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CONSEQUENCES OF THE MEROMORPHIC EQUIVALENCE OF STANDARD MATRIX DIFFERENTIAL EQUATIONS

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Abstract

In this article we investigate the question how meromorphic differential equations can be simplified by meromorphic equivalence. In the case of equations of block size 1, which generalizes the case of distinct eigenvalues, we identify a class of equations which are simplest possible in the sense that they carry the smallest number of parameters whithin their equivalence classes. We also discuss conditions under which individual equations can be simplified. Particular attention is paid to the requirement that the involved transformations can be explicitly computed.

1. Introduction

Meromorphic equivalence has always been an important tool for the discussion of systems of linear differential equations in the complex plane. Among the most prominent examples are the contiguous relations for the hypergeometric functions. In recent years W. B. Jurkat's book ([**Jur**]) stimulated many research efforts for a systematic investigation of meromorphic equivalence. A main goal is the identification of representatives under this equivalence relation which should then be the central objects of further function —theoretic inquiries. For many purposes it is important that this treatment is as explicit as possible. In a previous article ([**JZ1**]) W. B. Jurkat and the author studied these questions for differential equations of arbitrary dimension and block size 1 (definitions

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can be found in section 2). They showed that each equation is equivalent to an equation of a particularly simple form, a so-called standard equation. These standard equations allow a thorough and explicit study of all possible transformations.

It is a natural question in this context to ask: How can these standard equations be further simplified by means of meromorphic equivalence? The main goal of the present article is to show that *standard equations are simplest possible* in the sense that they carry the smallest number of parameters within their equivalence classes. They also allow a natural parametrization and are therefore particularly well suited for further function —theoretic studies, e.g. how connection coefficients depend on the parameters of an equation.

In section 2 we define the standard equations and introduce the concept of direct transformations and equivalence. Direct equivalence is closely related to meromorphic equivalence but turned out to simplify the discussion considerably since it avoids the use of constant similarities. These constant similarities would introduce unnecessarily complicated normalizations of the differential equations (compare [**JZ2**, Proposition 7]).

Section 3 contains the definition of piecewise algebraic functions and a summary of their central properties. These piecewise algebraic functions occur naturally in the explicit formulae for the direct transformations and the transformed differential equations (Proposition 1). They provide the basic key for section 4 which forms the central part of this paper. It is devoted to the discussion of the number of parameters of a standard equation (Lemma 2) and the fact that the totality of standard equations cannot be obtained from a subclass with fewer parameters (Theorem 1). We can even discuss whether individual equations allow simplifications (Theorem 2), e.g. whether an equivalent equation exists for which more parameters vanish. This discussion reveals *new invariants* for standard equations.

All of these results also apply to natural subclasses of our standard equations, e.g. to the important cases of equations with fixed formal invariants (isoformal) resp. with fixed monodromy (isomonodromy). This transfer is carried out in section 5 (Corollary 1 and Remark 5). In this context we compute the number of parameters for isoformal equations in Lemma 3.

Although the formulae for the transformations are in general made up from several pieces we prove in section 6 that for the vast majority of equations one system of rational formulae suffices (Theorem 3).

Finally we show in section 7 that the previous results carry over from

direct to meromorphic equivalence after an additional normalization of the standard equations (Theorems 1', 2', 3').

The reader is assumed to be familiar with the basic facts of the general theory of the meromorphic differential equations which can be found in [**Jur**]. Besides that we rely on the results of the articles [**JZ1**] and [**JZ2**].

2. Standard Equations and Direct Equivalence

This section contains a discussion of the differential equations and their transformations which are the basis for our investigations. A detailed treatment can be found in [**JZ1**].

A *meromorphic differential equation* is a linear system of the form $X'(z) = A(z)X(z)$ where $A(z)$ can be expanded in a convergent Laurent-series in a neighborhood of infinity with finite singular part and $X(z)$ denotes a fundamental solution matrix (all matrices have dimensions $n \times n$ with $n \geq 2$). For abbreviation we will denote such a differential equation by $A(z)$ as long as no confusion is possible.

When we replace $X(z)$ by $T(z)Y(z)$ where T is a *meromorphic transformation* (i.e. T and T^{-1} are meromorphic at infinity) then $Y(z)$ is a fundamental solution matrix for the equation $B = T^{-1}AT - T^{-1}T'$. We say that A and B are *meromorphically equivalent*.

Definition. The differential equation $X' = AX$ is called a *standard equation* if

(i) $A(z) = \sum_{k=0}^{r} A_k z^{k-1}$ with $r \in \mathbb{N}$ (*Poincaré rank*) and $A_r \neq 0$,

(ii) it possesses a formal solution $H(z)$ of the form $F(z)z^{\Lambda'}e^{Q(z)}$ where $Q(z) = \operatorname{diag}(q_1(z), \ldots, q_n(z))$ is a polynomial matrix without constant term, $\Lambda' = \operatorname{diag}(\lambda'_1, \ldots, \lambda'_n)$ is a complex matrix (called *formal monodromy*) whose diagonal elements satisfy $\lambda'_j \not\equiv \lambda'_k (\mathrm{mod}\, 1)$ whenever $q_j = q_k$ for $j \neq k$ (we summarize the conditions for Q and Λ' by saying that the equation has *block size one*) and the formal series $F(z)$ satisfies $F(z) = I + \sum_{j=1}^{\infty} F_j z^{-j}$,

(iii) the eigenvalues of A_0 which are not equal, are already incongruent $(\mathrm{mod}\, 1)$.

Remark 1 ([**JZ1**, Remarks 1 and 2])**.** The formal solution $(I + \sum_{j=1}^{\infty} F_j z^{-j})z^{\Lambda'}e^{Q(z)}$ is uniquely associated with A and denoted by

$H_A = F_A z^{\Lambda'_A} e^{Q_A}$ (the subscript $._A$ stands for a quantity that is uniquely determined by A). The special form of $F(z)$ implies that the Poincaré rank of A is minimal. Such a standard equation also possesses an actual solution $E(z)z^M$ where $E(z)$ is entire with $\det E(z) \neq 0$ $(\forall\, z \in \mathbb{C})$ and M is a lower triangular Jordan canonical form of A_0 (called *actual monodromy*). It is uniquely determined if we prescribe the ordering of the Jordan blocks. For that purpose we choose all complex z with $0 \leq \mathrm{Re}(z) < 1$ as a system of representatives modulo 1 and require:

(i) the eigenvalues of M are arranged such that the corresponding representatives are non-increasing (where here and in the sequel complex numbers are ordered lexicographically with real part first),

(ii) the sizes of consecutive Jordan blocks for the same eigenvalue do not increase.

This unique monodromy M will be denoted by M_A.

Definition. The standard equations A and B are said to be *directly equivalent*, if a meromorphic transformation T exists satisfying $H_A = TH_B$. Such a T is called *a direct transformation* from A to B.

Remark 2 ([**JZ1**, Remark 6])**.** Two standard equations A and B are meromorphically equivalent if and only if a permutation of A is directly equivalent to some B^* which is diagonally similar to B. The occuring permutation is uniquely determined by Q_A, Λ'_A, Q_B, Λ'_B. The direct transformation is the central part of every meromorphic transformation but has the advantage to avoid the possible constant matrices that can be applied to H_B and create the necessity to normalize $A(z)$ further. Therefore we will concentrate mainly on direct equivalence.

3. Piecewise Algebraic Functions

The coefficients of a direct transformation depend algebraically on the coefficients of the differential equation to which it is applied. This behavior implies the properties that are essential for the present reduction theory. Hence we will now introduce the concept of piecewise algebraic functions, give some of their basic properties and describe how they arise in the context of direct transformations.

Definition ([**JZ2**])**.** Let X be a non-empty subset of $\mathbb{C}^N (N \in \mathbb{N})$. A set of the form $\{x \in X : g(x) \neq 0,\, g_j(x) = 0 \text{ for } j = 1, \ldots, m\}$ with $m \in \mathbb{N}_0$, $g, g_j \in \mathbb{Q}[x]$ is called a *P-set* in x relative to X. A *PA-set*

relative to X consists of the finite union of P-sets relative to X. Its (algebraic) *dimension over* $\mathcal{F}$ (for an arbitrary field $\mathcal{F}$ with $\mathbb{Q} \subseteq \mathcal{F} \subseteq \mathbb{C}$) is by definition the maximal number of algebraically independent components for each of its elements (where the empty set has dimension -1). For $\mathcal{F} = \mathbb{Q}$ we simply use the term dimension. In the case of a PA-set $F \subseteq X \times \mathbb{C}^M$ ($M \in \mathbb{N}$) we say that F is a *PA-relation* relative to X.

A *multivalued function* $f : \mathbb{C}^N - \to \mathbb{C}^M$ associates with every $x \in \mathbb{C}^N$ the finite set $f(x) \subseteq \mathbb{C}^M$ (which may be empty). Its *domain* is defined as $D_f = \{x \in \mathbb{C}^N : f(x) \neq \emptyset\}$ and we also introduce the *image* of a set $A \subseteq \mathbb{C}^N$ as $f(A) = \bigcup_{x \in A} f(x)$ and the *preimage* of a set $B \subseteq \mathbb{C}^M$ as $f^{-1}(B) = \{x \in \mathbb{C}^N : f(x) \cap B \neq \emptyset\}$. Furthermore we say that $F = \{(x, y) : y \in f(x)\} \subseteq \mathbb{C}^N \times \mathbb{C}^M$ is the *graph* of f. If the graph F is a PA-relation relative to X then f is called a *PA-function* relative to X. Finally, a *PR-function* relative to X is defined to be a PA-function relative to X where each set $f(x)$ contains at most one element.

It is important to observe that *f is a PR-function relative to X if and only if there exists a decomposition of X into PA-sets $S_0, \ldots, S_m$ ($m \in \mathbb{N}_0$) relative to X such that f is defined on $X \neg S_0$* (difference of sets) *and the components of f allow rational representations on each S_j for $1 \leq j \leq m$* (decompositions are always disjoint).

A first important property of PA-functions is the fact that *the composition of PA-functions $f : X - \to \mathbb{C}^M$ and $g : Y - \to \mathbb{C}^{M'}$* —defined as $g(f(x)) = \bigcup_{y \in f(x)} g(y)$— with $\emptyset \neq Y \subseteq \mathbb{C}^M$ and $f(x) \subseteq Y$ for all $x \in X$ *is again a PA-function relative to X.*

Before we can relate these definitions to our direct transformation we must explain which parameters we will use to represent our standard equations. For that purpose we fix the Poincaré rank r of $A(z)$ and combine the entries of the matrices $A_{r-1}, \ldots, A_0$ and of the diagonal of A_r in some fixed way in order to obtain a parameter vector in $\mathbb{C}^{rn^2+n}$. The totality of parameter vectors corresponding to standard equations forms the *parameter space* X. Sometimes it is useful to extend the parameter vectors by including the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the monodromy matrix in the same order as they occur on the diagonal of M_A. This leads to the *parameter space* $X' \subseteq \mathbb{C}^{rn^2+2n}$, and to every standard equation there corresponds exactly one parameter vector in X resp. in X'.

Proposition 1. *Let diagonal matrices K and K' with integer entries and equal traces be given. To every standard equation A there exists at most one directly equivalent standard equation B with $M_A = M_B + K$, $Q_A = Q_B$, $\Lambda'_A = \Lambda'_B + K'$ and the function which*

maps the parameter vector of A onto the parameter vector of B is a PR-function ($\mathbb{C}^{rn^2+2n} \dashrightarrow \mathbb{C}^{rn^2+2n}$) relative to X'. On the other hand, every standard equation B which is directly equivalent to A can be obtained by an appropriate choice of K and K'.

Proof: This is essentially Corollary 2 and a consequence of Proposition 2 in [**JZ1**] with the difference that we no longer include the entries of Q or Λ' into the parameter vectors. Due to Proposition 6 of [**JZ2**] we know that Q and Λ' can be computed from A by means of PR-functions. Hence we can compose the two PR-functions to obtain a PR-function in the parameter vectors of X'. ■

Remark 3. In [**JZ1**, Corollary 2] and [**JZ2**, Theorem 9] we also gave an explicit procedure to compute these PR-functions, but this is of minor importance for our present investigations.

Although the eigenvalues of M_A coincide with those of A_0 and hence depend algebraically on the parameters of A, the parameters of B are not PA-functions relative to X since we had to demand the ordering of these eigenvalues in connection with the matrix K. But we can say that the parameter vectors in X' are a part of a PA-function in the parameter vector in X.

The closer study of the implications of Proposition 1 relies on some properties of PA-functions. For their formulation we need the following

Definition. For a set $S \subseteq \mathbb{R}^N$ ($N \in \mathbb{N}$) we define the *p-dimensional outer Hausdorff measure* ($p \in \mathbb{R}$, $p \geq 0$) as $H_p(S) = \sup_{\varepsilon>0} \inf \sum_{j=1}^{\infty} d^p(A_j)$ where $S \subseteq \bigcup_{j=1}^{\infty} A_j$ and the diameters $d(A_j) < \varepsilon$ for all j. Moreover S is called *σ_p-finite* if $S = \bigcup_{j=1}^{\infty} S_j$ with $H_p(S_j) < \infty$ for all j; if in addition $H_p(S) > 0$ then we say that S has *precise dimension p*.

We also use these real concepts for our complex sets by splitting each variable into its real and imaginary part. Therefore, the dimensions are doubled when we compare them to what we would expect from counting the parameters.

Lemma 1. *Let f be a PA-function ($\mathbb{C}^N \dashrightarrow \mathbb{C}^M$) relative to X. For given $A \subseteq X$ the following hold ($\mathcal{F}$ a field with $\mathbb{Q} \subseteq \mathcal{F} \subseteq \mathbb{C}$, $p \in \mathbb{R}$, $p \geq 0$):*

(i) *the algebraic dimension over $\mathcal{F}$ of $f(A)$ is not greater than the algebraic dimension over $\mathcal{F}$ of A,*

(ii) $H_p(A) = 0 \Rightarrow H_p(f(A)) = 0$,

(iii) *A is σ_p-finite $\Rightarrow$ $f(A)$ is σ_p-finite.*

Proof: See [**JZ2**, Proposition 2]. ■

From this lemma it follows in particular that, if f possesses an inverse that is a PA-function (relative to $f(X)$), then f preserves the precise dimension of a set. The existence of such an inverse is equivalent to the fact that f^{-1} $(\{y\})$ is a finite set $(\forall y \in \mathbb{C}^M)$ according to [**JZ2**, end of section 2].

4. On the Number of Parameters

After we have made the necessary preparations we can now explain that our standard equations cannot be reduced further by means of direct equivalence. For that purpose we start by counting the parameters.

Lemma 2. *The parameter spaces X and X' for the standard equations with fixed Poincaré rank r have precise dimension $2(rn^2 + n)$.*

Proof: The standard equations incorporate in particular those equations where A_r has distinct eigenvalues and A_0 has incongruent eigenvalues. Their parameter vectors in X form an open, non-empty subset of $\mathbb{C}^{rn^2+n}$. Therefore, $H_{2(rn^2+n)}(X) > 0$ holds. Furthermore, $\mathbb{C}^{rn^2+n}$ is $\sigma_{2(rn^2+n)}$-finite which is then also true for every subset.

If we associate with every parameter vector in X the vector itself and the eigenvalues of A_0 in any possible order, then we obtain a PA-function $(\mathbb{C}^{rn^2+n} \longrightarrow \mathbb{C}^{rn^2+2n})$ relative to X ([**JZ2**, Lemma 3]). The image of X under this function is $\sigma_{2(rn^2+n)}$-finite due to Lemma 1(iii), and this also holds for X' as a subset of this image. In addition, $H_{2(rn^2+n)}(X') = 0$ is impossible since this would imply $H_{2(rn^2+n)}(X) = 0$ by Lemma 1(ii) as X is the projection of X' and projections are PA-functions ([**JZ2**, Lemma 4(i)]). ■

It is worthwhile to notice that Theorem 9 in [**JZ2**] shows that the set X is a Borel-set and therefore measurable. Of course, the same holds for X'.

Remark 4. Let us consider a collection of standard equations whose parameter vectors constitute a set $S \subseteq X$; at the same time they yield a set $S' \subseteq X'$ when we include the eigenvalues of the monodromy matrix.

Then the proof of Lemma 2 shows that S is the image of S' under a PA-function, that S' is a subset of the image of S under a PA-function and that S satisfies $H_p(S) = 0$ resp. is σ_p-finite resp. has precise dimension p if and only if S' satisfies $H_p(S') = 0$ resp. is σ_p-finite resp. has precise dimension p. Moreover, both sets have the same algebraic dimension over any field $\mathcal{F}$ with $\mathbb{Q} \subseteq \mathcal{F} \subseteq \mathbb{C}$.

Now we can formulate precisely what we mean when we say that our standard equations cannot be reduced further.

Theorem 1. *Let $S \subseteq X$ satisfy $H_p(S) = 0$ (resp. be σ_p-finite resp. have precise dimension p for some $p \geq 0$) and denote by $S^* \subseteq X$ the parameter vectors of all standard equations which are directly equivalent to those whose parameter vectors belong to S. Then $H_p(S^*) = 0$ (resp. S^* is also σ_p-finite resp. has precise dimension p).*

Proof: We will prove the theorem in an equivalent version namely with X replaced by X' (see Remark 4). According to Proposition 1 any standard equation B which is directly equivalent to a given A leads to two diagonal matrices $K = M_A - M_B$ and $K' = \Lambda'_A - \Lambda'_B$ with integer entries and equal traces. By the σ-subadditivity of H_p it is sufficient to consider a fixed choice of K and K'. But then at most one B exists which is directly equivalent to A. We learn from Proposition 1 that the parameter vector of B is a PR-function relative to X' of the parameter vector of A, and due to Lemma 1(ii) such a PR-function maps nullsets onto nullsets. The other parts of the claim follow in exactly the same way by Lemma 1(iii) and the remark following it. ■

An immediate consequence of Theorem 1 is that we cannot obtain all standard equations by means of direct equivalence from a subcollection whose parameter vectors form a set S with $H_{2(rn^2+n)}(S) = 0$. Therefore if the totality of all standard equations can be generated from S then S has precise dimension $2(rn^2 + n)$. It is in this sense that our standard equations allow no further reduction and are simplest possible.

Of course, Theorem 1 also applies to natural subclasses of our standard equations and we will devote section 5 to the discussion of such a case.

But it is also interesting to investigate individual equations and ask whether they can be simplified. For that purpose we must first explain what the term "simplified" should mean. One natural interpretation is that we use the transcendency degree of the parameters of the equation and consider an equation to be simpler than another if its parameters have a lower transcendency degree.

Theorem 2. *If the standard equations A and B are directly equivalent*

then their parameters have equal transcendency degrees over $\mathcal{F}$ (where $\mathcal{F}$ is any field with $\mathbb{Q} \subseteq \mathcal{F} \subseteq \mathbb{C}$).

Proof: Again we can use Remark 4 to find out that it makes absolutely no difference in the statement whether we use the parameters from X or those from X'. Thus we consider the parameters from X' and notice that the parameters of B are PR-functions in those of A. Hence Lemma 1(i) applies and the transcendency degree over $\mathcal{F}$ of the parameters of B cannot by greater than that of the parameters of A. But since the situation is symmetric in A and B the two transcendency degrees must in fact be equal. ■

This theorem leads to the interesting observation that for every field $\mathcal{F}$ ($\mathbb{Q} \subseteq \mathcal{F} \subseteq \mathbb{C}$) we obtain an invariant under direct equivalence, namely the transcendency degree over $\mathcal{F}$ of the parameters of the standard equation.

One consequence of Theorem 2 is the fact that standard equations, all of whose parameters (from X) are algebraically independent over $\mathbb{Q}$, can never be directly transformed into standard equations where one or more parameters vanish.

5. Isoformal Equations

In this section we will apply Theorem 1 to those subclasses of standard equations which are obtained by fixing the formal invariants.

Every standard equation A is formally and directly transformed by F_A to $Q'_A(z) + \Lambda'_A z^{-1}$. The only direct transformations connecting such diagonal standard equations are z^K with arbitrary, diagonal K possessing integer entries. Therefore we should introduce the diagonal matrix Λ^*_A whose eigenvalues have real parts in $[0, 1)$ such that $\Lambda'_A - \Lambda^*_A$ has integer entries. Then $Q_A(z)$ and Λ^*_A form a *complete system of formal direct invariants for standard equations*. They are easily computable as can be seen from Theorem 9 in [**JZ2**]. The fact that they remain unchanged under direct transformations suggests that we should concentrate our attention to sets of equations for which these invariants agree. Such equations are called *isoformal*. This is an equivalence relation which yields equivalence classes of isoformal equations in the parameter spaces X. They are the objects of our further studies.

First we aim at counting the parameters of such an equivalence class. For that purpose we define $d(j,k) = \max\{\deg(q_j(z) - q_k(z)),\, 0\}$ for $1 \leq j, k \leq n$.

Here the *degrees* are defined as follows: $\deg(T^*)$ is the maximal occuring power of z in the Laurent expansion of T^* at infinity and $\deg_0(T^*)$

the last occuring power of z in the Laurent expansion of T^* at 0. These definitions can be used for any formal or convergent Laurent-series with finite singular part when we define $\deg(O) = -\infty$, $\deg_0(O) = +\infty$.

Lemma 3. *A given equivalence class of isoformal equations has precise dimension*

$$2 \cdot \sum_{j,k=1}^{n} d(j,k).$$

Proof: Since Λ'_A can assume only countably many values we may consider Λ'_A to be fixed. Then we will construct a system of $\sum_{j,k=1}^{n} d(j,k)$ complex, completely unrestricted parameters for the given isoformal equations such that these parameters together with the coefficients of the diagonal entries in Q_A and Λ'_A are PR-functions in the parameter vectors from X and vice versa. According to Lemma 1 this suffices to prove Lemma 3.

First we apply a fixed permutation to all isoformal equations in order to assure that Q_A has an iterated block structure ([**Jur**, p. 92]), i.e. $d(j,k)$ is increasing with $|j-k|$. This will simplify the following considerations which are essentially based on the construction of the formal solution by a successive block diagonalization as described in [**Was**, p. 52-54]. For that purpose we order the distinct values $d(j,k)$ as $r \geq d_1 > \cdots > d_t \geq 0$ ($t \in \mathbb{N}$) and define $d_0 = +\infty$, $d_{t+1} = 0$. The block structure of k-th level ($1 \leq k \leq t$) is now obtained by combining exactly those indices j, j' into one diagonal block for which $d(j,j') < d_k$ holds. Then the process in [**Was**, p. 52-54] yields the factorization $F_A(z) = F_1(z) \cdot \ldots \cdot F_{t+1}(z)$ where each $F_j(z)$ is a formal power series which is diagonal in the block structure of $(j-1)$-st level (for $1 < j \leq t+1$) and whose diagonal blocks in the block structure of j-th level are I (for $1 \leq j \leq t$). We put $F_j(z) = I + \sum_{k=1}^{\infty} F_{jk} z^{-k}$ for $1 \leq j \leq t+1$. In [**Was**, p. 52-54] F_j is used to transform a differential equation $A_{j-1}(z)$ into the equation $A_j(z)$ (take $A_0(z) = A$ and $A_{t+1}(z) = Q'_A + \Lambda'_A z^{-1}$) where $A_j(z)$ has the following structure ($A_{j-1}(z)$ analogously): it is diagonal in the block structure of j-th level; each individual diagonal block consists of a formal series of degree at most -2 plus a standard equation whose formal solution contains exactly one block of Q_A with elements $q_k, q_{k'}$, satisfying $d(k,k') < d_j$. We write $A_j(z) = \sum_{k=-\infty}^{r} A_{jk} z^{k-1}$. With these notations we learn from [**Was**, p. 52-54]:

(i) The entries of F_{jk} are rational functions with integer coefficients of the entries of the $A_{j-1,\nu}$ with $\nu \geq d_j - k$ $(k = 1, 2, \dots)$.

(ii) The matrix A_{jk} is completely determined by the matrices $A_{j-1,\nu}$ with $\nu \geq k$ and the matrices $F_{j\mu}$ with $\mu \leq d_j - k$ $(k = 1, 2, \dots)$.

Now we can explain what our new parameters shall be, namely those entries of F_{jk} which belong to diagonal blocks in the block structure of level $j-1$, but not in the block structure of level j $(1 \leq j \leq t, 1 \leq k \leq d_j)$. These are $\sum_{j,k=1}^{n} d(j,k)$ complex parameters that depend rationally on the original parameter vectors in X by (i) and (ii). To show the converse and the fact that these new parameters are completely unrestricted we define $\tilde{F}_j(z) = I + \sum_{k=1}^{d_j} F_{jk} z^{-k}$ $(1 \leq j \leq t+1)$ with arbitrary values for our new parameters and all other entries being zero. Then we see that $\tilde{F}_j \cdot \ldots \cdot \tilde{F}_{t+1} z^{\Lambda'_A} e^{Q_A}$ satisfies the differential equation $B_{j-1}(z)$ which has the same structure as $A_{j-1}(z)$ (see above) for $1 \leq j \leq t+1$. Now we define the standard equation $A(z)$ by requiring that the degree of $B_0(z) - A(z)$ is at most -2. The entries of the coefficients of A depend rationally (with integer coefficients) on the entries of the F_{jk}, Q_A and Λ'_A. When we formally solve the equation A we obtain the equations $A_j(z)$ and the formal series F_j as above. Using (i) and (ii) we conclude that changing $A_{j-1}(z)$ by an additive term of degree not exceeding -2 (i.e. changing only coefficients $A_{j-1,k}$ with $k \leq -1$) does not change the matrices A_{jk} with $k \geq 0$ nor F_{jk} with $k \leq d_j$. Hence the leading terms of F_j are exactly the initially given matrices F_{jk}, i.e. $F_j(z) - \tilde{F}_j(z)$ has degree at most $-d_j - 1$. ■

It is not surprising that this dimension agrees with twice the number of "free" actual invariants given in [**Jur**, p. 123].

Now we can immediately apply Theorem 1 and find

Corollary 1. *An equivalence class of isoformal equations can only be obtained from a subset by direct equivalence if this subset has precise dimension*

$$2 \cdot \sum_{j,k=1}^{n} d(j,k).$$

Remark 5. An analogous result holds for equivalence classes of *isomonodromy equations*, viz. equations A, B where $M_A - M_B$ is a diagonal matrix with integer entries. The precise dimension of such an equivalence class is $2(n + rn^2 + k - \sum_{j=1}^{k} a_j g_j)$ where $\lambda_{1,\dots,}\lambda_k$ $(k \in \mathbb{N})$ are the distinct eigenvalues of M_A and a_j (resp. g_j) denotes the algebraic

(resp. geometric) multiplicity of the eigenvalue λ_j $(1 \leq j \leq k)$. This can be seen by discussing how linear combinations of eigenvectors and principal vectors corresponding to the eigenvalue λ_j can be used to enforce that certain components of these vectors are 0 or 1.

6. Uniform Rational Formulae

From Proposition 1 we learned that we can compute the parameters of the transformed equation as PR-functions of the parameters of the original equation (all parameter vectors from X') provided that we fix the integral changes K and K' of the actual and formal monodromy. This gives the impression that we deal with an abundance of cases in each of which different formulae apply. Fortunately we will now show that for almost all equations one and the same formula can be used.

Theorem 3. *Let diagonal matrices K and K' with integer entries and equal traces be given. Then there exists a subset S of $\mathbb{C}^{rn^2+n}$ and a vector f of rn^2+2n rational functions from $\mathbb{Q}(y_1, \ldots, y_{rn^2+2n})$ with the following properties:*

(i) *The set $S^\neg \subseteq \mathbb{C}^{rn^2+n}$ is $\sigma_{2(rn^2+n)-2}$-finite.*

(ii) *If the parameter vector $a \in X$ of the standard equation A belongs to S then there exists a standard equation B which is directly equivalent to A with $M_A = M_B + K$, $Q_A = Q_B$, $\Lambda'_A = \Lambda'_B + K'$ and the parameter vectors a' (resp. b') $\in X'$ of A (resp. B) are related by $b' = f(a')$.*

Proof: The set S will consist of exactly those vectors in $\mathbb{C}^{rn^2+n}$ whose elements are algebraically independent over $\mathbb{Q}$. It has the following three properties:

(a) Each vector in S is the parameter vector of a standard equation since the eigenvalues of A_0 and the diagonal matrix A_r are algebraically independent (for the rest of the proof always over $\mathbb{Q}$), in particular incongruent modulo 1 and distinct. This leads immediately to the properties which we required for standard equations.

(b) The complement of S contains exactly the vectors with algebraically dependent elements and is therefore the countable union of sets $\{x \in \mathbb{C}^{rn^2+n} : p(x) = 0\}$ with $p \in \mathbb{Z}[x]\neg\{0\}$. Each of these sets is $\sigma_{2(rn^2+n)-2}$-finite ([**JZ2**, Remark 6 and Proposition 1]), and then the same holds for their countable union.

(c) Finally we want to prove that all equations with parameter vectors in S can be transformed using one and the same rational

formula which proves part (ii) of the claim. For that purpose we construct the formula in finitely many steps. This is done by first changing M_A to $M_A - K$ by means of the procedure described in [**JZ1**, Proposition 1]. We divide K into individual steps as in the proof of Proposition 1 in [**JZ1**] and discuss such a single step. It requires the computation of $E_0 = E(0)$ in the actual solution at 0. Notice that E_0 satisfies $A_0 = E_0 M_A E_0^{-1}$ and therefore is made up from the eigenvectors of A_0. We can always require that the last component in each eigenvector is either 0 or 1. Thus for each eigenvalue λ we seek an eigenvector solving the equation

$$A_0 \begin{bmatrix} \vec{t} \\ \alpha \end{bmatrix} = \begin{bmatrix} & \tilde{A} & \vec{a} \\ * & \dots & * \end{bmatrix} \begin{bmatrix} \vec{t} \\ \alpha \end{bmatrix} = \lambda \begin{bmatrix} \vec{t} \\ \alpha \end{bmatrix}$$

with $\alpha \in \{0, 1\}$.

Can λ be an eigenvalue of $\tilde{A}$? Then λ would depend algebraically on the elements of $\tilde{A}$; furthermore we learn from $\det(A_0 - \lambda I) = 0$ that a_{nn} depends algebraically on the other elements of A and on λ which then would yield an algebraic dependence among the elements of A_0 in contradiction to our choice of the set S. Hence $\tilde{A} - \lambda I$ is invertible and we obtain $\vec{t}$ as a rational function of A and λ by applying Cramer's rule to $\tilde{A}\vec{t} + \vec{a}\alpha = \lambda\vec{t}$. Since $\vec{t}$ is unique, $\alpha = 0$ is impossible because in this case all multiples of $\vec{t}$ had to be solutions as well; hence we conclude $\alpha = 1$ for all eigenvectors. Once

$$E_0 = \begin{bmatrix} \vec{t_1} & & \vec{t_n} \\ 1 & \dots & 1 \end{bmatrix}$$

is computed we notice that the components of all vectors $\vec{t_1}, \dots, \vec{t_n}$ must be algebraically independent since $A_0 (= E_0^{-1} M_A E_0)$has n^2 algebraically independent components. Now we want to factor it as $E_0 = UL$ ($\operatorname{diag} U = I$) which is done by applying the Gauss elimination process to E_0 starting in the (n, n)-position ([**JZ2**, Proposition 5]). The first step puts E_0 to the form

$$\begin{bmatrix} \vec{t_1} - \vec{t_n} & \dots & \vec{t_{n-1}} - \vec{t_n} & \vec{0} \\ 1 & \dots & 1 & 1 \end{bmatrix} = \begin{bmatrix} & T & & \vec{0} \\ 1 & \dots & 1 \end{bmatrix}$$

and the elements of T are still algebraically independent. Hence the Gauss elimination can be carried out without permutation of rows and yields the elements of U and L as rational functions in the elements of E_0 (notice that for the factorization $E_0 = LU$ it is more convenient to choose the first row of E_0 to consist of $1's$ only). Finally the

construction in Proposition 1 of [**JZ1**] requires the computation of the elements $r_{21}, \dots, r_{n1}$ of the matrix $U^{-1}F_1$ where we use the notation $(I + \sum_{k=1}^{\infty} F_k z^{-k}) z^{\Lambda'} e^{Q(z)}$ for the formal solution of $A(z)$. Fortunately $r_{21}, \dots, r_{n1}$ are determined by U^{-1} and the elements in the first column of F_1 which are located below the diagonal. Since A_r has distinct eigenvalues we know from the proof of Lemma 3 that these elements in F_1 are rational functions of the parameters of $A(z)$. Hence one step in the proof of Proposition 1 in [**JZ1**] leads to a new standard equation whose parameters (in X') are obtained as $rn^2 + 2n$ rational functions (with coefficients in $\mathbb{Q}$) from the parameters (in X') of the original equation. Furthermore the parameters (in X) of the new equation are again algebraically independent due to Theorem 2. Iterating this process we obtain a standard equation $\tilde{A}$ with $M_A = M_{\tilde{A}} + K$, $Q_A = Q_{\tilde{A}}$ and $\Lambda'_A = \Lambda'_{\tilde{A}} + K$ (since no permutations occured).

Hence we are left with the task to change $\Lambda_{\tilde{A}'}$ to $\Lambda'_B = \Lambda'_{\tilde{A}} + K - K'$ without altering the actual monodromy. For that purpose we write $K - K'$ as a finite sum of matrices $K(\mu, \nu)$ $(1 \le \mu, \nu \le n,\ \mu \ne \nu)$ where $K(\mu, \nu) = \operatorname{diag}(k_1, \dots, k_n)$ with $k_\mu = 1$, $k_\nu = -1$, $k_j = 0$ $(j \ne \mu, \nu)$. According to the example in section 6 of [**JZ1**] a direct transformation with $K = O$, $K' = K(\mu, \nu)$ exists if $a_{\nu\mu}^{(r-1)} \ne 0$; but this is certain due to the algebraic independence. Hence such a simple change is possible and yields also rational formulae (with coefficients in $\mathbb{Q}$). We are therefore in the same situation as before and can iterate this procedure until the desired equation B is obtained. Composing all these rational functions we obtain the claimed vector f. ■

Example. As an example we consider the case $r = 1$, $n = 2$, $K = O$ and $K' = \operatorname{diag}(-1, 1)$. Then the differential equations under consideration have the form

$$A(z) = A_1 + A_0 z^{-1} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + \begin{bmatrix} a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} z^{-1}.$$

As the set $S \subseteq \mathbb{C}^6$ in Theorem 3 we take

$$S = \{a_1 \ne a_2,\ a_5 \ne 0,\ A_0 \text{ has eigenvalues } \lambda_1,\ \lambda_2 \text{ which are either equal or incongruent (mod 1)}\}$$

whose complement is σ_{10}-finite. The eigenvalues λ_1, λ_2 are ordered as in the unique monochromy M_A. Equations A, whose parameters belong to S, are standard equations with $Q(z) = \operatorname{diag}(a_1 z, a_2 z)$ and $\Lambda' = \operatorname{diag}(a_3, a_6)$. Their parameter vectors $a' \in \mathbb{C}^8$ have the form

$$a' = (a_1, a_2, a_3, a_4, a_5, a_6, \lambda_1, \lambda_2).$$

Using the direct transformation

$$T(z) = \begin{bmatrix} z + \dfrac{a_3 - a_6 - 1}{a_1 - a_2} & -\dfrac{a_1 - a_2}{a_5} \\ \dfrac{a_5}{a_1 - a_2} & 0 \end{bmatrix}$$

the equation A is transformed into

$$B(z) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + \begin{bmatrix} a_3 - 1 & -\dfrac{(a_1 - a_2)^2}{a_5} \\ \dfrac{a_5(a_6 - a_3 + 1 - a_4 a_5)}{(a_1 - a_2)^2} & a_6 + 1 \end{bmatrix} z^{-1}.$$

This is the desired change $K = O$, $K' = \mathrm{diag}(-1, 1)$, and the parameter vector b' of the transformed equation B has the form

$$b' = \left(a_1, a_2, a_3 - 1, -\frac{(a_1 - a_2)^2}{a_5}, \frac{a_5(a_6 - a_3 + 1 - a_4 a_5)}{(a_1 - a_2)^2}, a_6 + 1, \lambda_1, \lambda_2 \right)$$

which gives the unique rational formula f of Theorem 3.

7. Meromorphic Equivalence

In this final section we will transfer our previous results to meromorphically equivalent equations.

By Remark 2 we know that a meromorphic transformation consists of a permutation and a diagonal similarity in addition to a direct transformation. The diagonal similarity can be used to normalize our standard equations further. One such normalization is given in Proposition 7 of [**JZ2**]. It relies on the determination of those entries of $A(z)$ which vanish identically and on requiring certain other entries to be monic (i.e. the highest occuring power of z has coefficient one). The choice of the terms that should be monic contains a certain degree of freedom, e.g. if none of the entries $a_{1j}(z)$ of $A(z)$ for $2 \leq j \leq n$ vanishes identically the normalization can consist of the requirement that exactly these entries should be monic. We will use this normalization throughout this section. If $a_{1j}(z)$ vanishes identically for some j $(2 \leq j \leq n)$ we use some other normalization, but its specific choice does not influence our further considerations.

In this way we obtain *normalized standard equations* in the sense of [**JZ1**, p. 314]. Their parameter vectors form the sets $Y \subseteq X \subseteq \mathbb{C}^{rn^2+n}$ resp. $Y' \subseteq X' \subseteq \mathbb{C}^{rn^2+2n}$.

Lemma 2'. *The parameter spaces Y and Y' for the normalized standard equations with fixed Poincaré rank r have precise dimension $2(rn^2+1)$.*

Proof: Let us first consider those normalized equations for which all entries $a_{1j}^{(r-1)}$ in A_{r-1} $(2 \le j \le n)$ are 1. Then the other rn^2+1 parameters in A are only restricted by the requirement that A is a standard equation. This is guaranteed if e.g. A_r has distinct eigenvalues and all eigenvalues of A_{-1} are incongruent modulo 1. Hence the parameter vectors of these equations form a set of precise dimension $2(rn^2+1)$ in Y and hence also in Y'.

In the remaining cases at least one entry $a_{1j}^{(r-1)}$ $(2 \le j \le n)$ is zero and this forces another parameter to be 0 or 1. Therefore the set of the corresponding parameter vectors is σ_{2rn^2}-finite. ■

Proposition 1'. *Let a permutation matrix P and diagonal matrices K, K' with integer entries and equal traces be given. For any normalized standard equation A there exists at most one meromorphically equivalent normalized standard equation B with $M_A = M_B + K$, $PQ_AP^{-1} = Q_B$, $P\Lambda'_AP^{-1} = \Lambda'_B + K'$ and the function which maps the parameter vector (from Y') of A onto the one of B is a PR-function relative to Y'. Furthermore every normalized standard equation B which is meromorphically equivalent to A is obtained by an appropriate choice of P, K and K'.*

Proof: This is a consequence of Proposition 2 and Corollary 2 in [**JZ1**] where the permutation can be applied to A in a preliminary step. Again we use that the entries of the diagonal elements in Q_A and Λ'_A are PR-functions relative to Y'. ■

With these informations we can now easily transfer Theorem 1, 2, and 3 to this situation.

Theorem 1'. *Let $S \subseteq Y$ satisfy $H_p(S) > 0$ (resp. be σ_p-finite resp. have precise dimension p) for some $p \ge 0$ and denote by $S^* \subseteq Y$ the parameter vectors of all normalized standard equations which are meromorphically equivalent to those whose parameter vectors belong to S. Then $H_p(S^*) = 0$ (resp. S^* is also σ_p-finite resp. has precise dimension p). In particular, if S^* is the totality of all normalized standard equations then S has precise dimension $2(rn^2+1)$.*

Theorem 2'. *If the normalized standard equations A and B are meromorphically equivalent then their parameters have equal transcendency degrees over $\mathcal{F}$ (where $\mathcal{F}$ is any field with $\mathbb{Q} \subseteq \mathcal{F} \subseteq \mathbb{C}$).*

Theorem 3'. *Let P, K and K' be given as in Proposition 1'. Then there exists a set $S' \subseteq \mathbb{C}^{rn^2+n}$ and a vector f of rn^2+2n rational functions from $\mathbb{Q}(y_1, \ldots, y_{rn^2+2n})$ with the following properties:*

(i) *The set $Y \neg S' \subseteq \mathbb{C}^{rn^2+n}$ is σ_{2rn^2}-finite.*

(ii) *If the parameter vector $a \in Y$ of the normalized standard equation A belongs to S' then $b' = f(a')$ —where a' is the parameter vector of A in Y'— is the parameter vector of the normalized standard equation B which is meromorphically equivalent to A and satisfies $M_A = M_B + K$, $PQ_AP^{-1} = Q_B$, $P\Lambda'_AP^{-1} = \Lambda'_B + K'$.*

Proof: Here we take S' as the set of exactly those vectors from $\mathbb{C}^{rn^2+n}$ whose corresponding differential equations have $a_{nj}^{(r-1)} = 1$ for $1 \leq j \leq n-1$ whereas all other rn^2+1 parameters are algebraically independent over $\mathbb{Q}$. Notice that such a vector is always the parameter vector of a normalized standard equation. Now we proceed along the same lines as in the proof of Theorem 3. The matrix E_0 is constructed in exactly the same way (notice that the normalizations do not restrict $a_{nn}^{(0)}$) and for $r \geq 2$ the remaining arguments remain valid since the normalization is carried out rationally and each new normalized equation must still contain rn^2+1 algebraically independent parameters which prevents $a_{nj}^{(r-1)}$ $(1 \leq j \leq n-1)$ from vanishing and therefore produces uniform rational formulae when we normalize them to 1. For $r = 1$ some elements in A_0 are normalized which forces us to discuss whether the parts $\vec{t}_1, \ldots, \vec{t}_n$ of the eigenvectors in E_0 still have algebraically independent components. For that purpose we consider the last row of

$$\begin{bmatrix} \tilde{A} & \vec{a} \\ 1 \ldots 1 & a_{nn} \end{bmatrix} \begin{bmatrix} \vec{t_j} \\ 1 \end{bmatrix} = \lambda_j \begin{bmatrix} t_j \\ 1 \end{bmatrix}$$

which yields $\sum_{k=1}^{n-1} t_{jk} + a_{nn} = \lambda_j$ $(1 \leq j \leq n)$.

Hence the eigenvalues depend algebraically on the n^2-n+1 parameters t_{jk} and a_{nn}, and the same is therefore true for the elements of A_0 $(= E_0^{-1} M_A E_0)$. But since A_0 contains n^2-n+1 algebraically independent elements, the t_{jk} must be algebraically independent, too. All other arguments carry immediately over to this situation. ■

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