Publicacions Matemàtiques, Vol **41** (1997), 127–134.

THE FIRST DERIVATIVE OF THE PERIOD FUNCTION OF A PLANE VECTOR FIELD

J.-P. Françoise

Abstract

The algorithm of the successive derivatives introduced in [**5**] was implemented in [**7**], [**8**]. This algorithm is based on the existence of a decomposition of 1-forms associated to the relative cohomology of the Hamiltonian function which is perturbed. We explain here how the first step of this algorithm gives also the first derivative of the period function. This includes, for instance, new presentations of formulas obtained by Carmen Chicone and Marc Jacobs in [**3**].

1. Introduction

Let X_ϵ be a vector field of the type:

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + \epsilon \sum_{2\leq i+j\leq n} a_{ij}x^iy^j\frac{\partial}{\partial x} + b_{ij}x^iy^j\frac{\partial}{\partial y}. \tag{1}$$

$((a_{ij}, b_{ij}) \in \mathbb{R}^2)$, ϵ is a parameter.

For a vector field X_ϵ corresponding to fixed values of (a_{ij}, b_{ij}), there is a neighborhood U of the origin $0 \in \mathbb{R}^2$ on which the flow of X_ϵ, solution of the differential system:

$$\begin{cases} \dot{x} = -y + \epsilon \displaystyle\sum_{2\leq i+j\leq n} a_{ij}x^iy^j \\ \dot{y} = x + \epsilon \displaystyle\sum_{2\leq i+j\leq n} b_{ij}x^iy^j \end{cases} \tag{2}$$

exists for all initial values which belong to this neighborhood.

There is furthermore a first return mapping L_ϵ defined on U. Given for instance, an initial point $(r, 0)$, $r > 0$ the solution of equation (2) with initial data $(r, 0)$ intersects again for the first time the x-axis at some point $(L_\epsilon(r), 0)$, $L_\epsilon(r) > 0$. We denote by $\Sigma = \{(x, 0) \in U\}$ the transversal section. By transversality, the mapping L_ϵ is analytic and it can be represented as a convergent Taylor series:

$$L_\epsilon(r) = r + L_2(r)\epsilon^2 + \cdots + L_k(r)\epsilon^k + \cdots \tag{3}$$

In [**5**], an algorithm was introduced to compute the first non-vanishing coefficient $L_{k0}(r)(\underline{a}, \underline{b})$ for a homogeneous perturbative part. It was later implemented in [**7**], [**8**].

Introduce now γ_ϵ the arc of the trajectory of X_ϵ between the two points $(r, 0)$ and $(L_\epsilon(r), 0)$ and T_ϵ the time along the flow between these two points. We give here a practical formula to compute the first derivative $\frac{\partial T_\epsilon}{\partial \epsilon}_{|\epsilon=0}$. This formula is based on the relative cohomology decomposition of 1-forms used in preceding articles [**5**], [**7**], [**8**].

2. The relative cohomology decomposition of 1-forms

We use complex coordinates $z = (1/\sqrt{2})(x + iy)$, $\bar{z} = (1/\sqrt{2})(x - iy)$; and the 1-form $\omega_\epsilon = i_{X_\epsilon}(dx \wedge dy)$. With these new notations, we use:

$$\omega_\epsilon = dH + \epsilon\omega_1 = dH + \epsilon \sum_{2 \le i+j \le d} A_{ij} z^i \bar{z}^j dz + \bar{A}_{ij} \bar{z}^i z^j d\bar{z}. \tag{4}$$

The complex coefficients A_{ij} of equation (4) are easily related to the real coefficients (a_{ij}, b_{ij}). The function H is $H : (z, \bar{z}) \mapsto H(z, \bar{z}) = z\bar{z}$. We remind without proof the following

Proposition 1. *Any polynomial 1-form ω can be decomposed into*

$$\omega = gdH + dR + (1/2)\psi(H)[zd\bar{z} - \bar{z}dz]$$

where g, R are polynomials ($R(0) = 0$) in $(z, \bar{z})$ and ψ is a polynomial in one variable.

To compute g and $\psi(H)$, take $d\omega = F(z, \bar{z})$. Write

$$F(z, \bar{z}) = \sum_{i \neq j} F_{ij} z^i \bar{z}^j + \sum_{i=j} F_{ij} (z\bar{z})^i,$$

then

$$g = \sum_{i \neq j} \frac{F_{ij} z^i \bar{z}^j}{i - j}.$$

To find ψ, take

$$\phi(t) = \sum_i F_{ii} t^i.$$

Then solve $t\psi'(t) + \psi(t) = \phi(t)$. Now get R by applying Poincaré's theorem to

$$d\omega - dg \wedge dH - \phi(H) dz \wedge d\bar{z}.$$

In the sequel, we describe the method given in [**5**] to get the successive derivatives of the return mapping.

A classical formula gives:

$$L_1(r, A_{ij}, \bar{A}_{ij}) = -\int_{H=r} \omega_1. \tag{5}$$

Assume that $L_1(r, A_{ij}, \bar{A}_{ij}) \equiv 0$ (as a function of r) then there is a polynomial g_1 such that

$$\omega_1 = g_1 dH + dR_1. \tag{6}$$

And we get

$$L_2(r, A_{ij}, \bar{A}_{ij}) = -\int_{H=r} g_1 \omega_1. \tag{7}$$

One can show inductively that given

$$L_k(r, A_{ij}, \bar{A}_{ij}) = -\int_{H=r} g_{k-1} \omega_1, \tag{8}$$

if $L_k(r, A_{ij}, \bar{A}_{ij}) \equiv 0$ (as a function of r), then there is a polynomial g_k such that

$$g_{k-1}\omega_1 = g_k dH + dR_k \tag{9}$$

and then

$$L_{k+1}(r, A_{ij}, \bar{A}_{ij}) = -\int_{H=r} g_k \omega_1. \tag{10}$$

As a consequence, we can compute the first non-zero coefficient $L_k(r, A_{ij}, \bar{A}_{ij})$ by building the sequence of polynomials $g_1, \dots, g_k, \dots$.

At each step k we have a 1-form $g_k\omega_1$. We must first compute the differential $d(g_k\omega_1) = F^k(z, \bar{z}) dz \wedge d\bar{z}$. We then split into two parts:

$$F^k(z, \bar{z}) = \sum_{i \neq j} F^k_{ij} z^i \bar{z}^j + \sum_{i=j} F^k_{ij} (z\bar{z})^i.$$

We find that

$$L_{k+1}(r, A_{ij}, \bar{A}_{ij}) = \sum F_{ll}^k r^l,$$

and we introduce

$$g_{k+1} = \sum_{i \neq j} \frac{F_{ij}^k z^i \bar{z}^j}{i-j}.$$

Next we compute $d(g_{k+1}\omega_1)$ and we repeat the process.

3. The first derivative of the period

We write $T_\epsilon = T_0 + \epsilon T_1 + O(\epsilon^2)(T_0 = 2\pi)$. Let $\varpi_0 = d\theta$ then $\varpi_0 \wedge dH = r d\theta \wedge dr = dx \wedge dy$ and $\int_{\gamma_\epsilon} \varpi_0 = 2\pi = T_O$.

Definition 2. A 1-form $\varpi_0 + \epsilon\varpi_1 + \cdots + \epsilon^k \varpi_k$ is said to be k-isochronous to X_ϵ if $(\varpi_0 + \epsilon\varpi_1 + \cdots + \epsilon^k \varpi_k)\ (X_\epsilon) = 1 + O(\epsilon^{k+1})$.

Proposition 3. *Let* $\varpi_0 + \epsilon\varpi_1$ *be a* 1*-isochronous* 1*-form, then* $T_\epsilon = \int_{\gamma_\epsilon} (\varpi_0 + \epsilon\varpi_1) + O(\epsilon^2) = 2\pi + \epsilon \int_{H=r^2} \varpi_1 + O(\epsilon^2)$.

Proof:

$$\int_{\gamma_\epsilon} (\varpi_0 + \epsilon\varpi_1) = \int_{\gamma_\epsilon} (\varpi_0 + \epsilon\varpi_1)(X_\epsilon)\, dt = \int_{\gamma_\epsilon} dt + O(\epsilon^2) = T_\epsilon + O(\epsilon^2). \quad \blacksquare$$

We now try to build a 1-isochronous form. This yields:

$$(\varpi_0 + \epsilon\varpi_1) \wedge (dH + \epsilon\omega_1) = (1 + O(\epsilon^2)) dx \wedge dy. \tag{11}$$

The 1-form ϖ_1 should be such that

$$\varpi_1 \wedge dH = -\varpi_0 \wedge \omega_1. \tag{12}$$

Now the relative cohomology decomposition of ω_1 displays:

$$\omega_1 = g dH + dR + (1/2)\psi(H)[z d\bar{z} - \bar{z} dz]. \tag{13}$$

This yields to a possible choice for ϖ_1:

$$\varpi_1 = -\left[g + (1/r) \left(\frac{\partial R}{\partial r} \right) \right] \varpi_0. \tag{14}$$

Now, we obtain with this choice of ϖ_1:

$$T_1 = \int_{H=r^2} \varpi_1 = -\int_{H=r^2} \left[g + (1/r) \left(\frac{\partial R}{\partial r} \right) \right] d\theta. \tag{15}$$

Now, we remark that the construction of g yields: $\int_{H=r^2} g \, d\theta = 0$.

We expand R:

$$R(z,\bar{z}) = \sum_{i \neq j} R_{ij} z^i \bar{z}^j + \sum_{i=j} R_{ii} (z\bar{z})^i$$

and get ultimately:

$$T_1(r) = -4\pi \sum_k k R_{kk} r^{(2k-2)}. \tag{16}$$

To summarize, the first derivative $\frac{\partial T_\epsilon}{\partial \epsilon}_{|\epsilon=0}$ is essentially given by the part in the Taylor development of R which depends only in $H = z\bar{z}$. An equivalent expression was already derived by C. Chicone and M. Jacobs in [**3**, Lemma 3.2, p. 455]. Our main point here was to clear up the link of their result with the relative cohomology of forms.

4. The case of an homogeneous perturbative part

To illustrate the preceding formula, we consider the situation where the perturbative part ω_1 is homogeneous of degree n. So we consider now the 1-form

$$\omega_\epsilon = dH + \epsilon\omega_1 = dH + \epsilon \sum_{2 \leq i+j=n} A_{ij} z^i \bar{z}^j dz + \bar{A}_{ij} \bar{z}^i z^j d\bar{z}. \tag{17}$$

We first compute $d\omega_1 = D(z,\bar{z}) dz \wedge d\bar{z}$,

$$D(z,\bar{z}) = \sum_{i+j=n} [-j A_{ij} z^i \bar{z}^{j-1} + j \bar{A}_{ij} \bar{z}^i z^{j-1}]. \tag{18}$$

There is a contribution to $\phi(H)$ only if $i = j - 1$, hence $n + 1 = 2j$. So n must be odd. Write $n = 2J - 1$, then $i = J - 1$ and

$$\phi(H) = -2J\,\mathrm{Im}(A_{J-1J})H^{(J-1)}. \tag{19}$$

So the first condition for a center is $\mathrm{Im}(A_{J-1J}) = 0$ [**5**]. We have now:

$$g = \sum_{i \neq j-1} [-jA_{ij}z^i\bar{z}^{j-1} + j\bar{A}_{ij}\bar{z}^i z^{j-1}]/(i - j + 1). \tag{20}$$

Note that g is homogeneous of degree $n - 1$. We have then to consider

$$\omega_1 - gdH - (1/2)\psi(H)[zd\bar{z} - \bar{z}dz] = \Omega. \tag{21}$$

Note that Ω is homogeneous of degree $n + 1$, then we obtain $\Omega = d[(i_C\Omega/(n + 1)]$ where $C = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}$ is the Euler vector field. So we can choose

$$R = [\omega_1(C) - 2gH]/(n + 1). \tag{22}$$

Now we have to isolate the monomials in R for which the exponent of z equals the exponent of $\bar{z}$. There are no such monomials in gH. Hence it is only necessary to look at those in $\omega_1(C)$. This yields

$$\omega_1(C) = \sum_{i+j=n} [A_{ij}z^{i+1}\bar{z}^j + \bar{A}_{ij}\bar{z}^{i+1}z^j]. \tag{23}$$

The monomials in $\omega_1(C)$ such that $i + 1 = j$ and $i + j = n$ exist only if $n = 2I + 1$ and we ultimately obtain

$$T_1(r) = 4\pi\,\mathrm{Re}(A_{II+1})r^2I. \tag{24}$$

In particular, this yields the "second" condition for a center to be isochronous

$$\mathrm{Re}(A_{II+1}) = 0, \tag{25}$$

if $n = 2I + 1$. This condition was previously obtained by [**9**]. In this article, the authors use the Cherkas transform and deal with the Abel Equation.

5. Final remarks on the isochronous centres

The search for isochronous centres is certainly one the priority subjects in the bifurcation theory of differential systems. To name few of the many important contributions to this field, we mention the articles of [**1**], [**2**], [**10**]. Recently, C. J. Christopher and J. Devlin [**4**] and independently B. Schuman [**11**] proved that the origin is never isochronous in non zero homogeneous hamiltonian perturbations. It may be of some interest to see appropriate extensions to several variables in the general setting provided by normal form theory. (cf. [**6**]).

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Institut de Mathematiques
Université Pierre et Marie Curie (Paris VI)
UFR 920 BP 172
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

e-mail: jpf@ccr.jussieu.fr

Primera versió rebuda el 30 de Novembre de 1996,
darrera versió rebuda el 5 de Febrer de 1997