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A NOTE ON INVERSE LIMITS OF CONTINUOUS IMAGES OF ARCS

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Abstract

The main purpose of this paper is to prove some theorems concerning inverse systems and limits of continuous images of arcs. In particular, we shall prove that if $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system of continuous images of arcs with monotone bonding mappings such that $\mathrm{cf}(\mathrm{card}(A)) \neq \omega_1$, then $X = \lim \mathbf{X}$ is a continuous image of an arc if and only if each proper subsystem $\{X_a, p_{ab}, B\}$ of $\mathbf{X}$ with $\mathrm{cf}(\mathrm{card}(B)) = \omega_1$ has the limit which is a continuous image of an arc (Theorem 18).

1. Inverse limits of hereditarily locally connected continua

An *arc (or ordered continuum)* is a Hausdorff continuum with exactly two non-separating points. Each separable arc is homeomorphic to the closed interval $I = [0, 1]$.

A space X is said to be an IOK (IOC) if there exists an ordered compact (connected) space K and a continuous surjection $f : K \to X$. Frequently, we will say that a space X is a continuous image of an arc if X is an IOC.

The cardinality of a set A will be denoted by $\mathrm{card}(A)$. We assume that $\mathrm{card}(A)$ is the initial ordinal number. The cofinality of a cardinal number m will be denoted by $\mathrm{cf}(m)$.

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A continuum X is said to be *hereditarily locally connected* if each subcontinuum of X is locally connected. A continuum X is said to be *finitely Suslinian* [**17**] if there do not exist open sets U and V, and an infinite collection $\mathcal{K}$ of pairwise disjoint subcontinua of X such that $Cl(U) \cap Cl(V) = \emptyset$ and $K \cap V \neq \emptyset$ and $K \cap U \neq \emptyset$ for each K in $\mathcal{K}$. Each finitely Suslinian continuum is hereditarily locally connected. A continuum X is *rim-finite (rim-countable)* if it has a basis $\mathcal{B}$ such that $\operatorname{card}(\operatorname{Bd}(U)) < \aleph_0$ ($\operatorname{card}(\operatorname{Bd}(U)) \leq \aleph_0$) for each $U \in \mathcal{B}$. Each rim-finite continuum is finitely Suslinian. Each hereditarily locally connected continuum is a continuous image of an arc [**11**, Theorem 3.4].

In the paper [**8**, Problem 9.10] the authors asked when the inverse limit of an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of hereditarily locally connected continua with monotone surjective bonding mappings p_{ab} is a continuous image of an arc.

If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system of hereditarily locally connected continua, then $X = \lim \mathbf{X}$ need not be a hereditarily locally continuum since each locally connected metric continuum of dimension 1 (= curve) is the limit of an inverse sequence of rim-finite continua with surjective monotone bonding mappings [**13**, Theorem 2.2].

In the present section we shall define a class of hereditarily locally connected continua such that each inverse limit of such spaces and monotone bonding mappings has a hereditarily locally connected limit.

In Appendix we review some definitions and known results needed in this section.

We say that an inverse system $\mathbf{Y} = \{Y_a, q_{ab}, B\}$ is a subsystem of $\mathbf{X} = \{X_a, p_{ab}, A\}$ if $B \subset A$, $Y_a = X_a$ and each q_{ab} is p_{ab}.

We start with the following theorem.

Theorem 1. *Let X be the limit of an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of hereditarily locally connected continua X_a such that the bonding mappings $p_{ab} : X_b \to X_a$ are monotone surjections. Then X is a hereditarily locally continuum if and only if each countable inverse subsystem of $\mathbf{X}$ has a hereditarily locally connected limit.*

Proof: By Theorem 29 X is homeomorphic to the limit of $\mathbf{X}_\sigma = \{X_\Delta, p_{\Delta\Gamma}, A_\sigma\}$, where A_σ is the family of all nonempty countable directed subsets of A. If X is hereditarily locally connected then each X_Δ is hereditarily locally connected as a monotone image of a hereditarily locally connected continuum X (see Lemma 28). Conversely, if each X_Δ is hereditarily locally connected, then X is hereditarily locally connected (Theorem 27). ■

The following theorem is a generalization of the well-known result of G. T. Whyburn [**20**, p. 81] which asserts that a metric continuum X is hereditarily locally connected if and only if each cyclic element (see Appendix) $Z \subseteq X$ is hereditarily locally connected.

Theorem 2. *A locally connected continuum X is hereditarily locally connected if and only if each cyclic element of X is hereditarily locally connected.*

Proof: By Theorems 31 and 29 there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric locally connected spaces such that p_{ab} are monotone and X is homeomorphic to $\lim \mathbf{X}$. Let us prove that each X_a is hereditarily locally connected. It suffices to prove that each cyclic element Z_a of X_a is hereditarily locally connected. By Lemma 34 there exists a cyclic element Z of $\lim \mathbf{X}$ such $p_a(Z) \supseteq Z_a$. Since $\lim \mathbf{X}$ is homeomorphic to X, Z is hereditarily locally connected. This means that $p_a(Z)$ is hereditarily locally connected. It follows that Z_a is hereditarily locally connected since $Z_a \subseteq p_a(Z)$. We infer that each X_a is hereditarily locally connected since X_a is a metric continuum. From Theorem 27 it follows that X is hereditarily locally connected. ∎

A surjective mapping $f : X \to Y$ is said to be *hereditarily monotone* [**5**, pp. 16–17] if for each subcontinuum K of X the restriction $f \mid K : K \to f(K)$ is monotone.

If $f : X \to Y$ and $g : Y \to Z$ are hereditarily monotone mappings, then $gf : X \to Z$ is hereditarily monotone [**5**, p. 29, (5.3)].

Lemma 3. *If Z is a cyclic element of X, then the canonical retraction $\rho : X \to Z$ is hereditarily monotone.*

Proof: Let $\rho : X \to Z$ (see Appendix, p. 13) and let K be any subcontinuum of X. Let us prove that the restriction $\rho_K : K \to \rho_K(K) \subseteq Z$ is monotone. Consider the following cases.

a) $K \subseteq Z$. Now ρ_K is the identity and is monotone.

b) $K \subseteq X \setminus Z$. It is clear that K is a subset of some component J of $X \setminus Z$. Let $\{z_J\} = \mathrm{Bd}(J)$. Now, $\rho_K(K) = \{z_J\}$. This means that $\rho_K^{-1}(z_J) = K$. Hence ρ_K is monotone.

c) $K \bigcap Z \neq \emptyset$ and $K \bigcap (X \setminus Z) \neq \emptyset$. In this case

$$K = \left(K \bigcap Z\right) \bigcup \left\{K \bigcap J : J \text{ is a component of } X \setminus Z,\ K \bigcap J \neq \emptyset\right\}.$$

We infer that

$$\rho_K(K) = \left(K \bigcap Z\right) \bigcup \{\{z_J\} : J \text{ is a component of } X \setminus Z,\ \{z_J\} = \mathrm{Bd}(J)\}.$$

If $x \in \rho_K(K)$ such that $x \neq z_J$ for all components J of $X \setminus Z$, then $\rho_K^{-1}(x) = \{x\}$. Hence $\rho_K^{-1}(x)$ is connected. If $x = z_J$ for some component J, then

$$\rho_K^{-1}(x) = \{z_J\} \bigcup \left\{K \bigcap J_i : \{z_J\} = \mathrm{Bd}(J_i),\ i \in I\right\}.$$

It suffices to prove that each set $Cl(J_i) \bigcap K$ is connected. Suppose that $Cl(J_i) \bigcap K$ is not connected. There exists a component L of $Cl(J_i) \bigcap K$ such that $z_J \notin L$. By virtue of the normality of X it follows that there exists a pair U, V of disjoint open sets such that $L \subseteq U$ and $z_J \in V$. We may assume that $ClU \subseteq J$ since J is open. For each point x of $K \setminus (U \bigcup V)$ there exists an open set U_x such that $x \in U_x$ and $U_x \bigcap U = \emptyset$. Let W be the union of V and all the sets U_x, $x \in K \setminus (U \bigcup V)$. It is clear that $U \bigcap W = \emptyset$ and $K \subseteq U \bigcup W$. This is impossible since K is connected. Hence, $Cl(J_i) \bigcap K$ is connected and ρ_K is monotone. ■

Theorem 4. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of continua and hereditarily monotone bonding mappings. Then the projections p_a, $a \in A$, are hereditarily monotone. Moreover, if each X_a is hereditarily locally connected, then $X = \lim \mathbf{X}$ is hereditarily locally connected.*

Proof: Let K be a subcontinuum of X. For each $a \in A$, $K_a = p_a(K)$ is a subcontinuum of X_a. We have the inverse system $K = \{K_a, p_{ab} \mid K_b, A\}$. From the definition of hereditarily monotone mapping it follows that each mapping $p_{ab} \mid K_b$ is a monotone mapping. This means that the projections $p_a \mid K$ are monotone [**2**, pp. 462–463]. From [**2**, Theorem 6.1.28] it follows that K is locally connected. Thus, X is hereditarily locally connected. ■

A surjective mapping $f : X \to Y$ is said to be *cyclically hereditarily monotone* if for each cyclic element Z of X the restriction $f \mid Z$ is hereditarily monotone.

Theorem 5. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of hereditarily locally connected continua X_a and cyclically hereditarily monotone bonding mappings p_{ab}. Then $X = \lim \mathbf{X}$ is hereditarily locally connected.*

Proof: By Theorem 2 it suffices to prove that each cyclic element Z of X is hereditarily locally connected. There exists an inverse system (Theorem 35) $\mathbf{Z} = \{Z_a, g_{ab}, A\}$ such that Z_a is a cyclic element of X_a, $g_{ab} = \rho_a \circ (f_{ab} \mid Z_b)$ for all $a \leq b \in A$, and Z is homeomorphic to $\lim \mathbf{Z}$. This means that g_{ab} is hereditarily monotone since ρ_a is hereditarily monotone (Lemma 3). From Theorem 4 it follows that Z is hereditarily locally connected. Thus, X is hereditarily locally connected. ■

A space X is in class $\mathcal{H}_m$ if X is a hereditarily locally connected continuum and X contains no non-degenerate metric subcontinuum. Each space in class $\mathcal{H}_m$ is rim-finite [**19**, Theorem 1].

Theorem 6. *Let* $\mathbf{X} = \{X_n, p_{mn}, \mathbb{N}\}$ *be an inverse sequence with monotone surjective bonding mappings. If each* X_n *is in class* $\mathcal{H}_m$*, then* $X = \lim \mathbf{X}$ *is in class* $\mathcal{H}_m$*. Moreover,* X *is rim-finite.*

Proof: Each X_n is a continuous image of an arc [**11**, Corollary 3.5] since each X_n is hereditarily locally connected. Thus, X is a continuous image of an arc (Theorem 14). Let us prove that X contains no non-degenerate metric subcontinuum. Suppose that Y is a non-degenerate metric subcontinuum of X. Then there exists a $n \in N$ such that $p_m(Y)$ is a non-degenerate metric subcontinuum of X_m for each $m \geq n$. This is impossible since X_m is in class $\mathcal{H}_m$. We infer that X contains no non-degenerate metric subcontinua. By virtue of Theorem 21 it follows that X is hereditarily locally connected. Moreover, from Theorem 1 of [**19**] it follows that X is rim-finite. ■

Theorem 7. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system with monotone surjective bonding mappings. If each* X_a *is in class* $\mathcal{H}_m$*, then* $X = \lim \mathbf{X}$ *is in class* $\mathcal{H}_m$*. Moreover,* X *is rim-finite.*

Proof: Apply Theorem 1 and Theorem 23. ■

A hereditarily locally connected continuum X is in class $\mathcal{H}_{zm}$ if each cyclic element Z of X is in class $\mathcal{H}_m$.

Theorem 8. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system with monotone surjective bonding mappings. If each* X_a *is in class* $\mathcal{H}_{zm}$*, then* $X = \lim X$ *is in class* $\mathcal{H}_{zm}$*.*

Proof: Let Z be a cyclic element of X. By Theorem 35 there exists an inverse system $(Z_\gamma, g_{\gamma\gamma'}, \Gamma)$ such that Z is homeomorphic to $\lim \operatorname{inv}(Z_\gamma, g_{\gamma\gamma'}, \Gamma)$, where Z_γ is a cyclic element of X_γ and $g_{\gamma\gamma'}$ is monotone. By Theorem 7 Z is in class $\mathcal{H}_m$. From Theorem 2 it follows that X is hereditarily locally connected. Hence, X is in class $\mathcal{H}_{zm}$. ■

2. Special classes of continuous images of arcs

Theorem 9 [**9**]. *Let X be a locally connected continuum such that for each pair of distinct points a, b in X, there exists a continuous onto map $f : X \to [c,d]$ such that $f(a) = c$ and $f(b) = d$ and $[c,d]$ is a non-metrizable arc. If X is rim-metrizable or rim-scattered or monotonically normal, then X is a continuous image of an arc.*

A locally connected continuum is said to be a *NTT-space* if for each pair of distinct points a, b in X, there exists a continuous onto map $f : X \to [c,d]$ such that $f(a) = c$ and $f(b) = d$ and $[c,d]$ is a non-metrizable arc.

Theorem 10. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of NTT-spaces with monotone surjective bonding mappings. Then $X = \lim \mathbf{X}$ is a NTT-space.*

Proof: It is known that X is locally connected continuum. Let x, y be a pair of distinct points of X. There exists an $a \in A$ such that $p_b(x) \neq p_b(y)$ for each $b \geq a$. Since X_b is a NTT-space there exists a non-metrizable arc $[c,d]$ and a surjective mapping $f : X_b \to [c,d]$ such that $f(x) = c$ and $f(y) = d$. Considering the mapping $fp_b : X \to [c,d]$ we infer that X is a NTT-space. ■

Theorem 11. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of NTT-spaces and monotone surjective bonding mappings. If $X = \lim \mathbf{X}$ is rim-metrizable or rim-scattered or monotonically normal, then X is a continuous image of an arc.*

Proof: By Theorem 10 X is a NTT-space. Apply Theorem 9. ■

Theorem 12. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed inverse system of spaces X_a such that for each pair x_a, y_a of points of X_a the subspace $X_a \setminus \{x_a, y_a\}$ is connected, $a \in A$. If each X_a is a continuous image of an arc and each p_{ab} is a monotone surjection, then $X = \lim \mathbf{X}$ is a continuous image of an arc if and only if there exists an $a \in A$ such that $p_b : X \to X_b$ is a homeomorphism for each $b \geq a$ (if and only if X is metrizable).*

Proof: By Theorem 2 of [**18**] each X_a is metrizable. We shall prove that for each pair x, y of points of X the subspace $Y = X \setminus \{x,y\}$ is connected. Suppose that Y is not connected. Then there exists a pair U, V of disjoint open subsets of X such that $Y = U \cup V$. Moreover, there exists an $a \in A$ such that $p_b(x) \neq p_b(y)$, $b \geq a$. The sets $U_a = \{x_a : x_a \in X_a, p_a^{-1}(x_a) \subset U\}$ and $V_a = \{x_a : x_a \in X_a, p_a^{-1}(x_a) \subset V\}$ are disjoint and open. Now we have $X \setminus \{p_a(x), p_a(y)\} = U_a \cup V_a$. This is impossible since $X \setminus \{p_a(x), p_a(y)\}$ is connected. Hence, $Y = X \setminus \{x, y\}$ is connected. It follows that if X is a continuous image of an arc, then it is metrizable ([**18**, Theorem 2]). From Theorem 26 it follows that there exists a $b \in A$ such that $p_c : X \to X_c$ is a homeomorphism for every $c \geq b$. Conversely, if such $b \in A$ exists, then X is a continuous image of an arc. ■

Theorem 13. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of spaces* X_a *such that for each pair* x_a, y_a *of points of* X_a *the subspace* $X_a \setminus \{x_a, y_a\}$ *is connected,* $a \in A$. *If each* X_a *is a continuous image of an arc and each* p_{ab} *is a monotone surjection, then* $X = \lim \mathbf{X}$ *is a continuous image of an arc if and only if there exists a countable subsystem* $\mathbf{Y}$ *of* $\mathbf{X}$ *such that* $\lim \mathbf{Y}$ *is homeomorphic to* X.

Proof: Consider the inverse system $\mathbf{X}_\sigma = \{X_\Delta, p_{\Delta\Gamma}, A_\sigma\}$ from Theorem 29 and apply Theorem 12. ■

3. Inverse systems and subsystems

Theorem 14 [**8**, Theorem 5.1]**.** *Let* $\mathbf{X} = \{X_n, p_{mn}, \mathbb{N}\}$ *be an inverse sequence with monotone surjective bonding mappings. If each* X_n *is the continuous image of an arc, then* $X = \lim \mathbf{X}$ *is the continuous image of an arc.*

Theorem 15 [**4**, Theorem 2.17]**.** *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a well-ordered inverse system such that* $\mathrm{cf}(A) \neq \omega_1$. *If the mappings* p_{ab} *are monotone surjections and if the spaces* X_a *are the continuous images of arcs, then* $X = \lim \mathbf{X}$ *is the continuous image of an arc.*

Remark 16. Theorem 15 is not true if $\mathrm{cf}(A) = \omega_1$. This is shown by the following example of Nikiel [**10**]. Let L denote the long interval [**2**, p. 297]. For each ordinal number α, $0 < \alpha < \omega_1$, let $f_\alpha : [0,1] \times L \to [0,1] \times [0, \alpha]_L$ be defined by

$$f(s,t) = \begin{cases} (s,t) & \text{if } t \leq_L \alpha \\ (s,\alpha) & \text{if } \alpha \leq_L t. \end{cases}$$

Each $X_\alpha = [0,1] \times [0,\alpha]_L$ is homeomorphic to $[0,1] \times [0,1]$ and it is a continuous image of an arc. Moreover, $w(X_a) = \aleph_0$. Let $f_{\alpha\beta} = f_\alpha \mid [0,1] \times [0,\beta]_L$, $\beta < \alpha$. We obtain an inverse system $\{X_\alpha, f_{\alpha\beta}, \alpha < \omega_1\}$ whose limit is $[0,1] \times L$ which is not a continuous image of an arc.

Theorem 17. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of compact spaces such that* $\operatorname{card}(A) > \aleph_0$. *There exists a transfinite sequence* $\{A_\alpha : \alpha < \operatorname{card}(A)\}$ *of directed subsets* A_α *of* A *such that:*

1. $\operatorname{card}(A_\alpha) < \operatorname{card}(A)$, $\alpha < \operatorname{card}(A)$,
2. $\alpha < \beta < \operatorname{card}(A)$ *implies* $A_\alpha \subseteq A_\beta$,
3. $A = \bigcup\{A_\alpha : \alpha < \operatorname{card}(A)\}$,
4. *each collection* $\{X_a, p_{ab}, A_\alpha\}$ *is an inverse system with limit* X_α,
5. *if* $\alpha < \beta < \operatorname{card}(A)$ *then there exists a mapping* $q_{\alpha\beta} : X_\beta \to X_\alpha$,
6. $\lim \mathbf{X}$ *is homeomorphic to* $\lim\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \operatorname{card}(A)\}$,
7. *if the mappings* p_{ab} *are monotone, then the mappings* $q_{\alpha\beta}$ *are monotone.*

Proof: The proof consists of several steps. Step 1 is from [**7**, pp. 238–239, Hilfssatz]. For the sake of the completeness we give the proof of Step 1.

Step 1: Let ν be any finite subset of A. There exists a $\delta(\nu) \in A$ such that $\delta \leq \delta(\nu)$ for each $\delta \in \nu$. For each $B \subseteq A$ there exists a set $F_1(B) = B \bigcup\{\delta(\nu) : \nu \subset B \text{ and } \nu \text{ is finite}\}$. Put

$$F_{n+1} = F_1(F_n(B)),$$

and

$$F_\infty(B) = \bigcup\{F_n(B) : n \in \mathbb{N}\}.$$

It is clear that

$$F_1(B) \subseteq F_2(B) \subseteq \ldots \subseteq F_n(B) \subseteq \ldots$$

The set $F_\infty(B)$ is directed since each finite subset ν of $F_\infty(B)$ is contained in some $F_n(B)$ and, consequently, $\delta(\nu)$ is contained in $F_\infty(B)$. If B is finite, then $\operatorname{card}(F_\infty(B)) = \aleph_0$. If $\operatorname{card}(B) \geq \aleph_0$, then we have $\operatorname{card}(\{\delta(\nu) : \nu \in B\}) \leq \operatorname{card}(B)\aleph_0$. We infer that $\operatorname{card}(F_1(B)) \leq \operatorname{card}(B)\aleph_0$. Similarly, $\operatorname{card}(F_n(B)) \leq \operatorname{card}(B)\aleph_0$. This means that $\operatorname{card}(F_\infty(B)) \leq \operatorname{card}(B)\aleph_0$. Thus

$$\operatorname{card}(F_\infty(B)) \leq \operatorname{card}(B)\aleph_0.$$

Suppose that $\operatorname{card}(A) > \aleph_0$. Put $\Omega = \operatorname{card}(A)$. Hence, $A = \{a_\alpha : \alpha < \Omega\}$. Put $B_\alpha = \{a_\mu : \mu < \alpha < \Omega\}$. We have a transfinite sequence $\{B_\alpha : \alpha < \Omega\}$ such that

a) $\operatorname{card}(B_\alpha) < \operatorname{card}(A)$,
b) $\alpha < \beta < \Omega$ implies $B_\alpha \subseteq B_\beta$,
c) $A = \bigcup\{B_\alpha : \alpha < \Omega\}$.

Put $A_\alpha = F_\infty(B_\alpha)$.

Step 2: Assertions 1-3 follow from Step 1.

Step 3: Assertion 4 follows from the fact that each A_α is directed subset of A.

Step 4: Let us prove 5. From assertion 2 it follows that there exists a continuous mapping $q_{\alpha\beta} : X_\beta \to X_\alpha$ since each point $x \in X_\beta$ induces a collection $\{x_a : a \in A_\alpha\}$ which satisfies $p_{ab}(x_b) = x_a$, i.e., $\{x_a : a \in A_\alpha\}$ is a point of X_α.

Step 5: It is obvious that there exists a mapping $H : \lim \mathbf{X} \to \lim\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \mathrm{card}(A)\}$ since each $x = (x_a : a \in A) \in \lim \mathbf{X}$ induces a collection $\{x_a : a \in A_\alpha\}$ on each A_α. Thus we have the mappings $H_\alpha : \lim \mathbf{X} \to X_\alpha$, for each $\alpha < \mathrm{card}(A)$. The mappings H_α induce a continuous mapping $H : \lim \mathbf{X} \to \lim\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \mathrm{card}(A)\}$. It remains to prove that H is 1-1 and onto. Let us prove that H is 1-1. Let x, $y \in \lim \mathbf{X}$ and $x \neq y$. There exists an $a \in A$ such that $x_a \neq y_a$. From Step 1 it follows that there is an A_α such that $a \in A_\alpha$. Now, $x_a \neq y_a$ implies $H_\alpha(x) \neq H_\alpha(y)$ (see Step 5). This means that $H(x) \neq H(y)$. Hence H is 1-1. In order to complete the proof it suffices to prove that H is onto. Let $y = (y_\alpha : \alpha < \mathrm{card}(A))$ be any point of $\lim\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \mathrm{card}(A)\}$. Then $y_\alpha \in X_\alpha$. Thus, y_α is a thread in X_α, i.e., $y = (x_a, a \in A_\alpha)$. We infer that for each $a \in A$ there exists a point $x_a \in X_a$. It is readily to seen that $p_{ab}(x_b) = x_a$. Thus, $(x_a : a \in A)$ is a thread in $\lim \mathbf{X}$ such that $H(x) = y$.

Step 6: Let us prove 7. If the mappings p_{ab} are monotone, then from Lemma 28 it follows that the mappings $q_{\alpha\beta}$ are monotone. ■

Now we shall prove the main theorem of this section which is a generalization of Theorem 15.

Theorem 18. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of continuous images of arcs with monotone bonding mappings. If* $\mathrm{cf}(\mathrm{card}(A)) \neq \omega_1$*, then $X = \lim \mathbf{X}$ is a continuous image of an arc if and only if each proper subsystem $\{X_a, p_{ab}, B\}$ of $\mathbf{X}$ with* $\mathrm{cf}(\mathrm{card}(B)) = \omega_1$ *has the limit which is a continuous image of an arc.*

Proof: The "only if part". If X is a continuous image of an arc, then for each subsystem $\{X_a, p_{ab}, B\}$ there exists a natural projections $f_a : X \to \lim\{X_a, p_{ab}, B\}$. Hence, $\lim\{X_a, p_{ab}, B\}$ is a continuous image of an arc.

The "if" part. By Theorem 17 there exists a well-ordered inverse system $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \operatorname{card}(A)\}$ such that X is homeomorphic to $\lim\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \operatorname{card}(A)\}$. If $\operatorname{cf}(\operatorname{card}(A)) \leq \omega_0$, then we have an inverse subsequence of $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \operatorname{card}(A)\}$ which is a cofinal subsystem of $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \operatorname{card}(A)\}$. By Theorem 14 X is a continuous image of an arc. Now, suppose that $\operatorname{cf}(\operatorname{card}(A)) > \omega_1$. By Theorem 17 it suffices to prove that each subsystem of $\{X_a, p_{ab}, B\}$ of $\mathbf{X} = \{X_a, p_{ab}, A\}$ has the limit which is a continuous image of an arc. We shall use the transfinite induction on $\operatorname{card}(B)$. If $\operatorname{card}(B) \leq \omega_0$, then we use Theorem 14. If $\operatorname{card}(B) = \omega_1$, then $\lim\{X_a, p_{ab}, B\}$ is a continuous image of an arc by assumption of Theorem. Let now $\{X_a, p_{ab}, B\}$ be a subsystem of $\{X_a, p_{ab}, A\}$ such that $\operatorname{card}(B) > \omega_1$. Suppose that Theorem is true for each subsystem of the cardinality $< \operatorname{card}(B)$. By Theorem 17 there exists a well-ordered inverse system $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \operatorname{card}(B)\}$ such that $\lim\{X_a, p_{ab}, B\}$ is homeomorphic to $\lim\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \operatorname{card}(B)\}$. Since each X_α is the limit of a subsystem of the cardinality $< \operatorname{card}(B)$, we have the inverse system $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \operatorname{card}(B)\}$ which satisfies the conditions of Theorem 15. Thus, $\lim\{X_a, p_{ab}, B\}$ is a continuous image of an arc. By the transfinite induction the proof is complete. ■

Corollary 19. *Let X be a locally connected continuum. The following conditions are equivalent:*

a) *X is a continuous image of an arc.*
b) *If $f : X \to Y$ is a continuous mapping and* $\operatorname{cf}(w(Y)) = \omega_1$, *then Y is a continuous image of an arc.*

Proof: a) $\Rightarrow$ b). Obvious.

b) $\Rightarrow$ a). By Theorem 31 there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that X_a are metric locally connected continua, p_{ab} are monotone mappings and X is homeomorphic to $\lim \mathbf{X}$. If $\mathbf{Y} = \{X_a, p_{ab}, B\}$ is any subsystem of $\{X_a, p_{ab}, A\}$ with $\operatorname{cf}(\operatorname{card}(B)) = \omega_1$, then there exists a natural projection $P : X \to \lim \mathbf{Y}$. By b) it follows that $\lim \mathbf{Y}$ is a continuous image of an arc since $w(\lim \mathbf{Y}) = \aleph_1$. Applying Theorem 18 we complete the proof. ■

Corollary 20. *Let X be a locally connected continuum such that $w(X) > \aleph_{\omega_1}$. The following conditions are equivalent:*

a) *X is a continuous image of an arc.*
b) *If $f : X \to Y$ is a continuous mapping and $w(Y) = \aleph_1$, then Y is a continuous image of an arc.*

4. Appendix

Theorem 21 [**15**, Theorem 4]**.** *If X is a connected IOK and X contains no nondegenerate metric subcontinuum, then X is hereditarily locally connected.*

Theorem 22 [**17**, Theorem]**.** *If X is a continuum containing no nondegenerate metric subcontinuum, then X is finitely Suslinian if and only if X is a continuous image of an arc.*

We say that $\mathbf{X} = \{X_a, p_{ab}, A\}$ is σ-directed if for each sequence $a_1, a_2, \dots, a_k, \dots$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Theorem 23 [**8**, Theorem 9.8]**.** *If X is the limit of a σ-directed inverse system of finitely Suslinian continua, then X is finitely Suslinian.*

Theorem 24 [**8**, Theorem 9.9]**.** *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed inverse system of rim-finite continua with surjective bonding mappings. Then $X = \lim \mathbf{X}$ is a rim-finite continuum.*

Theorem 25. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed inverse system of compact spaces with surjective bonding mappings and limit X. Let Y be a metric compact space. For each surjective mapping $f : X \to Y$ there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $g_b : X_b \to Y$ such that $f = g_b p_b$.*

Proof: Let $\mathcal{B}$ be a countable basis of Y and let $\mathcal{V}$ be a collection of all finite subfamilies of $\mathcal{B}$ which cover X. Clearly, $\mathrm{card}(\mathcal{V}) = \aleph_0$. Hence, $\mathcal{V} = \{\mathcal{V}_n : n \in \mathbb{N}\}$. For each $\mathcal{V}_n$ $f^{-1}(\mathcal{V}_n) = \left\{f^{-1}(U) : U \in \mathcal{V}_n\right\}$ is a covering of X. There exists an $a(n) \in A$ such that for each $b \geq a(n)$ there is a cover $\mathcal{V}_{nb}$ of X_b with $p_b^{-1}(\mathcal{V}_{nb}) \prec f^{-1}(\mathcal{V}_n)$. From the σ-directedness of A it follows that there is an $a \in A$ such that $a \geq a(n)$, $n \in \mathbb{N}$. Let $b \geq a$. We claim that $f(p_b^{-1}(x_b))$ is degenerate. Suppose that there exists a pair u, v of distinct points of Y such that u, $v \in f(p_b^{-1}(x_b))$. Then there exists a pair x, y of distinct points of $p_b^{-1}(x_b)$ such that $f(x) = u$ and $f(y) = v$. Let U, V be a pair of disjoint open sets of Y such that $u \in U$ and $v \in V$. Consider the covering $\{U, V, Y \setminus \{u, v\}\}$. There exists a covering $\mathcal{V}_n \in \mathcal{V}$ such that $\mathcal{V}_n \prec \{U, V, X \setminus \{u, v\}\}$. We infer that there is a covering $\mathcal{V}_{nb}$ of X_b such that $p_b^{-1}(\mathcal{V}_{nb}) \prec f^{-1}(\mathcal{V}_n)$. It follows that $p_b(x) \neq p_b(y)$ since x and y lie in the disjoint members of the covering $f^{-1}(\mathcal{V}_n)$. This is impossible since x, $y \in p_b^{-1}(x_b)$. Thus, $f(p_b^{-1}(x_b))$ is degenerate. Now we define $g_b : X_b \to Y$ by $g_b(x_b) = f(p_b^{-1}(x_b))$. It is clear that $g_b p_b = f$. Let us prove that g_b is continuous. Let U be open in Y. Then $g_b^{-1}(U)$ is open since $p_b^{-1}(g_b^{-1}(U)) = f^{-1}(U)$ is open and p_b is quotient (as a closed mapping). ■

Theorem 26. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed inverse system of compact spaces with limit X. A closed subspace Y of X is metrizable if and only if there exists an $a \in A$ such that $p_b \mid Y : Y \to p_b(Y)$ is a homeomorphism for each $b \geq a$.*

Proof: Consider the inverse system $\mathbf{Y} = \{p_a(Y), p_{ab} \mid p_b(Y), A\}$ with limit Y and the identity mapping $i : Y \to Y$. Apply Theorem 25. ■

Theorem 27 [**3**, Corollary 3]**.** *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed inverse system of hereditarily locally connected continua X_a. Then $X = \lim \mathbf{X}$ is hereditarily locally connected.*

The following lemma follows from Theorem 10 and Corollary on p. 69 of [**14**]. See also [**12**, Lemma 3.5].

Lemma 28. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with monotone bonding surjections, $X = \lim \mathbf{X}$, Y be a compact space and $m_a : Y \to X_a$, $a \in A$, be monotone surjections such that $m_a = p_{ab} m_b$ for any a, $b \in A$, $a \leq b$. Moreover, let $m : Y \to X$ denote the map induced by m_a, $a \in A$. Then m is also a monotone surjection. Moreover, each projection $p_a : X \to X_a$, $a \in A$, is a monotone surjection.*

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system. For each infinite subset Δ_0 of $(A, \leq)$ we define sets Δ_n, $n = 0, 1, \ldots$, by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x, y) : x, y \in \Delta_n\}$, where $m(x, y)$ is a member of A such that $x, y \leq m(x, y)$. Let $\Delta = \bigcup \{\Delta_n : n \in \mathbb{N}\}$. It is clear that $\mathrm{card}(\Delta) = \mathrm{card}(\Delta_0)$. Moreover, Δ is directed by $\leq$ [**8**, Lemma 9.2]. For each directed set $(A, \leq)$ we define

$$A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A,\ \mathrm{card}(\Delta) \leq \aleph_0 \quad \text{and } \Delta \text{ is directed by } \leq\}.$$

Then A_σ is σ-directed by inclusion [**8**, Lemma 9.3]. If $\Delta \in A_\sigma$, let $\mathbf{X}^\Delta = \{X_b, p_{bb'}, \Delta\}$ and $X_\Delta = \lim \mathbf{X}^\Delta$. If Δ, $\Gamma \in A_\sigma$ and $\Delta \subseteq \Gamma$, let $p_{\Delta\Gamma} : X_\Gamma \to X_\Delta$ denote the map induced by the projections $p_\delta^\Gamma : X_\Gamma \to X_\delta$, $\delta \in \Delta$, of the inverse system X^Γ. Now, we have the following theorem.

Theorem 29 [**8**, Theorem 9.4]**.** *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system, then $\mathbf{X}_\sigma = \{X_\Delta, p_{\Delta\Gamma}, A_\sigma\}$ is a σ-directed inverse system and $\lim \mathbf{X}$ and $\lim \mathbf{X}_\sigma$ are canonically homeomorphic.*

Theorem 30. *Let X be a compact space. There exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces X_a and surjective bonding mappings p_{ab} such that X is homeomorphic to* $\lim \mathbf{X}$.

Proof: See [**6**, pp. 152, 164]. ■

Theorem 31. *If X is a locally connected compact space, then there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric locally connected compact space, each p_{ab} is a monotone surjection and X is homeomorphic to* $\lim \mathbf{X}$. *Conversely, the inverse limit of such system is always a locally connected compact space.*

Proof: See [**6**, p. 163, Theorem 2]. ■

Theorem 32 [**11**, Corollary 2.9]**.** *If X is a hereditarily locally connected continuum, then there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metrizable hereditarily locally connected continuum, each p_{ab} is a monotone surjection and X is homeomorphic to* $\lim \mathbf{X}$.

Let X be a non-degenerate locally connected continuum. A subset Y of X is said to be a *cyclic element* of X if Y is connected and maximal with respect to the property of containing no separating point of itself. A cyclic element of locally connected continuum is again a locally connected continuum. We let

$$\mathbf{L}_X = \{Y \subset X : Y \text{ is a non-degenerate cyclic element of } X\}.$$

Lemma 33 [**8**, Lemma 2.2]**.** *If C is a connected subset of X and $Y \in \mathbf{L}_X$, then $C \bigcap Y$ is connected (possibly void).*

Lemma 34 [**8**, Lemma 2.3]**.** *If $f : X \to X'$ is a monotone surjection, then for each $Y' \in \mathbf{L}_{X'}$ there exists $Y \in \mathbf{L}_X$ such that $Y' \subseteq f(Y)$. In particular, $\mathbf{L}_X$ is non-empty if $\mathbf{L}_{X'}$ is non-empty.*

Let Z be a cyclic element of X. For each component J of $X \setminus Z$, let $\mathrm{Bd}(J) = \{z_J\}$. We define a mapping [**8**, p. 5] $\rho : X \to Z$ such that $\rho(x) = x$ if $x \in Z$ and $\rho(x) = z_J$ if $x \in J$.

The mapping ρ is a monotone continuous retraction. It is called the *canonical retraction* of X onto Z.

Theorem 35 [**8**, Theorem 2.7]**.** *Let Y be a cyclic locally connected continuum and $\mathcal{S}=(Y_\gamma, f_{\gamma\gamma'}, \Gamma)$ an inverse system such that $Y=\lim\operatorname{inv}\mathcal{S}$ and each bonding mapping $f_{\gamma\gamma'}$ is a monotone surjection. For each $\gamma\in\Gamma$, let Z_γ be either a cyclic element of Y_γ or a one-point subset of Y_γ. Let $\rho_\gamma : Y_\gamma \to Z_\gamma$ denote the canonical retraction if Z_γ is non-degenerate, and otherwise let ρ_γ be the constant map. Suppose that some Z_{γ_0} is non-degenerate, and that $Z_\gamma \subseteq f_{\gamma\gamma'}(Z_{\gamma'})$ for all $\gamma\le\gamma'\in\Gamma$. Let $g_{\gamma\gamma'} = \rho_\gamma \circ (f_{\gamma\gamma'} \mid Z_{\gamma'})$ for all $\gamma\le\gamma'\in\Gamma$. Then each $g_{\gamma\gamma'} : Z_{\gamma'} \to Z_\gamma$ is a monotone surjection and $Y = \lim\operatorname{inv}(Z_\gamma, g_{\gamma\gamma'}, \Gamma)$.*

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