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FUNCTIONS OF CLASS C^k WITHOUT DERIVATIVES

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Abstract

We describe a general axiomatic way to define functions of class C^k, $k \in \mathbf{N} \cup \{\infty\}$ on topological abelian groups. In the category of Banach spaces, this definition coincides with the usual one. The advantage of this axiomatic approach is that one can dispense with the notion of norms and limit procedures. The disadvantage is that one looses the derivative, which is replaced by a local linearizing factor. As an application we use this approach to define C^∞ functions in the setting of graded/super manifolds.

Introduction

In this paper we want to generalize the notion of functions of class C^k to a context in which quotients and/or limits are not defined. The motivation for this generalization is the desire to have a *single* definition that can be applied as well to functions on open sets in Banach spaces, as to functions on super domains with commuting and anti-commuting coordinates. There are two "obstructions" to copy the standard definition of differentiable functions to the graded setting. In the first place, a difference quotient like $\Delta y/\Delta x$ is not well defined because for nilpotent Δx the inverse $1/\Delta x$ is not defined. In the second place there is no consensus about the topology on the basic graded ring $\mathcal{A}$ which replaces the real numbers $\mathbf{R}$. The "canonical" choice of the DeWitt topology is highly non-Hausdorff, prohibiting unique limits.

To explain the basic idea of the approach advocated in this paper, consider a function $f : U \to F$, where U is a convex open subset of E and where E and F are Banach spaces. Denoting by $\mathrm{Hom}(E, F)$ the set of all continuous linear maps from E to F, one can easily show that such a function f is of class C^1 if and only if there exists a continuous function $\phi : U^2 \to \mathrm{Hom}(E, F)$ such that

$$\forall\, x, y \in U : f(x) - f(y) = \phi(x, y)(x - y).$$

Although $\phi(x, y)$ is not necessarily unique, the diagonal $\phi(x, x)$ *is* unique: it is the (Fréchet) derivative of f.

What we will do is take this theorem and turn it into the definition for functions of class C^1; the definition of functions of class C^k then is by induction. This idea is not new and can be traced back at least to Caratheodory ([**Ca**]). Our generalization is to allow E and F to be any abelian topological group, not necessarily Banach spaces. In this generalization, one looses an important property of differentiable functions: the derivative. As said above, for functions f of class C^1 defined on Banach spaces, the derivative can be defined as the diagonal $\phi(x, x)$, which is unique. In the general context of abelian topological groups, even the diagonal need not be unique and thus we cannot speak of a properly defined derivative. This treatement is far more general than needed for an intrinsic definition of graded manifolds, but since there is strictly no more effort involved, we could not resist the temptation. Moreover, it helps focussing the attention on the essential ingredients.

To outline the contents of this paper, in Section 1 we give the main definition of our approach to functions of class C^k, whereafter in Section 2 we prove some properties of the functions so defined. Finally in Section 3 we apply our method to the setting of graded vector spaces.

1. The main definitions

Definition 1.1. A *cat* is a category $\mathcal{C}$ of abelian groups, stable under direct products, with the following properties. In the first place each object $E \in \mathcal{C}$ has a topology for which addition is a continuous operation. In the second place, for every two objects $E, F \in \mathcal{C}$ there exists a third object $\mathrm{Shom}(E, F) \in \mathcal{C}$ consisting of continuous homomorphisms from E to F. The elements of $\mathrm{Shom}(E, F)$ will be called *smooth homomorphisms.*

Remark 1.2. The fact that $\mathrm{Shom}(E, F)$ is an object in $\mathcal{C}$ implies that it has a topology for which addition of homomorphisms is continuous.

Examples 1.3. (i) Let $\mathcal{C}_V$ denote the category of normed vector spaces (over $\mathbf{R}$ or $\mathbf{C}$). We define $\mathrm{Shom}(E, F)$ to be the set of all continuous linear maps (over $\mathbf{R}$ or $\mathbf{C}$) from E to F, equipped with the operator norm. Since addition of homomorphisms is continuous in this norm, we conclude that $\mathrm{Shom}(E, F) \in \mathcal{C}_V$ and that $\mathcal{C}_V$ is a cat. An interesting subcategory of $\mathcal{C}_V$ is $\mathcal{C}_B$ in which all objects are Banach spaces; $\mathcal{C}_B$ is also a cat.

(ii) More generally, let $\mathcal{C}_A$ be a category of abelian groups in which each object has a topology for which addition is continuous. We equip $\mathrm{Hom}(E, F)$ with the weak*-topology, and we define the sets $\mathrm{Shom}(E, F) \subset \mathrm{Hom}(E, F)$ to consist of all *continuous* homomorphisms from E to F, equiped with the induced topology. It turns out that addition of homomorphisms is continuous in this topology, so $\mathcal{C}_A$ becomes a cat.

(iii) A variation on the previous example is a category $\mathcal{C}_M$ of modules over a commutative ring $\mathcal{A}$ (with $\mathcal{A}$ belonging to $\mathcal{C}_M$), all equipped with a topology in which the bi-additive operations of addition and multiplication by elements of $\mathcal{A}$ are continuous. Here we define $\mathrm{Shom}(E, F)$ to be the $\mathcal{A}$-module of all continuous $\mathcal{A}$-module homomorphisms from E to F, equipped with the (induced) weak*-topology. The reader can easily verify that $\mathrm{Shom}(E, F)$ belongs to $\mathcal{C}_M$ and thus that these definitions turn $\mathcal{C}_M$ into a cat.

Remark 1.4. The cat $\mathcal{C}_V$ of normed vector spaces over $\mathbf{C}$ is an example of a cat in which $\mathrm{Shom}(E, F)$ does not consist of *all* continuous (additive) homomorphisms.

Remark 1.5. The example $\mathcal{C}_B$ of Banach spaces cannot be extended in any natural way to Fréchet spaces since the set of all continuous morphisms between two Fréchet spaces is in general not again a Fréchet space.

Discussion 1.6. What we now propose to do is to define sets $C^k(U, F)$, $k \in \mathbf{N} \cup \{\infty\}$ of functions of class C^k defined on a subset $U \subset E$ to F, E and F objects in a fixed cat. More precisely, we will define the whole family

$$\mathcal{F}_k = \{C^k(U, F) \mid F \in \mathcal{C} \quad \text{and} \quad \exists E \in \mathcal{C} : U \subset E\}$$

at the same time, i.e., we will define $C^k(U, F)$ for all F and all subsets U in any E at the same time.

Definition 1.7. Given a function $f : U \to F$, a (local) linearizing factor for f on U is a pair (V, ϕ) with $V \subset U$ open in U and $\phi : V^2 \to \mathrm{Shom}(E, F)$ satisfying

$$\forall\, x, y \in V : f(x) - f(y) = \phi(x, y)(x - y).$$

Nota Bene 1.8. We do not require the sets $U \subset E$ in $C^k(U, F)$ to be open. In most applications one will restrict attention to open subsets, but in the present axiomatic approach it is nowhere needed that they are open. On the other hand, the domain of definition of a local linearizing factor is always supposed to be open *within* U.

Definition 1.9. The families $\mathcal{F}_k$, $k \geq 0$ are defined by the conditions:

(C0) $C^0(U, F)$ is the set of all continuous functions $U \to F$;

(C1) $f \in C^k(U, F)$, $k > 0$ if and only if $f \in C^0(U, F)$ and if there exists a family (V_i, ϕ_i) of linearizing factors for f such that $\cup_i V_i = U$ and $\phi_i \in C^{k-1}(V_i^2, \mathrm{Shom}(E, F))$;

(C2) $\mathcal{F}_k$ is maximal with respect to (C1).

Remarks 1.10. (i) Condition (C1) says that a continuous function f is of class C^k if and only if there exist local linearizing factors for f of class C^{k-1}. Note that we explicitly *do not* require that the local linearizing factors ϕ in property (C1) are unique.

(ii) Property (C2) should be interpreted in the following way: if $\widehat{\mathcal{F}}_k = \{\widehat{C}^k(U, F)\}$ is another family satisfying property (C1), then necessarily for all F and all $U \subset E$ we have $\widehat{C}^k(U, F) \subset C^k(U, F)$. Of course property (C2) is unnecessary for $k \neq \infty$ since then the recursion with respect to k ends at C^0.

(iii) It follows immediately from the definitions that we have the inclusions

$$C^0(U, F) \supset \cdots \supset C^k(U, F) \supset C^{k+1}(U, F) \supset \cdots \supset \bigcap_{k \in \mathbf{N}} C^k(U, F) \supset C^\infty(U, F).$$

The reader should be warned however that the inclusion $\cap_{k \in \mathbf{N}} C^k(U, F) \subset C^\infty(U, F)$ is not automatic (if true at all) due to the non-uniqueness of the linearizing factors ϕ.

Example 1.11. Consider the category $\mathcal{C}_V$ of normed vector spaces (Example 1.3.(i)), let $f \in C^k(U, F)$ be a function of class C^k ($k \geq 1$) on an open subset $U \subset E$ in our axiomatic setting and define the function $f' : U \to \mathrm{Shom}(E, F)$ by:

$$f'(x) = \phi(x, x),$$

where ϕ is a local linearizing factor for f defined in x. Since the function ϕ is of class C^{k-1}, it is in particular continuous and hence f' is continuous. We claim that the function f' is well defined, i.e., does not depend upon a choice for ϕ but is uniquely defined by f. To see this, suppose that

(V, ϕ) and $(\widehat{V}, \widehat{\phi})$ are both local linearizing factors for f. It follows that for $x \in V \cap \widehat{V}$, all $h \in E$ sufficiently small and all $0 < t < 1$ we have:

$$(\phi(x + th, x) - \widehat{\phi}(x + th, x))(h) = 0.$$

The result now follows by taking the limit $t \to 0$ and using the continuity of ϕ and $\widehat{\phi}$. Finally we claim that f' is the standard Fréchet derivative of the function f. For this we compute:

$$\begin{aligned}\frac{||f(x+h) - f(x) - f'(x)h||}{||h||} &= \frac{||\phi(x+h, x)h - \phi(x, x)h||}{||h||} \\ &\leq ||\phi(x+h, x) - \phi(x, x)||.\end{aligned}$$

Again the result follows because of the continuity of ϕ. What we have shown is that a function f which is of class C^k in our axiomatic setting, is of class C^1 in the ordinary sense. But then it follows immediately by induction that f is of class C^k in the ordinary sense (the case $k = \infty$ included).

If we restrict our attention to the cat $\mathcal{C}_B$, we can easily show the converse, i.e., that a function which is of class C^k in the ordinary sense is also of class C^k in our axiomatic sense. For this, let f be a continuous function of class C^k in the ordinary sense defined on an open subset $U \subset E$ and define the function ϕ by:

$$\phi(x, y) = \int_0^1 f'(sx + (1-s)y)\, ds,$$

where f' is the ordinary Fréchet derivative of f. This integral is well defined for any function f' defined on an open convex set and taking values in a Fréchet space, and thus in particular for this f' which takes values in a Banach space. It follows immediately that ϕ is a (local) linearizing factor of class C^{k-1} for f. Since any open U can be covered by convex open subsets, f is of class C^k in our axiomatic setting. Note that this argument also shows that in the cat $\mathcal{C}_B$ one has the equality $\cap_{k\in\mathbf{N}} C^k(U, F) = C^\infty(U, F)$ in the axiomatic approach (use property (C2), or the recursion argument explained below).

Remark 1.12. Let E be a Banach space and let $M \subset E$ be an embedded submanifold of class C^ℓ, $\ell \leq \infty$. A little reflection will show that, due to the local nature of the linearizing factors, the axiomatically defined sets $C^k(M, F)$, $k \leq \ell$ correspond exactly to the set of C^k functions $f : M \to F$ in the usual sense.

The recursion argument. An axiomatic definition of functions of class C^k is very nice, but without interest if one cannot use it to prove propositions concerning such functions. In our axiomatic setting, the main tool to use in proofs concerning properties of functions of class C^k is the recursion argument which we will now explain.

If $f : U \to F$ is a function of class C^k, it defines a whole tree of continuous functions: at each branching level, the number of branches is determined by the number of local linearizing factors. The continuous linearizing factors themselves define the next level of branching. For any $k \in \mathbf{N} \cup \{\infty\}$ the tree is of length $k + 1$.

Now suppose that we have such a tree of continuous functions of length $k + 1$, each level consisting of local linearizing factors for the functions of the preceding level. We claim that the functions of level i are of class C^{k-i}. For suppose the claim is false, then we could add the continuous functions of level i to the spaces C^{k-i} and obtain bigger spaces $\widehat{C}^{k-i}$ still satisfying property (C1). Since this contradicts the maximality (C2), the conclusion follows. (N.B. If $k < \infty$, the tree is of finite length $k + 1$ and a simple induction argument on i will also suffice.)

Now suppose we want to prove that a given continuous function $f = f_0$ is of class C^k. According to the above argument, it suffices to find a tree of continuous functions of length $k+1$ in which each level consists of local linearizing factors for the continuous functions of the preceding level. In some cases the construction of such a tree can be explicit, but in most cases the construction will be by induction on the level i. In any case we will say that we use *the recursion argument* to prove that f is of class C^k.

2. Some properties of $C^k(U, F)$

Proposition 2.1. *If $\mathcal{C}$ is a cat and $k \in \mathbf{N} \cup \{\infty\}$ then:*

(i) *All constant functions belong to $C^k(E, F)$,*

(ii) $\mathrm{Shom}(E, F) \subset C^k(E, F)$,

(iii) *If $f \in C^k(U, F)$ and $\widehat{U} \subset U$, then $f_{|\widehat{U}} \in C^k(\widehat{U}, F)$,*

(iv) *$C^k(U, F)$ is an abelian group under pointwise addition of functions.*

Proof: (i) The constant functions certainly are continuous. The result then follows from the recursion argument, taking at every next level the zero function on the whole of U.

(ii) If $f \in \mathrm{Shom}(E, F)$, then it is continuous by definition of Shom. Taking for ϕ the *constant* function $\phi(x, y) = f$ on U and taking at each next level the zero function, the result follows by the recursion argument.

(iii) Taking restrictions certainly preserves continuity. For $f \in C^k(U,F)$ there exist local linearizing factors (V,ϕ) of class C^{k-1}. If $f_r = f_{|\widehat{U}}$, we define ϕ_r by $\phi_r = \phi_{|(V\cap\widehat{U})^2}$. The result follows again by the recursion argument.

(iv) Since addition is continuous, it follows that $C^0(U,F)$ is an abelian group under pointwise addition of functions. For $g, h \in C^k(U,F)$ there exist (V,χ) and (V,θ), local linearizing factors for g and h of class C^{k-1}. By taking pairwise intersections, we may indeed assume that the domains of definition for ϕ and χ coincide. This enlarges the family of local linearizing factors for g and h, but does not affect the smoothness class by the result (iii). If the function $f \in C^0(U,F)$ is defined as $f = g + h$, we define the local linearizing factor (V,ϕ) by $\phi = \chi + \theta$. The result now follows by the recursion argument. ■

Remark 2.2. The result (ii) of the above proposition justifies our name for the elements of $\mathrm{Shom}(E,F)$: all its elements are homomorphisms of class C^∞.

Discussion 2.3. The next question of interest concerning functions of class C^k is whether this property is stable under composition of functions. Since this can not be done without some additional hypotheses concerning the topology of the spaces $\mathrm{Shom}(E,F)$, we introduce several conditions on these topologies.

Notations 2.4. For convenience sake, we will denote by $C^{-1}(U,F)$ the set (abelian group) of all set-theoretic maps $\phi : U \to F$. Associated to any n-additive map $\Psi : \prod_{i=1}^n F_i \to F_0$ we will define n-additive maps $\Psi_j : \mathrm{Hom}(E,F_j) \times \prod_{i\neq j} F_i \to \mathrm{Hom}(E,F_0)$ by the formula:

$$\Psi_j(f_1,\dots,f_{j-1},\phi_j,f_{j+1},\dots,f_n)(e) = \Psi(f_1,\dots,f_{j-1},\phi_j(e),f_{j+1},\dots,f_n).$$

To any map $\Psi : \prod_{i=1}^n F_i \to F_0$ we will associate a map $\Psi_\star : \prod_i C^{-1}(U,F_i) \to C^{-1}(U,F_0)$ by:

$$\Psi_\star(\phi_1,\dots,\phi_n)(e) = \Psi(\phi_1(e),\dots,\phi_n(e)).$$

For any space $E' \in \mathcal{C}$ we define the canonical injections $\iota : C^{-1}(U,F) \to C^{-1}(U \times E', F)$ by:

$$(\iota(f))(x,e') = f(x).$$

We also define the maps $\omega : C^{-1}(U,F) \times C^{-1}(U,F') \to C^{-1}(U, F \times F')$ (concatenation at the target space) by:

$$(\omega(f,f'))(x) = (f(x), f'(x)).$$

Finally we define the maps $\mathcal{L}, \mathcal{R} : \mathrm{Hom}(E, E) \to \mathrm{Hom}(\mathrm{Hom}(E, E), \mathrm{Hom}(E, E))$ of left and right multiplication by:

$$\begin{aligned}\mathcal{L}(A)(B) &= AB\\ \mathcal{R}(A)(B) &= BA.\end{aligned}$$

Remark 2.5. The maps Ψ_j should be seen as something like the partial derivatives of the n-additive map Ψ. For $n = 1$ the map Ψ_1 is just the restriction of $\Psi_\star$ to homomorphisms.

Definition 2.6. Let $\mathcal{C}$ be a cat and let $\Psi : \prod G_i \to \mathrm{Hom}(E, F)$ be a map in which each G_i is either an object in $\mathcal{C}$ or the full set of homomorphisms between two objects in $\mathcal{C}$. Since we do not require any $\mathrm{Hom}(E', F')$ to carry a topology, continuity of Ψ is not defined. Nevertheless, we will say that Ψ is continuous if the following *two* conditions are satisfied:

(i) if we take in the domain of definition of Ψ only smooth homomorphisms, then the image is also a smooth homomorphism, and
(ii) when restricted to smooth homomorphism spaces, Ψ is continuous.

Remark 2.7. Restriction to smooth homomorphism spaces in condition (ii) above means that all (restricted) G_j belong to the cat $\mathcal{C}$, as does the target space $\mathrm{Shom}(E, F)$.

Definitions 2.8. We now present some conditions on the topology of the spaces Shom in a cat $\mathcal{C}$ with generic objects E, E', F, F' and G.

(T-ι) All canonical injections $\iota : \mathrm{Hom}(E, F) \to \mathrm{Hom}(E \times E', F)$ are continuous.
(T-ω) All canonical isomorphisms $\omega : \mathrm{Hom}(E, F) \times \mathrm{Hom}(E, F') \to \mathrm{Hom}(E, F \times F')$ are continuous.
(T-$\mathcal{M}$) All canonical multiplications $\mathcal{M} : \mathrm{Shom}(F, G) \times \mathrm{Shom}(E, F) \to \mathrm{Shom}(E, G)$ ($\mathcal{M}(A, B) = A \circ B = AB$) are continuous.
(T-$\mathcal{LR}$) The maps $\mathcal{L}, \mathcal{R} : \mathrm{Shom}(E, E) \to \mathrm{Shom}(\mathrm{Shom}(E, E), \mathrm{Shom}(E, E))$ are continuous.
(T$_n$) For all continuous n-additive maps Ψ satisfying a predefined set of algebraic conditions, the associated maps Ψ_j, $j = 1, \dots, n$ are continuous and satisfy the same set of algebraic conditions.

Implicit in T-$\mathcal{LR}$ are the assumptions that for $A \in \mathrm{Shom}(E,E)$ the maps $\mathcal{L}(A)$ and $\mathcal{R}(A)$ preserve $\mathrm{Shom}(E,E) \subset \mathrm{Hom}(E,E)$ and that they are smooth.

Examples 2.9. The cats $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_M$ and $\mathcal{C}_V$ of Example 1.3 all satisfy T-ι, T-ω, T-$\mathcal{M}$ and T_n for all $n \in \mathbf{N}$. For the cat $\mathcal{C}_A$, and for the cats $\mathcal{C}_B$ and $\mathcal{C}_V$ over $\mathbf{R}$, there are no additional algebraic conditions in T_n. For the cats $\mathcal{C}_B$ and $\mathcal{C}_V$ over $\mathbf{C}$ the additional algebraic condition is n-linearity over $\mathbf{C}$. (N.B. A continuous additive map is $\mathbf{R}$-linear, but not necessarily $\mathbf{C}$-linear.) Likewise, for the cat $\mathcal{C}_M$ the algebraic condition is to be n-linear over the ring $\mathcal{A}$.

Nota Bene 2.10. In the sequel we will never mention again the set of algebraic conditions alluded to in condition T_n. However, whenever we invoke this condition, it should be understood that the n-additive map in question should satisfy these unmentioned additional algebraic conditions. This concerns in particular the maps ι, $\mathcal{M}$, $\mathcal{L}$, $\mathcal{R}$ and ω of conditions T-ι, T-$\mathcal{M}$, T-$\mathcal{LR}$ and T-ω.

Remark 2.11. If a cat $\mathcal{C}$ satisfies T_1, then left composition of a smooth homomorphism with a continuous homomorphism will be a smooth homomorphism. It follows that there cannot be a big difference between continuous homomorphisms and smooth homomorphisms. Moreover, the conditions T_1 and T-$\mathcal{M}$, although not the same, are very close. If the sets $\mathrm{Shom}(E,F)$ consist of *all* continuous morphisms, then T-$\mathcal{M}$ implies T_1.

Lemma 2.12. *Let $\mathcal{C}$ be a cat satisfying T_1 and let $\Psi : F_1 \to F_0$ be a continuous homomorphism. Then $\Psi_\star : C^k(U,F_1) \to C^k(U,F_0)$.*

Proof: Since Ψ is continuous, the result is certainly true for $k = 0$. For $f \in C^k(U,F)$, let ϕ be a local linearizing factor of class C^{k-1}. It follows that $\Psi_1 \circ \phi$ is a local linearizing factor for $\Psi \circ f$. The result now follows by the recursion argument and T_1. ■

Remark 2.13. The above lemma should be interpreted as saying that, if $f : U \to F_1$ is of class C^k, then $\Psi \circ f : U \to F_0$ is also of class C^k, i.e., as saying that left composition with a continuous homomorphism does not alter the smoothness class of a function. A similar interpretation holds for the next lemmas.

Lemma 2.14. *If $\mathcal{C}$ satisfies T_1 and T-ι, then $\iota : C^k(U,F) \to C^k(U \times E', F)$.*

Proof: The result is true for $k = 0$ since the canonical projection $E \times E' \to E$ is continuous ($E \times E'$ has the product topology). For $f \in C^k(U,F)$, let (V,ϕ) be a local linearizing factor of class C^{k-1}. According to the case $k = 0$ we have the canonical injection $\hat{\iota} : C^0(V^2, \mathrm{Shom}(E,F)) \to C^0((V \times E')^2, \mathrm{Shom}(E,F))$. Moreover, by T-$\iota$ the map $\check{\iota} : \mathrm{Shom}(E,F) \to \mathrm{Shom}(E \times E', F)$ is continuous. A simple calculation then shows that the map $\check{\iota}_\star(\hat{\iota}(\phi))$ is a continuous local linearizing factor for $\iota(f)$ on $V \times E'$. We now can start the induction needed for the recursion argument because by Lemma 2.12 we know how to deal with $\check{\iota}_\star$ and by the above argument we know how to deal with $\hat{\iota}(\phi)$. Application of the recursion argument finishes the proof. ■

Lemma 2.15. *If $\mathcal{C}$ satisfies T_1 and T-ω, then*

$$\omega : C^k(U,F) \times C^k(U,F') \to C^k(U, F \times F').$$

Proof: By definition of the product topology, the result is true for $k = 0$. For $f \in C^k(U,F)$ and $f' \in C^k(U,F')$ let (V,ϕ) and (V,ϕ') be local linearizing factors of class C^{k-1}. As in the proof of Lemma 2.1.(iv), we may assume that the domains of definition of ϕ and ϕ' coincide. According to the case $k = 0$ we have the map $\hat{\omega} : C^0(V^2, \mathrm{Shom}(E,F)) \times C^0(V^2, \mathrm{Shom}(E,F')) \to C^0(V^2, \mathrm{Shom}(E,F) \times \mathrm{Shom}(E,F'))$. Moreover, by T-$\omega$ the map $\check{\omega} : \mathrm{Shom}(E,F) \times \mathrm{Shom}(E,F') \to \mathrm{Shom}(E, F \times F')$ is continuous. A simple calculation then shows that the map $\check{\omega}_\star(\hat{\omega}(\phi,\phi'))$ is a continuous local linearizing factor for $\omega(f,f')$ on V. We now can start the induction needed for the recursion argument because by Lemma 2.12 we know how to deal with $\check{\omega}_\star$ and by the above argument we know how to deal with $\hat{\omega}(\phi,\phi')$. Application of the recursion argument finishes the proof. ■

Lemma 2.16. *Let $\mathcal{C}$ be a cat satisfying T_1, T_n and T-ι, and let $\Psi : \prod_i F_i \to F_0$ be a continuous n-additive map. Then $\Psi_\star : \prod_i C^k(U,F_i) \to C^k(U,F_0)$.*

Proof: The proposition is obviously true for $k = 0$ since by assumption Ψ is continuous. Now let f_i be n functions of class C^k and (V,ϕ_i) local linearizing factors of class C^{k-1} for the f_i. Analogous to the proofs of Lemmas 2.1.(iv) and 2.15, by taking n-fold intersections we may assume that the domains of definition of the ϕ_i coincide. We then compute:

$$\begin{aligned}
&\Psi_\star(f_1,\dots,f_n)(x) - \Psi_\star(f_1,\dots,f_n)(y)\\
&\qquad\equiv \Psi(f_1(x),\dots,f_n(x)) - \Psi(f_1(y),\dots,f_n(y))\\
&\qquad= \sum_{i=1}^{n} \Psi(f_1(x),\dots,f_{i-1}(x), f_i(x)-f_i(y), f_{i+1}(y),\dots,f_n(y))\\
&\qquad= \sum_{i=1}^{n} \Psi_i\big(f_1(x),\dots,f_{i-1}(x),\phi_i(x,y),f_{i+1}(y),\dots,f_n(y)\big)(x-y)\\
&\qquad= \Big(\sum_{i=1}^{n}(\Psi_{i\star}(f_1,\dots,f_{i-1},\phi_i,f_{i+1},\dots,f_n)(x,y)\Big)(x-y).
\end{aligned}$$

To obtain the last line of this calculation, we have extended the functions f_j to depend upon x *and* y. It follows from Lemma 2.14 that all the extended f_j are of class C^k and thus of class C^{k-1}, as are the ϕ_j. By Proposition 2.1.(iv) we know how to deal with sums, hence we can apply the recursion argument to finish the proof, just because of T_n. ∎

Corollary 2.17. *In the cat $\mathcal{C}_M$ (and likewise in $\mathcal{C}_V$ and $\mathcal{C}_B$), $C^k(U,\mathcal{A})$ is a commutative ring and $C^k(U,F)$ is a module over $C^k(U,\mathcal{A})$.*

Proof: By assumption, the multiplication $\mathcal{A}\times F \to F$ is a continuous bi-additive map. It then follows immediately from Lemma 2.16 that pointwise multiplication satisfies $C^k(U,\mathcal{A})\times C^k(U,F)\to C^k(U,F)$. The result now follows by taking $F=\mathcal{A}$ and realizing that $\mathcal{A}$ is a commutative ring. ∎

Proposition 2.18. *If a cat $\mathcal{C}$ satisfies the conditions T_1, T_2, T-ι, T-$\mathcal{M}$ and T-ω, then for all $f\in C^k(U,F)$, $U\subset E$, and all $g\in C^k(V,G)$, $V\subset F$ we have: if $g\circ f$ is defined, then $g\circ f\in C^k(U,G)$.*

Proof: The proposition is certainly true for $k=0$. Let f and g be as above of class C^k, let (V',χ) be local linearizing factor of class C^{k-1} for g and let (U',ϕ) be local linearizing factor of class C^{k-1} for f. By taking, if needed, intersections with $g^{-1}(V')$, we may assume that $f(U')\subset V'$. We then compute:

$$\begin{aligned}
(g\circ f)(x)-(g\circ f)(y) &= \chi(f(x),f(y))(f(x)-f(y))\\
&= \chi(f(x),f(y))(\phi(x,y)(x-y))\\
&= \big(\chi(f(x),f(y))\circ\phi(x,y)\big)(x-y)\\
&= \mathcal{M}\big(\chi(f(x),f(y)),\phi(x,y)\big)(x-y)\\
&= \big(\mathcal{M}_\star(\chi\circ\widetilde{f},\phi)(x,y)\big)(x-y),
\end{aligned}$$

where we have defined $\widetilde{f} : (U')^2 \to F^2$ by $\widetilde{f}(x,y) = (f(x), f(y))$. By Lemmas 2.14 and 2.15 $\widetilde{f}$ is again of class C^k. It thus follows that $\mathcal{M}_\star(\chi \circ \widetilde{f}, \phi)$ is a continuous local linearizing factor on U' for $g \circ f$. By Lemma 2.16 we know how to deal with $\mathcal{M}_\star$ and by the above argument we know how to deal with the composition $\chi \circ \widetilde{f}$. We thus can apply the recursion argument to conclude. ■

Discussion 2.19. Once we know that composition of functions preserves the smoothness class, the next question is whether one can prove some kind of inverse function theorem. Now this is far too much to ask without a more detailed knowledge of the cat in question. What we *can* do is reduce the statement of the inverse function theorem to a statement concerning topology only. To be able to speak about inverses, we define the set $D\mathcal{I} \subset \mathrm{Shom}(E,E)$ by:

$$D\mathcal{I} = \{A \in \mathrm{Shom}(E,E) \mid A \text{ invertible}\},$$

and we define the map $\mathcal{I} : D\mathcal{I} \to \mathrm{Hom}(E,E)$ by: $\mathcal{I}(A) = A^{-1}$.

Proposition 2.20. *Let $\mathcal{C}$ be a cat satisfying T_1, T_2, T-ι, T-$\mathcal{LR}$, T-$\mathcal{M}$ and T-ω. If $\mathcal{I}$ is continuous for all E, then $\mathcal{I}$ is of class C^∞ for all E.*

Proof: Note first that continuity of $\mathcal{I}$ implies that $\mathcal{I}(\mathcal{DI}) \subset \mathrm{Shom}(E,E)$. We then start with the obvious calculation of $\mathcal{I}(x) - \mathcal{I}(y)$ yielding:

$$\mathcal{I}(x) - \mathcal{I}(y) = -\big(\mathcal{L}(\mathcal{I}(x)) \circ \mathcal{R}(\mathcal{I}(y))\big)(x-y),$$

which suggests as linearizing factor on the whole of $\mathcal{DI}$ the map:

$$\begin{aligned} \phi(x,y) &= -\mathcal{M}(\mathcal{L}(\mathcal{I}(x)), \mathcal{R}(\mathcal{I}(y))) \\ &= -\mathcal{M}_\star(\mathcal{L} \circ \mathcal{I}, \mathcal{R} \circ \mathcal{I})(x,y) \\ &= -\mathcal{M}_\star(\mathcal{L}_\star(\mathcal{I}), \mathcal{R}_\star(\mathcal{I}))(x,y) \\ \iff \qquad \phi &= -\mathcal{M}_\star(\mathcal{L}_\star(\mathcal{I}), \mathcal{R}_\star(\mathcal{I})). \end{aligned}$$

From the first line we see that ϕ is continuous; to obtain the second line, we have used Lemma 2.14 to see $\mathcal{L}(\mathcal{I}(x))$ and $\mathcal{R}(\mathcal{I}(y))$ as functions of (x,y). But then we can start the recursion argument, using Lemma 2.16. ■

Proposition 2.21. *Let $\mathcal{C}$ be a cat satisfying all conditions of Proposition 2.20, including the condition on $\mathcal{I}$. Let (V,ϕ) be a local linearizing* *factor for some* $f \in C^k(U, E)$ *satisfying:* $\forall x, y \in V : \phi(x, y) \in D\mathcal{I}$. *If* $f : V \to f(V)$ *is a homeomorphism, then* $f^{-1} \in C^k(f(V), E)$.

Proof: Denoting the inverse function by $g = f^{-1}$, the fact that ϕ is a linearizing factor for f on V implies:

$$g(p) - g(q) = \mathcal{I}(\phi(g(p), g(q)))(p - q).$$

It follows that $\mathcal{I}(\phi(g(p), g(q)))$ is a continuous linearizing factor for g. Since $\mathcal{I}$ is of class C^∞ and ϕ is of class C^{k-1}, we can apply the recursion argument to conclude. ■

Remark 2.22. The above proposition is the easy part of an inverse function theorem. The hard part is of the form "$\phi(x_o, x_o)$ invertible $\Longrightarrow$ f is a local homeomorphism".

It should be noted that the existence of a local linearizing factor as demanded in Proposition 2.21 is not guaranteed (remember, V should be *open* in U; else it would be trivial, taking V a single point). However, if $D\mathcal{I} \subset \mathrm{Shom}(E, E)$ is *open*, then a condition like $\phi(x_o, x_o) \in D\mathcal{I}$, together with the continuity of any linearizing factor ϕ on a neighbourhood of x_o, implies the existence of a local linearizing factor $\phi_{|V}$ on $V \ni x_o$ with the required properties.

3. An application to graded manifolds

The idea behind graded (or super) manifolds is to incorporate the notion of anti-commuting variables, inspired by the anti-commuting nature of fermion fields in physics. We want to follow this idea here to its extreme conclusion and replace the standard field of reals $\mathbf{R}$, on which all of differential geometry is founded, by a non-commutative ring $\mathcal{A}$. Vector spaces will be replaced by modules over $\mathcal{A}$ and we will use our axiomatic approach to define smooth functions on open subsets of such modules. These in turn then can serve as the local model for graded manifolds. Since it is outside the scope of this paper to provide all details of graded manifolds, this section will be rather sketchy; more details can be found in [**Tu**].

The algebra $\mathcal{A}$. Let X be an infinite dimensional vector space over $\mathbf{R}$; we define $\mathcal{A}$ as the full exterior algebra over X:

$$\mathcal{A} = \Lambda X = \bigoplus_{k \in \mathbf{N}} \Lambda^k X.$$

In $\mathcal{A}$ we define three subspaces $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{N}$ of even, odd and nilpotent elements respectively by : $\mathcal{A}_0 = \bigoplus_{k\in\mathbf{N}} \Lambda^{2k}X$, $\mathcal{A}_1 = \bigoplus_{k\in\mathbf{N}} \Lambda^{2k+1}X$ and $\mathcal{N} = \bigoplus_{k\in\mathbf{N}} \Lambda^{k+1}X$. With these definitions, one has the obvious equalities:

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 = \mathbf{R} \oplus \mathcal{N}.$$

Note that by definition of the exterior algebra one has indeed the property: $\forall n \in \mathcal{N}\ \exists k \in \mathbf{N} : n^k = 0$, justifying the name for $\mathcal{N}$. Note also that one has the property:

$$a_i \in \mathcal{A}_i \quad \text{and} \quad a_j \in \mathcal{A}_j \quad \Longrightarrow \quad a_i a_j = (-1)^{ij} a_j a_i \in \mathcal{A}_{i+j},$$

where $i + j$ should be read modulo 2. In particular, the elements of $\mathcal{A}_1$ anti-commute, and $\mathcal{A}_0$ is a commutative subalgebra of $\mathcal{A}$.

We will denote by $\mathbf{B}$ the canonical projection $\mathbf{B} : \mathcal{A} = \mathbf{R} \oplus \mathcal{N} \to \mathcal{A}/\mathcal{N} = \mathbf{R}$.

Definition 3.1. A graded module over $\mathcal{A}$ is a (left) module E over $\mathcal{A}$ with two distinguished subspaces E_0 and E_1, called its even and odd part, satisfying:

$$E = E_0 \oplus E_1 \quad \text{and} \quad \forall a \in \mathcal{A}_i\ \forall e \in E_j : ae \in E_{i+j}.$$

If E is a graded module over $\mathcal{A}$, we can turn it into a bi-module by defining the right action of $\mathcal{A}$ on E by:

$$a \in \mathcal{A}_i, e \in E_j \quad \Longrightarrow \quad ea = (-1)^{ij} ae.$$

We leave it to the reader to verify that this indeed defines a right action. Moreover, $\mathcal{A}$ itself is a graded module, the left and right action just being the left and right multiplication.

Definition 3.2. A special graded module is a graded module E together with a finite set of elements $e_1, \ldots, e_p \in E_0$ and $f_1, \ldots, f_q \in E_1$ such that E is (isomorphic to) the free module generated by the $p + q$ elements $e_1, \ldots, f_q$. For any special graded module we define a map $\mathbf{B} : E \to \mathbf{R}^{p+q}$ by:

$$\mathbf{B}\left(\sum_{i=1}^{p} x^i e_i + \sum_{j=1}^{q} \xi^j f_j\right) = \big(\mathbf{B}(x^1), \ldots, \mathbf{B}(x^p), \mathbf{B}(\xi^1), \ldots, \mathbf{B}(\xi^q)\big).$$

Remarks 3.3. (i) It can be shown that any finitely generated free graded $\mathcal{A}$-module E is a special graded module in the above sense. Moreover, the pair (p, q) is an invariant of E called its (graded) dimension. The algebra $\mathcal{A}$ itself is a special graded module of dimension $(1, 0)$.

(ii) The map $\mathbf{B}$, defined above in terms of coordinates with respect to a basis, can be given a more intrinsic definition in the following way. Define the set of "nilpotent" vectors $E_{\mathcal{N}}$ by $E_{\mathcal{N}} = \{e \in E \mid \exists n \in \mathcal{N} : n \neq 0$ and $ne = 0\}$. The quotient $E/E_{\mathcal{N}}$ is isomorphic to $\mathbf{R}^{p+q}$, and the map $\mathbf{B}$ translates as the canonical projection $\mathbf{B} : E \to E/E_{\mathcal{N}} \cong \mathbf{R}^{p+q}$.

Discussion 3.4. If E and F are special graded modules of dimensions (p_E, q_E) and (p_F, q_F) respectively, then $\mathrm{Hom}_{\mathcal{A}}(E, F)$ of (left) $\mathcal{A}$-module morphisms is a special graded module of dimension $(p_E\, p_F + q_E\, q_F, p_E\, q_F + q_E\, p_F)$; a morphism f is even if $f(E_i) \subset F_i$ and odd if $f(E_i) \subset F_{i+1}$. Apart from a topology, we thus have constructed a cat in which the objects are special graded modules over the algebra $\mathcal{A}$. However, it turns out that the cat $\mathcal{C}_{gr}$ that is at the basis of the theory of graded manifolds is a variation on the above.

The cat $\mathcal{C}_{gr}$. The objects in the cat $\mathcal{C}_{gr}$ are the even parts of special graded modules. If E_0 and F_0 are two objects in $\mathcal{C}_{gr}$, then the object $\mathrm{Shom}(E_0, F_0)$ is defined as the even part of $\mathrm{Hom}_{\mathcal{A}}(E, F)$. This is indeed a subset of the (additive) morphisms from E_0 to F_0 because of the definition of parity in $\mathrm{Hom}_{\mathcal{A}}(E, F)$.

To define the topology on an object E_0, consider the canonical projection $\mathbf{B} : E \to \mathbf{R}^{p+q}$. Define the topology on E as the coarsest topology for which this map $\mathbf{B}$ is continuous. Finally, equip E_0 with the relative topology; it is usually called "the DeWitt topology" ([**DW**]).

Remark 3.5. If E_0 is an object in $\mathcal{C}_{gr}$, an element $e \in E_0$ can be characterised by $p + q$ coordinates $(x^1, \ldots, x^p, \xi^1, \ldots, \xi^q)$ with $x^i \in \mathcal{A}_0$ and $\xi^j \in \mathcal{A}_1$. This simple observation has the important consequence that we would have been in serious trouble, had we chosen a finite dimensional vector space X in our definition of $\mathcal{A}$. To see why this is so, suppose X has dimension n. It follows that $\Lambda^k X = \{0\}$ for $k > n$. For simplicity we now assume that n is even, a similar argument will hold in case n is odd. Let E be a special graded module of dimension $(0, 1)$ and let $e \in E_1$ be a basis. Now let $a \in \Lambda^n X$ be non-zero and define the morphism $a : E \to E$ by $a(xe) = (ax)e$. This is a perfectly well defined even morphism in the $(1, 0)$ dimensional special graded module $\mathrm{Hom}_{\mathcal{A}}(E, E)$; with respect to its canonically induced basis, this morphism has coordinate a.

But now, what happens when we restrict this (even) morphism to the even part E_0 of E? Since xe is even if and only if x is odd, it follws that the induced map is identically zero because $x \in \mathcal{N}$ and thus $ax \in \bigoplus_{k>n} \Lambda^k X = \{0\}$! The inescapable conclusion is that $\mathrm{Hom}_{\mathcal{A}}(E, F)_0$ is not itself a subset of the (additive) morphisms from E_0 to F_0, but projects onto such a subset. It follows that this subset itself is *not* the even part of a special graded module. Whence the trouble.

When one uses a matrix element approach for linearizing factors instead of the morphism approach introduced here, the trouble does not disappear. It only surfaces in an additional non-uniqueness of the linearizing matrix elements, prohibiting a unique diagonal (which should be the matrix of partial derivatives).

Proposition 3.6. *$\mathcal{C}_{gr}$ is a well defined cat satisfying T-ι, T-$\mathcal{LR}$, T-$\mathcal{M}$, T-ω and T_n for all $n \geq 1$. Moreover, the set $D\mathcal{I} \subset \mathrm{Shom}(E_0, E_0)$ is open and the map $\mathcal{I}$ is continuous.*

Remark 3.7. The algebraic condition to be used in conjunction with condition T_n is that it should concern restrictions of maps which are n-linear in the sense of $\mathcal{A}$-modules.

Discussion 3.8. In view of the above proposition, we now can apply all results of the previous section to the cat $\mathcal{C}_{gr}$. Unfortunately, the non-Hausdorff character of the topology prohibits the existence of a derivative, i.e., the uniqueness of the diagonal of a linearizing factor. To see why, note that any set theoretic function $f : U \subset E_0 \to \mathcal{A}_0 \cap \mathcal{N}$ is continuous for the DeWitt topology. This implies that the limit argument used in Example 1.11 to prove the uniqueness of the diagonal $\phi(x, x)$ is not valid in $\mathcal{C}_{gr}$. Moreover, it is relatively easy to construct examples of functions that admit linearizing factors with different diagonals.

Besides the absence of a well defined derivative, there is a second drawback to the general approach of the previous section. The original idea of graded manifolds was (and still is) to extend the standard theory of manifolds with anti-commuting variables. In this context, extending obviously means that nothing should change if there are no such odd coordinates. In the cat $\mathcal{C}_{gr}$ we have indeed commuting (even) coordinates x^i and anti-commuting (odd) coordinates ξ^j (see Remark 3.5). But in the absence of odd coordinates (i.e., if $q = 0$) we do *not* recover the standard theory of C^k-functions on $\mathbf{R}^p$.

For C^∞-functions, both problems can be solved simultaneously. However, before we can explain the trick, we first have to analyze the structure of C^∞-functions in more detail. So let $f \in C^\infty(U, F)$ be a C^∞-function of a single even or odd variable y (i.e., the dimension of $E \supset U$ is either $(1, 0)$ or $(0, 1)$), and let $a \in \mathcal{N}$ be of the same parity as y. Repeated use of local linearizing factors around a point $y_o \in U$ shows the existence of functions $f^{(k)}$ defined on (decreasing) neighbourhoods

$V_k \ni y_o$, $k \in \mathbf{N}$ such that:

$$f(y+a) = \sum_{k\in\mathbf{N}} \frac{1}{k!} f^{(k)}(y) a^k, \tag{3.9}$$

where we have added the combinatorial factor $1/k!$ for convenience sake. Note that this sum is actually finite because a is nilpotent; if a (and thus y) is odd, it only has two terms because the anti-commutativity forces $a^2 = 0$. Note also that there is no restriction on the nilpotent element a: if y lies in a certain neighbourhood, then by definition of the DeWitt topology $y + a$ also lies in this neighbourhood. If y is odd, this has the important consequence that the expansion 3.9 is valid for all y in V_1 and all a in $\mathcal{N}_1$.

Next let $f \in C^\infty(U, F)$ be arbitrary with $E \supset U$ of dimension (p, q). The expansion 3.9 applied to the odd coordinates implies the existence of local C^∞-functions $f_{i_1,\dots,i_k}$, $0 \le k \le q$, $1 \le i_1 < \cdots < i_k \le q$ of the even coordinates x only such that:

$$f(x^1, \dots, x^p, \xi^1, \dots, \xi^q) = \sum_{k=0}^{q} \sum_{1\le i_1<\cdots<i_k\le q} f_{i_1,\dots,i_k}(x^1, \dots, x^p)\, \xi^{i_1} \cdots \xi^{i_k}.$$

In view of the above expansion, the two disadvantages can be described by saying that (i) the functions $f_{i_1,\dots,i_k}$ are not unique, and (ii) they do not, in general, correspond to real valued functions of real coordinates.

Both these disadvantages (for C^∞-functions only!) can be neutralized in a single stroke: in condition (C1) we just add the condition that $f(x, \xi)$ has real coordinates whenever the original coordinates (x, ξ) are real (and thus in particular $\xi = 0$)! It should be noted that all results of the previous section remain valid with this modified condition (C1), with the obvious modifications to Proposition 2.1.(i)/(ii). The first immediate consequence of our additional assumption is that the restriction of a C^∞-function to real coordinates is a C^∞-function in the sense of the cat $\mathcal{C}_B$. In particular, the diagonal of a linearizing factor in points with real coordinates is the (unique) Fréchet derivative. From there it is an easy step to prove that the function $f^{(k)}$ in the expansion 3.9 is the k-th order derivative of the function f, provided the coordinate y is real. Since an even coordinate can be uniquely decomposed into a real part and a nilpotent part, formula 3.9 then tells us that the function f is completely determined by its restriction to real coordinates, which is equivalent to an ordinary real valued C^∞-function. No additional difficulties are involved in proving that the same result holds in case there are several even coordinates. We thus may conclude that the functions $f_{i_1,\dots,i_k}$ are indeed unique, that they have well defined derivatives and that they are in $1-1$ correspondence with ordinary real-valued C^∞-functions.

What we have done in the above discussion, is to deduce the usual definition of graded C^∞-functions from our axiomatic definition (using a slightly modified condition (C1)). To be more precise, we have "shown" an isomorphism between $C^\infty(U;F)$ and the tensor product $C^\infty(O;\mathbf{B}F) \otimes \bigwedge \mathbf{R}^q$, where O is the unique open subset of $\mathbf{R}^p$ such that $\mathbf{B}U = O$ (remember that $\mathbf{B}E_0 \cong \mathbf{R}^p$). Once we have this result, it is not hard to deduce an inverse function theorem in the cat $\mathcal{C}_{gr}$, using the inverse function theorem for real-valued functions. With these results, the rest of differentiable manifold theory follows by just copying the standard theory. The advantage of our definition of smooth superfunctions over others (e.g., see [**Ba80**] and [**Ro**] and references given therein) is that we can use *the same* definition for ordinary real-valued smooth functions as for smooth superfunctions, that we do not need a complicated new definition/construction, and that we do not need a Banach algebra structure on $\mathcal{A}$.

Remark 3.10. The kind of graded manifolds one gets with the above describe method is equivalent (although not canonically) to the one given by Kostant-Batchelor ([**Ko**], [**Ba79**]). Other kinds of graded/super manifolds exist; the reader is referred to [**BBH**] and references given therein for more details.

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