_C) \circ \delta_C \circ \\
 &\quad \circ (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 &= ([\partial_1(\sigma^{-1}) * \partial_4(\sigma)] \otimes \eta_C) \circ (C \otimes \lambda \otimes C) \circ (C \otimes \delta_C) \circ \delta_C \circ \\
 &\quad \circ (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 &= ([\partial_2(\sigma^{-1}) * \partial_3(\sigma)] \otimes \eta_C) \circ (C \otimes \lambda \otimes C) \circ (C \otimes \delta_C) \circ \delta_C \circ \\
 &\quad \circ (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 &= (\sigma^{-1} \otimes \sigma \otimes \eta_C) \circ (\mu_C \otimes \tau_C^C \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes \tau_C^C \otimes C) \circ
 \end{aligned}$$

$$\begin{aligned}
 & \circ (\delta_C \otimes [(\lambda \otimes \lambda) \circ \tau_C^C \circ \delta_C] \otimes \delta_C) \circ (C \otimes \delta_C) \circ \delta_C \circ \\
 & \circ (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 & = (\sigma \otimes \sigma^{-1} \otimes \eta_C) \circ (C \otimes \tau_C^C \otimes C) \circ \\
 & \circ (C \otimes [\mu_C \circ (C \otimes \lambda) \circ \delta_C] \otimes [\mu_C \circ (\lambda \otimes C) \circ \\
 & \circ \delta_C] \otimes C) \circ (\delta_C \otimes \delta_C) \circ \delta_C \circ \\
 & \circ (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
 & = \eta_C \circ \sigma
 \end{aligned}$$

because σ is a 2-cocycle and $\mathbf{H}$ a cocommutative Hopf algebra.

If $[(\mathbf{B}, \rho_B)], [(\mathbf{B}', \rho_{B'})] \in \mathbf{N}_C(\mathbf{H})$ then the morphism $(f \otimes g) \circ \delta_C : C \rightarrow B \otimes B'$ factors through the equalizer $i_{BB'}$, where f and g are the morphisms defined in Proposition 9 for $(\mathbf{B}, \rho_B)$ and $(\mathbf{B}', \rho_{B'})$ respectively. Let $h : C \rightarrow B \otimes B'$ be this factorization. The morphism h is in $\text{Reg}(C, B \otimes B')$ with inverse the factorization of the morphism $(f^{-1} \otimes g^{-1}) \circ \delta_C$ through the equalizer $i_{BB'}$, and satisfying $h \circ \eta_C = \eta_{BB'}$. Moreover, $\bar{\sigma}_f * \bar{\sigma}_g = \bar{\sigma}_h$. ■

Remark. If $\mathcal{C} = K\text{-Mod}$ (K a field), this result has been obtained by Sweedler for the $\mathbf{H}$ -module algebra $(K, \varepsilon_H \otimes K)$ in [21].

12. Definition. A monoid $\mathbf{A} = (A, \eta_A, \mu_A)$ is said to be Azumaya if and only if:

- i) A is a progenerator in $\mathcal{C}$.
- ii) The morphism of monoids $\chi_A : A \otimes A \rightarrow [A, A]$; $\chi_A := [A, \mu_A \circ (A \otimes \mu_A) \circ (\tau_A^A \otimes A)] \circ \alpha_A(A \otimes A)$ is an isomorphism.

13. Definition. On the set of $\mathbf{H}$ -module monoid isomorphism classes of $\mathbf{H}$ -module Azumaya monoids we define the following equivalence relation:

$$(\mathbf{A}, \varphi_A) \sim (\mathbf{B}, \varphi_B) \iff \mathbf{AE}(M)^{\text{op}} \cong \mathbf{BE}(N)^{\text{op}}$$

for some progenerators $\mathbf{H}$ -modules (M, φ_M) and (N, φ_N) .

The set of equivalence classes of $\mathbf{H}$ -module Azumaya monoids forms a group under the operation induced by the tensor product, $(\mathbf{AB}, \varphi_{A \otimes B} = (\varphi_A \otimes \varphi_B) \circ (C \otimes \tau_A^C \otimes B) \circ (\delta_C \otimes A \otimes B))$. The unit element is the class of the $\mathbf{H}$ -module Azumaya monoid:

$$\begin{aligned}
 & (E(M)^{\text{op}}, \varphi_{E(M)} = [M, \varphi_M \circ (C \otimes \beta_M(M)) \circ (\tau_C^M \otimes [M, M]) \circ \\
 & \circ (\varphi_M \otimes C \otimes [M, M]) \circ (\tau_C^M \otimes C \otimes [M, M]) \circ (M \otimes \tau_C^C \otimes [M, M]) \circ \\
 & \circ (M \otimes C \otimes \lambda \otimes [M, M]) \circ (M \otimes \delta_C \otimes [M, M]) \circ \alpha_M(C \otimes [M, M]))
 \end{aligned}$$

for some progenerator $\mathbf{H}$ -module (M, φ_M) ; and the opposite of $(\mathbf{A}, \varphi_A)$ is $(A^{\text{op}}, \varphi_A)$. This group is denoted by $\mathbf{BM}(\mathcal{C}, \mathbf{H})$.

If $\mathbf{1} = (1, 1, \tau^K, 1)$ is the trivial Hopf algebra in $\mathcal{C}$, then we define the Brauer group of Azumaya monoids in $\mathcal{C}$ as $\mathbf{BM}(\mathcal{C}, \mathbf{1})$ and we will denote it by $\mathbf{B}(\mathcal{C})$.

Examples.

- 1) If $\mathcal{C}$ is the category of modules over a commutative ring R , then $\mathbf{B}(\mathcal{C})$ is the Brauer group of R defined by Auslander and Goldman in [4].
- 2) If $\mathcal{C}$ is the category of sheaves of θ -modules, $\mathbf{B}(\mathcal{C})$ is the Brauer group defined by Auslander in [3].
- 3) If $\mathcal{C}$ is the category of (R, σ) -Mod with σ an idempotent noetherian kernel functor in R -Mod, López and Villanueva obtain, in [17], a homomorphism $Br(R, \sigma) \rightarrow \mathbf{B}((R, \sigma)\text{-Mod})$ which is an isomorphism if R is a noetherian ring, where $Br(R, \sigma)$ is the relative Brauer group introduced by Oystaeyen and Verschoren in [22].

14. Definition. We denote by $\mathbf{BM}_{\text{inn}}(\mathcal{C}, \mathbf{H})$ the subgroup of $\mathbf{BM}(\mathcal{C}, \mathbf{H})$ built up with the equivalence classes that can be represented by an $\mathbf{H}$ -module Azumaya monoid with inner action.

([2, 16]).

15. Theorem. $\mathbf{BM}_{\text{inn}}(\mathcal{C}, \mathbf{H}) \cong \mathbf{B}(\mathcal{C}) \oplus H^2(\mathbf{H}, K)$.

Proof:

The sequence

$$1 \longrightarrow \mathbf{B}(\mathcal{C}) \xrightarrow{i} \mathbf{BM}_{\text{inn}}(\mathcal{C}, \mathbf{H}) \xrightarrow{\Pi} \mathbf{N}_{\mathcal{C}}(\mathbf{H}) \longrightarrow 1$$

is split exact, where the morphism i is given by $i[(A)] = [(A, \varepsilon_C \otimes A)]$ and the morphism Π is given by $\Pi[(A, \varphi_A)] := (\Pi(\mathbf{A}), \rho_{\Pi(A)})$ with $\Pi(\mathbf{A}) := \text{Ig}(m, n)$

$$\Pi(A) \xrightarrow{j} A \otimes C \xrightleftharpoons[n]{m} [A, A \otimes C]$$

where

$$m = [A, \mu_A \otimes C] \circ \alpha_A(A \otimes C)$$

$$n = [A, (\mu_A \otimes C) \circ (A \otimes (\varphi_A \circ \tau_C^A) \otimes C) \circ (\tau_A^A \otimes \delta_C)] \circ \alpha_A(A \otimes C)$$

([2, 17]).

If H is a finitely generated projective, commutative and cocommutative Hopf algebra over a commutative ring K , this result generalizes the one obtained by Beattie in [5], and if the action of H over A is inner, the description of $\Pi(A)$ is due to Beattie and Ulbricht ([6]). ■

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J. N. Alonso Álvarez:
Departamento de Matemáticas
Universidad de Vigo
Lagoas-Marcosende
E-36271 Vigo
SPAIN

J. M. Fernández Vilaboa:
Departamento de Álgebra
Universidad de Santiago de Compostela
E-15771 Santiago de Compostela
SPAIN

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