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THE VORONOVSKAYA THEOREM FOR SOME LINEAR POSITIVE OPERATORS IN EXPONENTIAL WEIGHT SPACES

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Abstract

In this note we give the Voronovskaya theorem for some linear positive operators of the Szasz-Mirakjan type defined in the space of functions continuous on $[0,+\infty)$ and having the exponential growth at infinity.

Some approximation properties of these operators are given in [**3**], [**4**].

1. Preliminaries

1.1. Let $R_0 := [0,+\infty)$, $N := \{1,2,\dots\}$, $N_0 := N\cup\{0\}$ and let $w_r(\cdot)$, $r>0$, be the weight function defined on R_0 by the formula

$$w_r(x) := e^{-rx}. \tag{1}$$

Similarly as in [**1**] we denote by C_r, $r>0$, the space of real-valued functions f defined on R_0 and such that $w_r f$ is a uniformly continuous and bounded function on R_0. The norm in C_r is defined by

$$\|f\|_r := \sup_{x\in R_0} w_r(x)|f(x)|.$$

For a fixed $r>0$ let

$$C_r^2 := \{f\in C_r : f', f''\in C_r\}.$$

1.2. In [**3**] were introduced the following operators of the Szasz-Mirakjan type for functions $f\in C_r$, $r>0$,

$$L_n^{(1)}(f;x) := \sum_{k=0}^{\infty} p_{n,k}(x)\, f\left(\frac{2k}{n}\right), \tag{2}$$

$$L_n^{(2)}(f;x) := \sum_{k=0}^{\infty} p_{n,k}(x)\frac{n}{2}\int_{I_{n,k}} f(t)\,dt, \tag{3}$$

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$x \in R_0$, $n \in N$, where

$$p_{n,k}(x) := \frac{1}{\cosh nx} \frac{(nx)^{2k}}{(2k)!}, \quad k \in N_0, \tag{4}$$

$\sinh x$, $\cosh x$, $\tanh x$ are the elementary hyperbolic functions and $I_{n,k} := \left[\frac{2k}{n}, \frac{2k+2}{n}\right]$, $k \in N_0$.

In [**4**] were introducend the operators

$$L_n^{(3)}(f;x) := \frac{f(0)}{1+\sinh nx} + \sum_{k=0}^{\infty} q_{n,k}(x)\, f\left(\frac{2k+1}{n}\right), \tag{5}$$

$$L_n^{(4)}(f;x) := \frac{f(0)}{1+\sinh nx} + \sum_{k=0}^{\infty} q_{n,k}(x) \frac{n}{2} \int_{I_{n,k}^*} f(t)\, dt, \tag{6}$$

$x \in R_0$, $n \in N$, where

$$q_{n,k}(x) := \frac{1}{1+\sinh nx} \frac{(nx)^{2k+1}}{(2k+1)!}, \tag{7}$$

and $I_{n,k}^* := \left[\frac{2k+1}{n}, \frac{2k+3}{n}\right]$ for $k \in N_0$.

We observe that the above operators are linear positive operators well-defined on every space C_r, $r > 0$, and

$$L_n^{(i)}(1;x) = 1, \quad 1 \le i \le 4, \tag{8}$$

for all $x \in R_0$ and $n \in N$.

In [**3**] and [**4**] it was proved that $L_n^{(i)}$, $1 \le i \le 4$, are operators from C_r into C_s for every fixed $s > r > 0$ provided n is large enough. Moreover in [**3**], [**4**] some approximation properties of there operators were given. In particular in [**3**], [**4**] we proved the following

Theorem A. *Suppose that r, s, n_0 are fixed numbers such that $s > r > 0$, $n_0 \in N$ and $n_0 > r\left(\ln \frac{s}{r}\right)^{-1}$. If $f \in C_r$, then there exists a positive constant $M_1 \equiv M_1(n_0, r, s)$ depending only on n_0, r, s such that for all $x \in R_0$, $n_0 < n \in N$ and $1 \le i \le 4$*

$$w_s(x) \left| L_n^{(i)}(f;x) - f(x) \right| \le M_1\, \omega\left(f, C_r; \sqrt{\frac{x+1}{n}}\right),$$

where $\omega(f; C_r; \cdot)$ is the modulus of continuity of f, i.e.,

$$\omega(f; C_r; t) := \sup_{0<h\le t} \|f(\cdot + h) - f(\cdot)\|_r .$$

2. Auxiliary results

In this part we shall give some properties of the operators $L_n^{(i)}$. Let

$$
\begin{aligned}
S(nx) &:= \frac{\sinh nx}{1+\sinh nx}, \\
T(nx) &:= \frac{\cosh nx}{1+\sinh nx}, \qquad (9)\\
V(nx) &:= 1-\tanh nx,
\end{aligned}
$$

for $n \in N$ and $x \in R_0$. By elementary calculations from (2)-(8) and (9) we obtain

Lemma 1. *For all $x \in R_0$ and $n \in N$ we have*

$$L_n^{(1)}(t-x;x) = -xV(nx),$$

$$L_n^{(1)}\left((t-x)^2;x\right) = \left(2x^2-\frac{x}{n}\right)V(nx)+\frac{x}{n},$$

$$L_n^{(1)}\left((t-x)^4;x\right) = \left(8x^4-\frac{12x^3}{n}+\frac{4x^2}{n^2}-\frac{x}{n^3}\right)V(nx)+\frac{3x^2}{n^2}+\frac{x}{n^3},$$

$$L_n^{(2)}(t-x;x) = -x\,V(nx)+\frac{1}{n},$$

$$L_n^{(2)}\left((t-x)^2;x\right) = \left(2x^2-\frac{3x}{n}\right)V(nx)+\frac{x}{n}+\frac{4}{3n^2},$$

$$L_n^{(2)}\left(t-x)^4;x\right) = \left(8x^4-\frac{28x^3}{n}+\frac{32x^2}{n^2}-\frac{21x}{n^3}\right)V(nx)+\frac{12x}{n^3}+\frac{16}{5n^4},$$

$$L_n^{(3)}(t-x;x) = x\left(T(nx)-1\right),$$

$$L_n^{(3)}\left((t-x)^2;x\right) = x^2\left(S(nx)-2\,T(nx)+1\right)+\frac{x}{n}\,V(nx),$$

$$
\begin{aligned}
L_n^{(3)}\left((t-x)^4;x\right) &= x^4\left(7\,S(nx)-8\,T(nx)+1\right)+\frac{12x^3}{n}\left(T(nx)-S(nx)\right)\\
&\quad+\frac{x^2}{n^2}\left(7\,S(nx)-4\,T(nx)\right)+\frac{x}{n^3}\,T(nx),
\end{aligned}
$$

$$L_n^{(4)}(t-x;x) = x\left(T(nx)-1\right)+\frac{1}{n}S(nx),$$

$$
\begin{aligned}
L_n^{(4)}\left((t-x)^2;x\right) &= x^2\left(S(nx)-2T(nx)+1\right)\\
&\quad+\frac{2x}{n}\left(T(nx)-S(nx)\right)+\frac{4}{3n^2}S(nx),
\end{aligned}
$$

$$
\begin{aligned}
L_n^{(4)}\left((t-x)^4;x\right) &= x^4\left(7S(nx)-8T(nx)+1\right)+\frac{28x^3}{n}\left(T(nx)-S(nx)\right)\\
&\quad+\frac{x^2}{n^2}\left(35\,S(nx)-32\,T(nx)\right)+\frac{17x}{n^3}\,T(nx).
\end{aligned}
$$

Using Lemma 1, we shall prove two lemmas.

Lemma 2. *For every fixed $x_0 \in R_0$ one has*

$$\lim_{n\to\infty} nL_n^{(i)}(t-x_0;x_0) = \begin{cases} 0 & \text{if } i=1,3, \\ 1 & \text{if } i=2,4, \end{cases} \tag{10}$$

and

$$\lim_{n\to\infty} nL_n^{(i)}((t-x_0)^2;x_0) = x_0 \quad \text{for} \quad 1 \le i \le 4. \tag{11}$$

Proof: We shall prove only (10) and (11) for $i=3$, because the proof for $i=1,2,4$ is analogous.

By Lemma 1 and (9) we have

$$nL_n^{(3)}(t-x;x) = \frac{nx}{e^{2nx}(1+\sinh nx)} - \frac{nx}{(1+\sinh nx)},$$

$$nL_n^{(3)}((t-x)^2;x) = \frac{nx^2}{1+\sinh nx} - \frac{2nx^2}{e^{nx}(1+\sinh nx)} + \frac{x\cosh nx}{1+\sinh nx},$$

for every $x \in R_0$ and $n \in N$, which immediately yield (10) and (11). ∎

Lemma 3. *For every fixed $x_0 \in R_0$ there exists a positive constant $M_2(x_0)$, depending only on x_0, such that for all $n \in N$*

$$L_n^{(i)}\left((t-x_0)^4;x_0\right) \le M_2(x_0)n^{-2}, \quad 1 \le i \le 4. \tag{12}$$

Proof: For example we shall prove (12) for $L_n^{(1)}$. By (9) we have for $n \in N$, $p \in N$ and $x \in R_0$

$$0 \le x^p\, V(nx) = \frac{2x^p}{e^{2nx}+1} \le 2^{1-p}\, p!\, n^{-p}.$$

Applying the above inequality to the formula given in Lemma 1, we obtain

$$L_n^{(1)}\left((t-x_0)^4;x_0\right) \le \frac{47}{n^4} + \frac{3x_0^2}{n^2} + \frac{x_0}{n^3} \le M_2(x_0)n^{-2},$$

for every fixed $x_0 \ge 0$ and for all $n \in N$. ∎

The proof of (12) for $i=2,3,4$ is similar.

In the papers [**3**] (for $L_n^{(i)}$, $i=1,2$) and [**4**] (for $L_n^{(i)}$, $i=3,4$) we proved the following two lemmas.

Lemma 4. *Let $s > r > 0$ and let n_0 be a natural number such that*

$$n_0 > r \left(\ln \frac{s}{r} \right)^{-1}. \tag{13}$$

Then there exists a positive constant $M_3 \equiv M_3(r, s, n_0)$ depending only on r, s, n_0 such that for all $n > n_0$ and $i = 1, 2, 3, 4$

$$\left\| L_n^{(i)} \left(\frac{1}{w_r(t)}; \cdot \right) \right\|_s \le M_3.$$

Lemma 5. *Suppose that r, s and n_0 are a numbers as in Lemma 4. Then there exists a positive constant $M_4 \equiv M_4(r, s, n_0)$ depending only on r, s, n_0 such that for all $x \ge 0$, $n > n_0$ and $i = 1, 2, 3, 4$*

$$w_s(x)\, L_n^{(i)} \left(\frac{(t-x)^2}{w_r(t)}; x \right) \le M_3 \frac{x+1}{n}. \tag{14}$$

Applying the above lemmas, we shall prove

Lemma 6. *Suppose that x_0 is a fixed point on R_0 and $\varphi(\cdot; x_0)$ is a function belonging to a give space C_r, $r > 0$, such that $\lim_{t\to\infty} \varphi(t; x_0) = 0$, $\left(\lim_{t\to 0_+} \varphi(t; 0) = 0\right)$. Then*

$$\lim_{n\to\infty} L_n^{(i)} \left(\varphi(t; x_0); x_0 \right) = 0 \quad \text{for} \quad 1 \le i \le 4. \tag{15}$$

Proof: We shall prove (15) for $i = 1$, because the proof of (15) for $i = 2, 3, 4$ is analogous.

Choose $\varepsilon > 0$ and M_3 as in Lemma 4. Then by the properties of $\varphi(\cdot; x_0)$ there exist positive constants $\delta \equiv \delta(\varepsilon, M_3)$ and M_5 such that

$$w_r(t)\, |\varphi(t; x_0)| < \frac{\varepsilon}{2\, M_3} \quad \text{for } |t - x_0| < \delta,$$

$$w_r(t)\, |\varphi(t; x_0)| < M_5 \qquad \text{for } t \ge 0.$$

Denoting by $Q_{n,1} := \left\{ k \in N_0 : \left| \frac{2k}{n} - x_0 \right| < \delta \right\}$ and $Q_{n,2} := \left\{ k \in N_0 : \left| \frac{2k}{n} - x_0 \right| \ge \delta \right\}$, we get for $s > r$ and $n > n_0$ by (1)-(4) and Lemma 4

$$\begin{aligned} w_s(x_0) \left| L_n^{(1)} \left(\varphi(t; x_0); x_0 \right) \right| &\le w_s(x_0) \sum_{k=0}^{\infty} p_{n,k}(x_0) \left| \varphi\left(\frac{2k}{n}; x_0 \right) \right| \\ &= w_s(x_0) \sum_{k \in Q_{n,1}} p_{n,k}(x_0) \left| \varphi\left(\frac{2k}{n}; x_0 \right) \right| \\ &\quad + w_s(x_0) \sum_{k \in Q_{n,2}} p_{n,k}(x_0) \left| \varphi\left(\frac{2k}{n}; x_0 \right) \right| \\ &:= \sum_1 + \sum_2 \end{aligned}$$

and

$$\sum_1 < \frac{\varepsilon}{2M_3} w_s(x_0) \sum_{k=0}^{\infty} p_{n,k}(x_0) \left(w_r\left(\frac{2k}{n}\right)\right)^{-1} < \frac{\varepsilon}{2},$$

$$\sum_2 \le M_5 w_s(x_0) \sum_{k\in Q_{n,2}} p_{n,k}(x_0) \left(w_r\left(\frac{2k}{n}\right)\right)^{-1}.$$

Since $1 \le \delta^{-2}\left(\frac{2k}{n} - x_0\right)^2$ if $\left|\frac{2k}{n} - x_0\right| \ge \delta$, we have

$$\sum_2 \le M_5 \delta^{-2} w_s(x_0) \sum_{k\in Q_{n,2}} p_{n,k}(x_0) \left(w_r\left(\frac{2k}{n}\right)\right)^{-1} \left(\frac{2k}{n} - x_0\right)^2$$

$$\le M_5 \delta^{-2} w_s(x_0)\, L_n^{(1)}\left(\frac{(t-x_0)^2}{w_r(t)}; x_0\right),$$

wich by (14) and (13) yields

$$\sum_2 \le M_5 M_4 \frac{x_0+1}{n\delta^2} \quad \text{for all} \quad n > n_0.$$

It is obvious that for fixed numbers $\varepsilon > 0$, $\delta > 0$, $M_3 > 0$, $M_4 > 0$, $n_0 \in N$ and $x_0 \ge 0$ there exist a natural number $n_1 > n_0$ depending on the above parameters such that for all $n_1 < n \in N$

$$M_4\, M_5 \frac{x_0+1}{n\delta^2} < \frac{\varepsilon}{2}.$$

Hence we have

$$\sum_2 < \frac{\varepsilon}{2} \quad \text{for all} \quad n > n_1.$$

Consequently,

$$w_s(x_0) |L_n^{(1)}\left(\varphi(t;x_0); x_0\right)| < \varepsilon \quad \text{for} \quad n > n_1,$$

which proves that

$$\lim_{n\to\infty} w_s(x_0) L_n^{(1)}\left(\varphi(t;x_0); x_0\right) = 0.$$

From this and (1) assertion (15) follows for x_0 and $i = 1$. Thus the proof is completed. ■

3. Theorems of the Voronovskaya type

The Voronovskaya theorem for the Bernstein operators is given in [**2**]. We shall prove a similar theorem for the operators $L_n^{(i)}$.

Theorem 1. *Let $f \in C_r^2$ with some $r > 0$. Then*

$$\lim_{n\to\infty} n \left\{ L_n^{(i)}(f;x) - f(x) \right\} = \frac{x}{2} f''(x) \tag{16}$$

for every $x \in R_0$ and $i = 1, 3$.

Proof: Let $x_0 \geq 0$ be an arbitrary fixed point and $i = 1$. By the Taylor formula we have for $t \geq 0$

$$f(t) = f(x_0) + f'(x_0)(t-x_0) + \frac{1}{2} f''(x_0)(t-x_0)^2 + \psi(t;x_0)(t-x_0)^2, \tag{17}$$

where $\psi(\cdot;x_0)$ is a function belonging to the space C_r and $\lim_{t\to x_0}\psi(t;x_0) = 0$. By (2), (8) and (17) we get

$$\begin{aligned} L_n^{(1)}(f(t);x_0) &= f(x_0) + f'(x_0)\, L_n^{(1)}(t-x_0;x_0) \\ &+ \frac{1}{2} f''(x_0)\, L_n^{(1)}\left((t-x_0)^2;x_0\right) + L_n^{(1)}\left(\psi(t;x_0)(t-x_0)^2;x_0\right) \end{aligned} \tag{18}$$

for every $n \in N$. Using Lemma 2, we have

$$\begin{aligned} &\lim_{n\to\infty} n\, L_n^{(1)}(t-x_0;x_0) = 0, \\ &\lim_{n\to\infty} n\, L_n^{(1)}\left((t-x_0)^2;x_0\right) = x_0. \end{aligned} \tag{19}$$

By (2) and the Hölder inequality we have for every $n \in N$

$$\begin{aligned} &\left| L_n^{(1)}\left(\psi(t;x_0)(t-x_0)^2;x_0\right) \right| \\ &\qquad \leq \left\{ L_n^{(1)}\left(\psi^2(t;x_0);x_0\right) \right\}^{\frac{1}{2}} \left\{ L_n^{(1)}\left((t-x_0)^4;x_0\right) \right\}^{\frac{1}{2}}. \end{aligned} \tag{20}$$

Since for the function $\varphi(t;x_0) := \psi^2(t;x_0)$, $t \geq 0$, we have $\varphi(\cdot;x_0) \in C_{2r}$ and $\lim_{t\to x_0}\varphi(t;x_0) = 0$, we get by Lemma 6

$$\lim_{n\to\infty} L_n^{(1)}\left(\psi^2(t;x_0);x_0\right) \equiv \lim_{n\to\infty} L_n^{(1)}\left(\varphi(t;x_0);x_0\right) = 0. \tag{21}$$

Applying (21) and (12) to (20), we obtain

$$\lim_{n\to\infty} n L_n^{(1)}\left(\psi(t;x_0)(t-x_))^2;x_0\right) = 0. \tag{22}$$

Now we immediately obtain (16) for a given x_0 and $i = 1$ from (18) by (19) and (22). This proves the desired assertion for $i = 1$. ■

Similarly we can prove the following

Theorem 2. *Suppose that $f \in C_r^2$ with some $r > 0$. Then*

$$\lim_{n\to\infty} n\left\{L_n^{(i)}(f;x) - f(x)\right\} = f'(x) + \frac{x}{2}f''(x)$$

for every $x \in R_0$ and $i = 2, 4$.

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