Publ. Mat. **45** (2001), 69–77

PROPER HOLOMORPHIC MAPPINGS BETWEEN RIGID POLYNOMIAL DOMAINS IN $\mathbb{C}^{n+1}$

Bernard Coupet and Nabil Ourimi

Abstract

We describe the branch locus of proper holomorphic mappings between rigid polynomial domains in $\mathbb{C}^{n+1}$. It appears, in particular, that it is controlled only by the first domain. As an application, we prove that proper holomorphic self-mappings between such domains are biholomorphic.

1. Introduction

A domain $D \subset \mathbb{C}^{n+1}$ is called rigid polynomial if

$$D = \{(z_0, z) \in \mathbb{C}^{n+1} : r(z_0, z) = 2\operatorname{Re}(z_0) + P(z, \bar{z}) < 0\}$$

for some real polynomial $P(z) = P(z, \bar{z})$. We say that D is nondegenerate if its boundary $\{(z_0, z) \in \mathbb{C}^{n+1} : 2\operatorname{Re}(z_0) + P(z) = 0\}$ contains no nontrivial complex variety. When P is homogeneous these domains naturally appear as approximation of domains of finite type and may be considered as their homogeneous models. These ones are useful in studies of many problems for more general domains (see for instance [**7**]).

The main result of this paper describes the branch locus of proper holomorphic mappings between rigid polynomial domains in $\mathbb{C}^{n+1}$. Let $f : D \to \Omega$ be a holomorphic mapping between domains in $\mathbb{C}^{n+1}$. We will denote by $J_f(z_0, z)$ the Jacobian determinant of f and by $V_f = \{(z_0, z) \in D : J_f(z_0, z) = 0\}$ its branch locus. Our principal result is the following.

2000 *Mathematics Subject Classification.* 32H35.
Key words. Proper holomorphic mappings, rigid polynomial nondegenerate pseudoconvex domains.

Theorem 1. *Let D and Ω be rigid polynomial nondegenerate pseudoconvex domains in $\mathbb{C}^{n+1}$. Then there exists a finite number of complex algebraic varieties $\hat{B}_1, \dots, \hat{B}_N$ in $\mathbb{C}^n$ (irreducible) depending only on D such that the branch locus of any proper holomorphic mapping $f\colon D \to \Omega$ satisfies:*

$$V_f \subset \cup_{1 \le k \le N} \{(z_0, z) \in D : z \in \hat{B}_k\}.$$

Note that the integer N is bounded by the degree of the polynomial P.

In the bounded strongly pseudoconvex case, the branch locus is empty [**17**], and in the real analytic case, one gives a nice description using semi-analytic stratification of the boundary (as it was observed in [**9**], this argument works in the smooth case as well if the set of weakly pseudoconvex boundary point admits a nice stratification). On the other hand, Rudin [**18**], Bedford [**5**], Forstnerič [**15**], Barletta-Bedford [**4**], proved that the structure of the branch locus of a proper holomorphic mapping relies on properties of its automorphism group via factorization type theorems.

As an immediate application of Theorem 1, one has the following corollary.

Corollary 1. *Let D be a rigid polynomial nondegenerate pseudoconvex domain in $\mathbb{C}^{n+1}$. Then every proper holomorphic self-mapping $f\colon D \to D$ is a biholomorphism.*

For the case $n = 1$, this result was proved in [**13**] and [**11**].

Now, we recall some definitions and results that we will need for the proof of Theorem 1. A mapping in $\mathbb{C}^{n+1}$ is algebraic if there exists an irreducible algebraic set of dimension $n+1$ in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ which contains the graph of the map. Thus, this map may be extended to a possibly multiple valued map defined on the complement of an algebraic set in $\mathbb{C}^{n+1}$. Webster [**19**] proved that a locally biholomorphic mapping taking an algebraic nondegenerate hypersurface into another one is algebraic.

Let $f\colon D \to \Omega$ be a proper holomorphic mapping satisfying the assumption of Theorem 1. According to Coupet-Pinchuk [**14**], f is algebraic. Furthermore, if the cluster set of a boundary point $a \in \partial D$ contains a point $b \in \partial\Omega$, then f extends holomorphically to a neighborhood of a. Therefore, there exists an algebraic set $\hat{S} \subset \partial D$ such that f extends holomorphically to a neighborhood of any point from $\partial D \backslash \hat{S}$ and for all $p \in \hat{S}$, $\lim_{z \to p} |f(z)| = +\infty$. Then we get the following stratification of the boundary:

$$\partial D = S_h \cup \hat{S}$$

where S_h is the set of points p in ∂D such that f extends holomorphically in a neighborhood of p. Note that this result of Coupet-Pinchuk does not assume the pseudoconvexity of the domains.

2. Behavior of the mapping and its branch locus on the boundary

For an irreducible component W of V_f, we define

$$E_W := \overline{W} \cap \partial D.$$

Lemma 1. (1) *W extends across the boundary of D as a pure n-dimensional polynomial variety in $\mathbb{C}^{n+1}$.*

(2) *There exists an open dense subset $O_W \subset E_W$ such that for each $p \in O_W$:*

(i) *E_W is a polynomial submanifold in a neighborhood of p of dimension $2n-1$.*

(ii) *f is holomorphic in a neighborhood p.*

Proof: (1) Since W is an irreducible algebraic set in D of dimension n, there exists an irreducible polynomial h in $\mathbb{C}^{n+1}$ such that $W = \{Z = (z_0, z) \in D : h(Z) = 0\}$. If W does not extend across ∂D, the defining function r will be negative on $\hat{W} = \{Z \in \mathbb{C}^{n+1} : h(Z) = 0\}$. According to [**12**] (see Proposition 2, p. 76), there exists an analytic cover $\pi\colon \hat{W} \to \mathbb{C}^n$. Let $g_1, \dots, g_k$ be the branches of π^{-1} which are locally defined and holomorphic on $\mathbb{C}^n \backslash \sigma$, with $\sigma \subset \mathbb{C}^n$ an analytic set of dimension at most $n-1$. Consider the function $\hat{r}(w) = \sup\{r \circ g_1(w), \dots, r \circ g_k(w)\}$. Since π is an analytic cover, $\hat{r}$ extends as a plurisubharmonic on $\mathbb{C}^n$. Then it is constant; since it is negative. This contradicts the fact that the domain D is nondegenerate.

(2-i) We may assume that ∇h is not identically zero on W. Thus, h is a defining function of W. Let for example $\frac{\partial h}{\partial z_1}(p) \neq 0$ for some point $p \in W$. Applying the maximum principle to W, then there exists an open dense subset O_W of E_W such that for any $q \in O_W$, $\frac{\partial h}{\partial z_1}(q) \neq 0$. For a fixed $q \in O_W$, there exists a neighborhood U in $\mathbb{C}^{n+1}$ of q such that $\frac{\partial h}{\partial z_1}$ vanishes nowhere on U. Then $\tilde{W} = \{z \in U : h(z) = 0\}$ is a polynomial submanifold of U. Since W extends across the boundary of D as a variety, a useful consequence of this fact is that $\tilde{W}$ has dimension $2n-1$. Otherwise, the Hausdorff dimension of $\tilde{W}$ will be less or equal to $2n-2$. Then $\tilde{W} \backslash \tilde{W} \cap \partial D$ will be connected (see [**12**, p. 347]). This implies that $\tilde{W}$ cannot be separated by ∂D and contradicts (i).

(2-ii) Since f is algebraic, all its components f_j are also algebraic. Then there exist $n+1$ polynomial equations $P_j(z,w)=0$ satisfied by $w_j = f_j(z)$. Let be

$$P_j(z, f_j(z)) = a_j^{m_j}(z) f_j(z)^{m_j} + \cdots + a_j^1(z) f_j(z) + a_j^0(z),$$

where $m_j \in \mathbb{N}$ and a_j^k are holomorphic polynomials for all $k \in \{0,\dots,m_j\}$ and for all $j \in \{1,\dots,n+1\}$. We may assume that for all j, $a_j^{m_j} \not\equiv 0$ on W.

Since $\partial D = S_h \cup \hat{S}$, for all $p \in \hat{S}$ there exists $j \in \{1,\dots,n+1\}$ such that $a_j^{m_j}(p) = 0$. Then the polynomial function $a = \prod_{1\le j\le n+1} a_j^{m_j}$ vanishes identically on $\hat{S}$. Now, we prove that $S^h \cap O_W$ is a dense subset in O_W. Suppose by contradiction that $\hat{S}\cap O_W$ has an interior point. The uniqueness theorem implies that $a \equiv 0$ on $\mathbb{C}^{n+1}$. This implies that $a_j^{m_j} \equiv 0$ for a certain $j \in \{1,\dots,n+1\}$: a contradiction. This completes the proof of the lemma. □

The Levi determinant of D is defined by: $\Lambda_r\colon \mathbb{C}^{n+1} \to \mathbb{R}$ via

$$-\det \begin{bmatrix} 0 & r_{\overline{z}_j} \\ r_{z_j} & r_{z_j\overline{z}_j} \end{bmatrix}.$$

The set of weakly pseudoconvex points in ∂D is

$$\omega(\partial D) = \{(z_0, z) \in \mathbb{C}^{n+1} : 2\,\mathrm{Re}(z_0) = -P(z) \text{ and } \Lambda_r(z_0,z) = 0\}.$$

Since D is a rigid polynomial domain, $\Lambda_r(z_0,z)$ depends only on z. We write $\Lambda_r(z_0,z)$ as $\Lambda_r(z_0,z) = L(z) = L_1^{\alpha_1}\dots L_s^{\alpha_s}(z)$, where the L_j denote the irreducible components of the polynomial L and $\alpha_j \in \mathbb{N}$ for $j = 1,\dots,s$.

If p is a boundary point of D, we define the member $\tau(p)$, the vanishing order of Λ_r, to be the smallest nonnegative integer m such that there is a tangential differential operator T of order m on ∂D such that $T\Lambda_r(p) \neq 0$. It can easily be checked that $\tau(p)$ is independent of the choice of the defining function r. Note that the set $\{p \in \partial D : \tau(p) = 0\}$ is the set of strongly pseudoconvex boundary points. The function τ is uppersemicontinuous. In our case, it is bounded by the degree of the polynomial P.

We need the following important statement.

Lemma 2. *Let $f\colon D \to \Omega$ be a proper holomorphic mapping as in Theorem 1. Then for all $p \in S_h$, $\tau(p) \ge \tau(f(p))$ and the inequality holds if and only if f is branched at p.*

Proof: Let $p \in S_h$. Then f extends holomorphically to a neighborhood of p. By the Hopf lemma, $\nabla(\rho \circ f)(p) \neq 0$. Then $\rho \circ f$ is a local defining function of D in a neighborhood of p, and by the chain rule we have: $\Lambda_{\rho\circ f}(p) = |J_f(p)|^2 \Lambda_\rho(f(p))$. Hence, we are able to deduce the lemma (see [**10**]). $\square$

Remark 1. Note that the lemma above still remains true, if the domains are not pseudoconvex. The proof is as in [**16**]. It uses some results of Baouendi-Rothschild [**1**] and Baouendi-Jacobwitz-Treves [**3**] to show that the transversal component f_0 of f satisfies $\frac{\partial f_0}{\partial z_0}(p) \neq 0$ for all $p \in S_h$.

Proposition 1. *The closure $\overline{V}_f$ does not intersect the set $\partial D \backslash \omega(\partial D)$ of strongly pseudoconvex points in ∂D.*

Proof: As in the proof of Lemma 2, we have:

$$\Lambda_{\rho\circ f}(z) = |J_f(z)|^2 \Lambda_\rho(f(z)), \quad \forall\, z \in S_h$$

so $O_W \subset \omega(\partial D)$, which implies that $E_W \subset \omega(\partial D)$. $\square$

3. Stratification of the weakly pseudoconvex set

Here, we give a real analytic stratification of the weakly pseudoconvex set. For bounded pseudoconvex domains with real analytic boundary, Bedford [**6**] obtained a similar stratification.

Lemma 3. *There exists an algebraic stratification of $\omega(\partial D)$ as follows:*

$$\omega(\partial D) = \{(z_0, z) \in \partial D : z \in A_1 \cup A_2 \cup A_3 \cup A_4\}$$

with the following properties.

(a) *A_4 is an algebraic set of dimension $\leq 2n-3$.*
(b) *A_1, A_2 and A_3 are either empty or algebraic manifolds; A_2 and A_3 have dimension $2n-2$ and A_1 has dimension $2n-1$.*
(c) *A_2 and A_3 are CR manifolds with*

$$\dim_{\mathbb{C}} HA_2 = n-1$$

and

$$\dim_{\mathbb{C}} HA_3 = n-2.$$

(d) *τ is constant on every component of $\{(z_0, z) \in \partial D : z \in A_1\}$.*

Proof: Let $A = \{z \in \mathbb{C}^n : L(z) = 0\}$ and let $\hat{A}_1$ be the union of all components of A with dimension $2n-1$ (if there are any). We consider $A_1 = \mathrm{Reg}(\hat{A}_1) = \cup_k \{L_k = 0 \text{ and } L_j \neq 0 \text{ for } j \neq k\}$.

Next we let $\hat{A}_2$ be the union of all $2n-2$-dimensional components of $A \backslash A_1$. We see that we may write

$$\operatorname{Reg}(\hat{A}_2) = A_2 \cup A_3 \cup \hat{A}_3$$

where A_2 and A_3 are an open subsets of $\hat{A}_2$ with

$$\begin{aligned} \dim_{\mathbb{C}} HA_2 &= n-1 \\ \dim_{\mathbb{C}} HA_3 &= n-2 \end{aligned}$$

and $\dim_{\mathbb{R}} \hat{A}_3 \leq 2n-3$. Now, let

$$A_4 = A \backslash (A_1 \cup A_2 \cup A_3)$$

then, we have the desired stratification.

To show (d), we consider the complex tangential derivative along the boundary of D, i.e.,

$$T_j = \frac{\partial}{\partial z_j} - \frac{1}{2}\frac{\partial P}{\partial z_j}\frac{\partial}{\partial z_0}, \quad 1 \leq j \leq n.$$

For example, we prove that $\tau \equiv \alpha_1$ on $C_1 = \partial D \cap \{L_1 = 0 \text{ and } L_k \neq 0,\ k \neq 1\}$. Let $(z_0, z) \in C_1$, we have

$$\begin{aligned} T_j^m \Lambda_r(z_0, z) = T_j^m L(z) &= \frac{\partial^m L}{\partial z_j^m}(z) \\ &= \alpha_1 \ldots (\alpha_1 - m + 1)\left(\frac{\partial L}{\partial z_j}\right)^m L_1^{\alpha_1 - m}.L_2^{\alpha_2} \ldots L_s^{\alpha_s}(z). \end{aligned}$$

Since L_1 is irreducible, $D(L_1)(z) \neq 0$. Then there exists j such that $T_j^m L(z) = 0$ for all $m < \alpha_1$ and $T_j^{\alpha_1} L(z) \neq 0$. This finishes the proof of the lemma. □

Proof of Theorem 1: The analytic set A_2 contains finitely many components which we will denote by $B_1, B_2, \ldots, B_N$. Since $\dim_{\mathbb{R}} B_j = \dim_{\mathbb{R}} HB_j$, then for each j, B_j is an $n-1$-dimensional complex manifold.

We denote by $\Gamma_j = \{(z_0, z) \in \partial D : z \in A_j\}$ for $j = 1, \ldots, 4$. By considering dimension and CR dimension, we see that $\Gamma_3 \cap O_W$ and $\Gamma_4 \cap O_W$ are nowhere dense in O_W.

Next, we prove that $\Gamma_1 \cap O_W$ cannot contain an open subset of O_W. By contradiction, let suppose $p \in O_W \subset \Gamma_1$. We may choose a sequence $\{q_k\}_k \subset \Gamma_1 \cap \{J_f \neq 0\}$ such that $q_k \to p$. The mapping f is a local diffeomorphism in a neighborhood of all points q_k and the function τ is constant on Γ_1. Then, we have for all k

$$\tau(p) = \tau(q_k) = \tau(f(q_k)). \tag{1}$$

On the other hand, by Lemma 2,

$$\tau(p) > \tau(f(p)). \tag{2}$$

Since τ is uppersemicontinuous, then (1) and (2) together give a contradiction. We mention that the same argument has appeared in [**6**]. We conclude that $\Gamma_2 \cap O_W$ contains an open subset of Γ_2. Thus it contains an open subset of $\{(z_0, z) \in \partial D : z \in B_j\}$ for some j. For $k = 1, \ldots, N$, let $\hat{B}_j$ be the complex variety in $\mathbb{C}^n$ such that $\operatorname{Reg} \hat{B}_k = B_k$. Applying the maximum principle, we conclude that $W \subset \{(z_0, z) \in D : z \in \hat{B}_j\}$, and by irreducibility, $W = \{(z_0, z) \in D : z \in \hat{B}_j\}$. This completes the proof of Theorem 1. □

Remark 2. (i) Using the same argument of Bedford [**6**] (appeared also in [**16**]), we can prove that the branching multiplicity of the mapping f is bounded by a constant independent of f.

(ii) For a holomorphic function H between algebraic hypersurface M and M' (M is essentially finite at p_0), Baouendi-Rothschlid [**2**] showed that the multiplicity of its components is bounded by a constant depending only on M and M' and the points p_0 and $H(p_0)$.

4. Proper self-mappings

Here, we give the proof of Corollary 1. Since D is simply connected, it suffices to prove that V_f is empty. The variety V_f has a finite number of connected components independent of the mapping f, then there exists an integer k such that $V_{f^k} = V_{f^{k+1}}$. We may assume $k = 1$, that is $V_f = V_f^2$. Since $V_{f^2} = V_f \cup f^{-1}(V_f)$, it follows that $V_f \subseteq f(V_f)$, where $f(V_f)$ is a complex analytic variety of D by a theorem of Remmert. Hence, we have $V_f = f(V_f)$ because V_f has finitely many components. Assume that V_f is not empty. According to Lemma 1, there exists a boundary point $p \in \overline{V}_f \cap \partial D$, such that f extends holomorphically in a neighborhood of p. Note that for all k $f^k(p) \in \overline{V}_f$, since $V_f = f(V_f)$ as shown above. The sequence of numbers $\tau(f^k(p))$ is strictly decreasing and $\tau(p)$ is a finite integer, then there exists an integer k_0 such that $\tau(f^{k_0}(p)) = 0$, which implies that $f^{k_0}(p)$ is a strongly pseudoconvex boundary point, contradicting the fact that $f^{k_0}(p) \in \overline{V}_f \cap \partial D$. This proves that $V_f = \emptyset$ and completes the proof of Corollary 1 .

We would like to thank the referee for his useful remarks on this material.

References

[1] M. S. BAOUENDI AND L. P. ROTHSCHILD, Geometric properties of mappings between hypersurfaces in complex space, *J. Differential Geom.* **31(2)** (1990), 473–499.

[2] M. S. BAOUENDI AND L. P. ROTHSCHILD, Mappings of real algebraic hypersurfaces, *J. Amer. Math. Soc.* **8(4)** (1995), 997–1015.

[3] M. S. BAOUENDI, H. JACOBOWITZ AND F. TRÈVES, On the analyticity of CR mappings, *Ann. of Math. (2)* **122(2)** (1985), 365–400.

[4] E. BARLETTA AND E. BEDFORD, Existence of proper mappings from domains in $\mathbb{C}^2$, *Indiana Univ. Math. J.* **39(2)** (1990), 315–338.

[5] E. BEDFORD, Proper holomorphic mappings from strongly pseudoconvex domains, *Duke Math. J.* **49(2)** (1982), 477–484.

[6] E. BEDFORD, Proper holomorphic mappings from domains with real analytic boundary, *Amer. J. Math.* **106(3)** (1984), 745–760.

[7] E. BEDFORD AND S. I. PINCHUK, Convex domains with noncompact groups of automorphisms, *Mat. Sb.* **185(5)** (1994), 3–26.

[8] E. BEDFORD AND S. BELL, Proper self-maps of weakly pseudoconvex domains, *Math. Ann.* **261(1)** (1982), 47–79.

[9] S. BELL, Local boundary behavior of proper holomorphic mappings, in *"Complex analysis of several variables"* (Madison, Wis., 1982), Proc. Sympos. Pure Math. **41**, Amer. Math. Soc., Providence, R.I., 1984, pp. 1—7.

[10] D. BURNS, JR. AND S. SHNIDER, Geometry of hypersurfaces and mapping theorems in $\mathbb{C}^n$, *Comment. Math. Helv.* **54(2)** (1979), 199–217.

[11] A. CHAOUECH, Auto-applications holomorphes propres des domaines polynomiaux rigides de $\mathbb{C}^2$, *Publ. Mat.* **40(1)** (1996), 41–66.

[12] E. M. CHIRKA, *"Complex analytic sets"*, Mathematics and its Applications (Soviet Series) **46**, Kluwer Academic Publishers Group, Dordrecht, 1989.

[13] B. COUPET, Y. PAN AND A. SUKHOV, On proper holomorphic mappings from domains with **T**-action, *Nagoya Math. J.* **154** (1999), 57–72.

[14] B. COUPET AND S. PINCHUK, *Holomorphic equivalence problem for weighted homogeneous rigid domains in* $\mathbb{C}^{n+1}$, in collection of papers dedicated to B. V. Shabat, Nauka, Moscow (1997), 111–126.

[15] F. FORSTNERIČ, Proper holomorphic maps from balls, *Duke Math. J.* **53(2)** (1986), 427–441.

[16] X. HUANG AND Y. PAN, Proper holomorphic mappings between real analytic domains in $\mathbb{C}^n$, *Duke Math. J.* **82(2)** (1996), 437–446.

[17] S. I. PINCHUK, Holomorphic inequivalence of certain classes of domains in $\mathbb{C}^n$, *Mat. Sb. (N.S.)* **111(153)(1)** (1980), 67–94, 159.
[18] W. RUDIN, Proper holomorphic maps and finite reflection groups, *Indiana Univ. Math. J.* **31(5)** (1982), 701–720.
[19] S. M. WEBSTER, On the mapping problem for algebraic real hypersurfaces, *Invent. Math.* **43(1)** (1977), 53–68.

C.M.I.
39 rue Joliot-Curie
13453 Marseille Cedex 13
France
E-mail address: coupet@protis.univ-mrs.fr
E-mail address: ourimi@protis.univ-mrs.fr

Primera versió rebuda el 6 d'abril de 1999,
darrera versió rebuda el 30 d'octubre de 2000.