Publ. Mat. **47** (2003), 261–309

HAPPY FRACTALS AND SOME ASPECTS OF ANALYSIS ON METRIC SPACES

Stephen Semmes*

Dedicated to Leon Ehrenpreis and Mitchell Taibleson

Abstract

There has been a lot of interest and activity along the general lines of "analysis on metric spaces" recently, as in [**2**], [**3**], [**26**], [**40**], [**41**], [**46**], [**48**], [**49**], [**51**], [**82**], [**83**], [**89**], for instance. Of course this is closely related to and involves ideas concerning "spaces of homogeneous type", as in [**18**], [**19**], [**66**], [**67**], [**92**], as well as sub-Riemannian spaces, e.g., [**8**], [**9**], [**34**], [**47**], [**52**], [**53**], [**54**], [**55**], [**68**], [**70**], [**72**], [**73**], [**84**], [**86**], [**88**]. In the present survey we try to give an introduction to some themes in this general area, with selections related to several points of view. Let us also mention [**39**], [**93**], [**97**], [**98**], [**99**] for topics dealing with nonstandard analysis, where one might think of a continuous metric space as something like a nonstandard graph.

Contents

1. Graphs 263
2. Finitely-generated groups 265
3. Happy fractals 266
4. Lipschitz retracts 269
5. The Sierpinski gasket and carpet 274
6. Heisenberg groups 275

2000 *Mathematics Subject Classification.* 42-02.
Key words. Graphs, happy fractals, Lipschitz classes.

*This survey was prepared partially in connection with the trimester "Heat kernels, random walks, and analysis on manifolds and graphs" at the Centre Émile Borel, Institut Henri Poincaré, in the Spring of 2002. This trimester was organized by P. Auscher, G. Besson, T. Coulhon, and A. Grigor'yan, and the author was fortunate to be a participant. The proceedings will be published in the Contemporary Mathematics series of the American Mathematical Society, and a report on the trimester can be found in [**82**]. Another survey on related themes is [**83**]. The author is grateful to an unnamed reader for many helpful comments and suggestions.

7. Some happy fractals from Helsinki 279
8. More on Lipschitz functions 280
9. Lipschitz functions of order α 283
10. Some functions on the real line 284
11. Sums on general metric spaces 288
12. The Zygmund class on $\mathbf{R}$ 289
13. Approximation operators, 1 291
14. Approximation operators, 2 293
15. A kind of Calderón-Zygmund decomposition related to Lipschitz functions 295
16. A brief overview of "atoms" 296
References 302

As usual, to say that $(M, d(x, y))$ is a metric space means that M is a nonempty set and that $d(x, y)$ is a nonnegative real-valued function on $M \times M$ such that $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$ for all $x, y \in M$, and

$$d(x, z) \leq d(x, y) + d(y, z) \tag{0.1}$$

for all $x, y, z \in M$ (the triangle inequality). Here we shall make the standing assumption that

$$M \text{ has at least 2 elements,} \tag{0.2}$$

to avoid degeneracies.

If E is a nonempty subset of M, then $\operatorname{diam} E$ denotes the *diameter* of E, defined by

$$\operatorname{diam} E = \sup\{d(u, v) : u, v \in E\}. \tag{0.3}$$

Given x in M and a positive real number r, we let $B(x, r)$ and $\overline{B}(x, r)$ denote the open and closed balls in M with center x and radius r, so that

$$\begin{aligned} B(x, r) &= \{y \in M : d(x, y) < r\}, \\ \overline{B}(x, r) &= \{y \in M : d(x, y) \leq r\}. \end{aligned} \tag{0.4}$$

Sometimes there might be another metric space $(N, \rho(u, v))$ in play, and we may introduce a subscript as in $B_N(w, s)$ to indicate in which metric space the ball is defined.

Of course the n-dimensional Euclidean space $\mathbf{R}^n$ with the standard metric $|x - y|$ is a basic example of a metric space, which is always good to keep in mind. Metric spaces associated to connected graphs will be discussed in Section 1, and the special case of Cayley graphs from finitely-generated groups will be reviewed in Section 2. In Section 3 we consider some notions that apply to any metric space, concerning rectifiable paths in particular. Sections 4–7 deal with related notions and examples. The remaining sections deal with various general aspects of analysis on metric spaces.

1. Graphs

Suppose that we have a graph consisting of a nonempty set V of vertices and a set E of edges. An element of E can be described by an unordered pair of distinct elements of V; we do not wish to consider edges which form loops by themselves, or multiple edges between the same pair of vertices. Two vertices connected by an edge are said to be *adjacent*.

Let us assume that our graph is *connected*, which is to say that every pair of vertices can be connected by a finite path. The length of a path is defined to be the number of edges that the path traverses. Thus the length of a path is a nonnegative integer, which is 0 in the case of a path that consists of a single vertex and traverses no edges.

We define a metric $d(v, w)$ on V by taking $d(v, w)$ to be the length of the shortest path between v and w. It is easy to see that $(V, d(v, w))$ is indeed then a metric space.

Let us also assume that the graph is *locally finite*, which is to say that there are only finitely many vertices adjacent to a given vertex. For each p in V and each positive integer m one can show that there are only finitely many vertices whose distance to p is at most m.

In fact, let us assume that there is a nonnegative integer k such that for every vertex v in V there are at most k vertices w in V which are adjacent to v. If $k = 0$ then V contains only one vertex and there are no edges, and if $k = 1$ then V has either one or two vertices, with no edges if there is only one vertex and exactly one edge when there are two vertices. For simplicity let us assume that $k \geq 2$.

If p is an element of V and m is a nonnegative integer, then we define $A_m(p)$ to be the number of vertices v in V whose distance to p is exactly equal to m. Thus $A_0(p) = 1$, since p is the only vertex at distance 0 from itself, and $A_1(p) \leq k$, since $A_1(p)$ is the same as the number of vertices in V which are adjacent to p. For $m \geq 2$ we have that

$$A_m(p) \leq (k-1) A_{m-1}(p). \tag{1.1}$$

Indeed, suppose that v is an element of V whose distance to p is exactly equal to m. Then there is vertex w in V such that v is adjacent to w and the distance from w to p is exactly $m-1$. Since $m \geq 2$, there is also a vertex u in V such that w is adjacent to u and the distance from u to p is exactly $m-2$. The total number of vertices in V which are adjacent to w and which have distance to p equal to m is at most $k-1$, because there are at most k vertices which are adjacent to w at all, and u is adjacent to w and has distance to p equal to $m-2$. There are $A_{m-1}(p)$ vertices w whose distance to p is equal to $m-1$, and hence there are at most $(k-1)A_{m-1}(p)$ vertices whose distance to p is equal to m, as desired.

Thus $A_m(p)$ grows at most exponentially in m in general, and exponential growth is certainly possible, at least when $k \geq 3$. Of course there are many interesting situations where the growth is in fact bounded by a polynomial. In this survey we shall focus on situations with polynomial growth, and the *doubling condition* described in Section 3 gives a nice version of this which makes sense in any metric space.

Instead of looking at rates of growth in terms of $A_m(p)$, one also frequently considers the quantity

$$\sum_{j=0}^{m} A_j(p), \tag{1.2}$$

which is the same as the number of elements of V whose distance to p is at most equal to m.

As a basic example, fix a positive integer n, and consider the set $\mathbf{Z}^n$ of points in $\mathbf{R}^n$ with integer coordinates as a set of vertices. Two points v, w in $\mathbf{Z}^n$ can be defined to be adjaced if $v-w$ has $n-1$ coordinates equal to 0 and the remaining coordinate equal to ± 1. This is the same as saying that v, w are adjacent if and only if $|v-w| = 1$. In this case it is not difficult to determine the metric on $\mathbf{Z}^n$ coming from paths in the graph, namely

$$d(v,w) = \sum_{j=1}^{n} |v_j - w_j|, \tag{1.3}$$

where v_j, w_j denote the jth coordinates of v, w, respectively. This is often called the *taxicab metric*, and it satisfies the following comparison with the Euclidean distance:

$$|v-w| \leq d(v,w) \leq \sqrt{n}\,|v-w|. \tag{1.4}$$

The first inequality can be derived from the triangle inequality for the standard distance, since each step of size 1 in the graph metric is also a step of size 1 in the Euclidean distance. The second inequality is a consequence of the Cauchy-Schwarz inequality.

In this case the growth is polynomial, with the number of points at distance to a fixed point p less than or equal to r is on the order of r^n. Notice that this number does not depend on p, because of translation-invariance.

Concerning analysis and geometry on graphs and related matters, see [**7**], [**23**], [**77**], [**96**] and the article by Coulhon in [**2**], for instance.

2. Finitely-generated groups

A very interesting special case of graphs and their geometry comes from *Cayley graphs* of *finitely generated groups*. Let Γ be a group with a finite set F of generators. Thus every element of Γ can be expressed as a product of elements of F and their inverses, with the identity element viewed as an empty product of generators. For the Cayley graph of Γ we use Γ as the set of vertices, and define two elements γ_1, γ_2 of Γ to be adjacent if one of them can be written as the product of the other times an element of F, where the group operation is applied in that order. From this it follows that the graph is invariant under left-translations, which is to say that γ_1, γ_2 are adjacent if and only if $\alpha\,\gamma_1$, $\alpha\,\gamma_2$ for any α in Γ.

Every pair of elements of Γ can be joined by a path in the Cayley graph, because of the assumption that every element of Γ can be expressed as a product of generators and their inverses. If $d(\gamma_1, \gamma_2)$ denotes the distance function on Γ coming from the Cayley graph, then we have that

$$d(\alpha\,\gamma_1, \alpha\,\gamma_2) = d(\gamma_1, \gamma_2) \tag{2.1}$$

for all α, γ_1, and γ_2 in Γ, by left-invariance of the Cayley graph.

If γ is an element of Γ, then the number of elements of Γ which are adjacent to γ is at most twice the number of elements of F, by construction. As in the preceding section, this leads to a simple exponential bound on the growth of the Cayley graph of Γ. Exponential growth occurs for free groups with at least two generators, and more generally for nonelementary hyperbolic groups in the sense of Gromov, as in [**21**], [**22**], [**38**], [**42**], [**44**]. Hyperbolic groups have very interesting spaces at infinity associated to them which satisfy the doubling property described in the next section. In addition to the references already mentioned, see [**20**], [**45**], [**75**], [**76**] in this regard. Note that fundamental groups of compact Riemannian manifolds without boundary and with strictly negative sectional curvatures are nonelementary hyperbolic groups. Simply-connected symmetric spaces always have compact quotients by a well-known result of Borel [**14**], [**78**], and for symmetric spaces of noncompact type and rank 1 the sectional curvatures are strictly negative.

The graph associated to $\mathbf{Z}^n$ in the previous section is exactly its Cayley graph as group with the n standard generators, where each generator has one coordinate equal to 1 and the others equal to 0. This graph has polynomial growth, as we saw, and more generally it is a well-known result that the Cayley graph of a finitely-generated group has polynomial growth when the group is *virtually nilpotent*, which means that the group contains a nilpotent subgroup of finite index. A famous theorem of Gromov [**43**] states that the converse is true.

3. Happy fractals

Let us say that a metric space $(M, d(x, y))$ is a *happy fractal* if the following three conditions are satisfied. First, M is complete as a metric space. Second, there is a constant $C_1 > 0$ so that for each pair of points x, y in M there is a path in M connecting x to y with length at most $C_1\, d(x, y)$. Third, M satisfies the *doubling property* that there is a constant C_2 so that any ball B in M can be covered by a family of balls with half the radius of B and at most C_2 elements.

One might prefer the name *happy metric space*, since the metric space need not be fractal, as in the case of ordinary Euclidean spaces. There are plenty of examples which are more intricate and not fractal, such as domains or surfaces with cusps. There can be interesting fractal behavior at some kind of boundary, if not for the space itself.

Cantor sets and snowflake curves give examples of self-similar fractals which satisfy the doubling condition but are not happy fractals, because every curve of finite length in these spaces is constant. Some basic examples of happy fractals will be discussed in the next few sections.

It does not seem to be known whether every compact connected 4-dimensional topological manifold can be realized as a happy fractal, i.e., whether every compact Hausdorff topological space which is locally homeomorphic to the open unit ball in $\mathbf{R}^4$ has a topologically-equivalent metric in which it becomes a happy fractal. This is true for dimensions not equal to 4, since n-dimensional topological manifolds admit unique smooth structures when $n \leq 3$ and they admit unique Lipschitz structures when $n \geq 5$. See [**10**], [**28**], [**35**], [**71**], [**91**] concerning these topics.

In general dimensions there are plenty of questions about noncompact spaces. For instance, in this connection one might consider conditions of bounded local geometry, with the happy fractal aspect being concerned with larger scales. In dimension 4, let us recall a well-known result of

Quinn that every connected 4-dimensional topological manifold can be smoothed in the complement of a single point. Of course, near that point there can be a lot of complications, although there are also topological restrictions since that point is a topological manifold point.

To be more precise, a *path* in M which goes from a point x to a point y is a continuous mapping $p(t)$ defined on a closed interval $[a, b]$ in the real line and with values in M such that $p(a) = x$ and $p(b) = y$. If

$$a = t_0 < t_1 < t_2 < \cdots < t_m = b \tag{3.1}$$

is a partition of $[a, b]$, then we can associate to this partition the quantity

$$\sum_{j=1}^{m} d(p(t_j), p(t_{j-1})), \tag{3.2}$$

which is the approximation to the length of p corresponding to this partition. The *length* of the path is defined to be the supremum of (3.2) over all partitions of $[a, b]$. In general this can be infinite.

A standard observation is that the quantity (3.2) can only increase as points are added to the partition, because of the triangle inequality. Any two partitions admit a common refinement, for which the approximation to the length is then greater than or equal to the approximations to the length associated to the original refinements.

Suppose that the length of the path $p(t)$ is finite. Then the length of the restriction of p to any subinterval of $[a, b]$ is also finite, and is less than or equal to the length of the whole path. Let us define a function $L(u, v)$ for $u, v \in [a, b]$, $u \le v$, to be the length of the restriction of $p(t)$ to $[u, v]$. Of course a constant path has length 0, which includes the case where the domain has one element. Note that

$$d(p(u), p(v)) \le L(u, v) \tag{3.3}$$

for all $u, v \in [a, b]$ with $u \le v$. If

$$a \le u \le v \le w \le b, \tag{3.4}$$

then it is not hard to verify that

$$L(u, w) = L(u, v) + L(v, w), \tag{3.5}$$

using the monotonicity properties of the length, and the possibility of taking refinements of the partitions in particular.

Fix $t \in [a, b]$. If $t > a$, then

$$\lim_{s \to t-} L(s, t) = 0. \tag{3.6}$$

This is equivalent to saying that

$$\lim_{s \to t-} L(a, s) = L(a, t). \tag{3.7}$$

From the definition we know that $L(a, s)$ is monotone increasing in s, so that the limit on the left side exists and is less than or equal to the right side. To show that equality holds, one can choose a partition of $[a, t]$ so that the approximation to the length of $p(u)$ along this partition is close to $L(a, t)$, and then check that $L(a, s)$ is greater than or equal to this approximation minus a small number when s is sufficiently close to t. This employs the continuity of $p(u)$ at t, to move the last point in the partition from t to s without making more than a small change to the approximation to the length.

If $t < b$, then

$$\lim_{s \to t+} L(t, s) = 0. \tag{3.8}$$

This is equivalent to

$$\lim_{s \to t+} L(s, b) = L(t, b), \tag{3.9}$$

which can be verified in the same manner as before.

Set $\lambda = L(a, b)$, and consider the real-valued function $\sigma(t)$ defined on $[a, b]$ by

$$\sigma(t) = L(a, t). \tag{3.10}$$

Thus $\sigma(t)$ is monotone increasing (and not necessarily strictly increasing), $\sigma(0) = 0$, $\sigma(b) = \lambda$, and $\sigma(t)$ is continuous by the preceding remarks.

There is a mapping $\widetilde{p}\colon [0, \lambda] \to M$ such that

$$\widetilde{p}(\sigma(t)) = p(t) \tag{3.11}$$

for all $t \in [a, b]$. In other words, if $s, t \in [a, b]$, $s < t$, and $\sigma(s) = \sigma(t)$, then $L(s, t) = 0$, so that p is constant along $[s, t]$, and (3.11) leads to a single value for p at $\sigma(s) = \sigma(t)$. Moreover, (3.3) implies that

$$d(\widetilde{p}(r), \widetilde{p}(w)) \leq |r - w| \tag{3.12}$$

for all $r, w \in [0, \lambda]$.

On the other hand, if $q\colon [c, d] \to M$ is a path such that

$$d(q(s), q(t)) \leq k|s - t| \tag{3.13}$$

for some constant k and all $s, t \in [c, d]$, then it is easy to check that the length of q on $[c, d]$ is at most $k|c - d|$. One can trade between k and $|c - d|$ by rescaling in the domain.

Thus there is a path in M from x to y with length less than or equal to a constant A if and only if there is a mapping $q\colon [0,1] \to M$ such that $q(0) = x$, $q(1) = y$, and (3.13) holds for all $s, t \in [0,1]$ with $k \le A$. Assuming that there is a path in M from x to y with finite length and that closed and bounded subsets of M are compact, one can use the Arzela-Ascoli theorem to find such a mapping q with k as small as possible, and this minimal k is the same as the length of the shortest path in M from x to y.

A well-known result in basic analysis states that if $(M, d(x,y))$ is a complete metric space, then a closed subset K of M is compact if and only if K is totally bounded, which means that for every $\epsilon > 0$ there is a finite family of balls in M with radius ϵ whose union contains K. Thus, if $(M, d(x,y))$ is complete, then closed and bounded subsets of M are compact if and only if all balls in M are totally bounded. It is easy to verify that the latter holds when M satisfies the doubling property. In short, closed and bounded sets are compact in a happy fractal (or happy metric space).

4. Lipschitz retracts

Suppose that $(M, d(x,y))$ is a metric space, and that A and E are subsets of M, with $E \subseteq A$. A mapping $\phi\colon A \to E$ is said to be a *Lipschitz retract* of A onto E if

$$\phi(x) = x \quad \text{for all } x \in E \tag{4.1}$$

and ϕ is Lipschitz, so that there is a constant $k \ge 0$ such that

$$d(\phi(y), \phi(z)) \le k\, d(y,z) \tag{4.2}$$

for all $y, z \in A$. Note that if M is complete and E is a closed subset of M, then one can always take A to be closed, because any Lipschitz mapping from A into E can be extended to a Lipschitz mapping from the closure of A into E, and with the same Lipschitz constant k.

Let us say that a complete metric space $(N, \rho(u,v))$ is a *Lipschitz extension space* with constant $s \ge 1$ if for every separable metric space $(M, d(x,y))$ and every mapping f from a subset Z of M into N which is Lipschitz with constant L, so that

$$\rho(f(x), f(y)) \le L\, d(x,y) \tag{4.3}$$

for all $x, y \in Z$, there is an extension of f to a Lipschitz mapping from M into N with constant $s\,L$.

Remark 4.4. If $(M, d(x,y))$ is a separable metric space and E is a subset of M, and if $(E, d(x,y))$ satisfies the Lipschitz extension property with constant s, then there is a Lipschitz retraction from M onto E with constant s, simply by extending the identity mapping on E.

The requirement above that N be complete is not really needed, since it can be derived from the extension property. The restriction to metric spaces M which are *separable* —i.e., which contain a countable dense subset— is made because we shall only be concerned with spaces that satisfy this condition, and because it permits one to avoid such things as transfinite induction. Specifically, one can make the following observation.

Lemma 4.5. *Let $(N, \rho(u,v))$ be a complete metric space. A necessary and sufficient condition for N to satisfy the Lipschitz extension property with constant s is that it satisfy this property in the special case where the metric space $(M, d(x,y))$ and the subset Z of M have the feature that $M \backslash Z$ is at most countable.*

Indeed, given arbitrary $(M, d(x,y))$, Z, f, and L as in the definition of the Lipschitz extension property, one can first use separability of M to find a subset M_0 of M such that M_0 contains Z, $M_0 \backslash Z$ is at most countable, and M_0 is dense in M. Under the restricted version of the Lipschitz extension property mentioned in the lemma, one can extend f to a Lipschitz mapping from M_0 to N with Lipschitz constant $s\,L$. The completeness of N then permits this mapping to be extended to one from all of M into N, with Lipschitz constant $s\,L$ still.

I learned the next lemma from M. Gromov, as well as the way it can be used.

Lemma 4.6. *Let $(N, \rho(u,v))$ be a complete metric space. A necessary and sufficient condition for $(N, d(x,y))$ to satisfy the Lipschitz extension property with constant $s = 1$ is that it satisfy this property in the special case where the metric space $(M, d(x,y))$ and the subset Z of M have the feature that $M \backslash Z$ contains only one element.*

Indeed, if one can extend a Lipschitz mapping to a set with one extra element, without increasing the Lipschitz constant, then one can repeat this to get extensions to sets with arbitrary finite numbers of additional elements, or even countably many additional elements, without increasing the Lipschitz constant. The preceding lemma then applies to deal with the general case.

Lemma 4.7. *Let $(N, \rho(u,v))$ be a complete metric space. Suppose that for every collection*

$$\{B_i\}_{i\in I} = \{\overline{B}_N(u_i, r_i)\}_{i\in I} \tag{4.8}$$

of closed balls in N such that I is at most countable and

$$\rho(u_i, u_j) \le r_i + r_j \quad \text{for all } i, j \in I \tag{4.9}$$

we have that

$$\bigcap_{k\in I} B_k \neq \emptyset. \tag{4.10}$$

Then $(N, \rho(u,v))$ satisfies the Lipschitz extension property with $s = 1$.

Note that the completeness of N corresponds in fact to the special case of the condition in the lemma where $\{B_i\}_{i\in I}$ is a sequence of closed balls which is decreasing in terms of inclusion and whose radii are tending to 0.

To prove the lemma, it is enough to obtain one-point extensions, as in Lemma 4.6. Let $(M, d(x,y))$, Z, f, and L be given as in the definition of the Lipschitz extension property, with $M\backslash Z$ containing exactly one element w. For each $z \in Z$, consider the closed ball

$$B_z = \overline{B}_N(f(z), L\, d(w,z)) \tag{4.11}$$

in N. If $z_1, z_2 \in Z$, then

$$\rho(f(z_1), f(z_2)) \le L\, d(z_1, z_2) \le L\, d(w, z_1) + L\, d(w, z_2). \tag{4.12}$$

In other words, this family of balls satisfies the condition (4.9) in Lemma 4.7. Although Z may not be at most countable, one can use the separability of M to obtain that there is a dense subset I of Z which is at most countable. The hypothesis of the lemma then implies that

$$\bigcap_{z\in I} B_z \neq \emptyset. \tag{4.13}$$

Fix a point α in this intersection, and set $f(w) = \alpha$. We have that

$$\rho(f(w), f(z)) = \rho(\alpha, f(z)) \le L\, d(w,z) \tag{4.14}$$

for all $z \in I$, precisely because $\alpha \in B_z$ for all $z \in I$. By continuity, (4.14) holds for all $z \in Z$. Thus we have an extension of f to $M = Z \cup \{w\}$ which is Lipschitz with constant L, as desired. This proves the lemma.

Corollary 4.15. *The real line* $\mathbf{R}$ *with the standard metric* $|x - y|$ *satisfies the Lipschitz extension property with* $s = 1$.

Of course this is well-known and can be established by other means, as in Section 8, but one can check that the hypothesis of Lemma 4.7 holds in this case. To be more precise, the B_i's are closed and bounded intervals in this case, and the condition (4.9) implies that every pair of these intervals intersects. The special geometry of the real line implies that the intersection of all of the intervals is nonempty.

Part of the point of this kind of approach is that it can be applied to tree-like spaces. As a basic scenario, suppose that $(T, \sigma(p, q))$ is a metric space which consists of a finite number of pieces which we shall call *segments*, and which are individually isometrically equivalent to a closed and bounded interval in the real line. We assume that any two of these segments are either disjoint or that their intersection consists of a single point which is an endpoint of each of the two segments. We also ask that T be connected, and that the distance between any two elements of T is the length of the shortest path that connects them. One may as well restrict one's attention to paths which are piecewise linear, and the length of the paths is easy to determine using the fact that each segment is equivalent to a standard interval (of some length).

So far these conditions amount to saying that T is a finite graph, with the internal geodesic distance. Now let us also ask that T be a *tree*, in the sense that any simple closed path in T is trivial, i.e., consists only of a single point.

The effect of this is that if p and q are elements of T, then there is a special subset $S(p, q)$ of T which is isometrically-equivalent to a closed and bounded interval in the real line, with p and q corresponding to the endpoints of this interval. In practice, with a typical picture of a tree, it is very easy to draw the set $S(p, q)$ for any choice of p and q. This set gives the path of minimal length between p and q (through the isometric equivalence mentioned before), and it satisfies a stronger minimality property, namely, any path in T connecting p and q contains $S(p, q)$ in its image.

Lemma 4.16. *Under the conditions just described,* $(T, \sigma(p, q))$ *satisfies the hypotheses of Lemma 4.7. As a result,* $(T, \sigma(p, q))$ *enjoys the Lipschitz extension property with* $s = 1$.

Clearly T is complete, and in fact compact. Now suppose that $\{B_i\}_{i \in I}$ is a family of closed balls in T. The condition (4.9) implies in this setting (and in any geodesic metric space) that any two of the B_i's intersect. (Note that the converse always holds.)

Let us call a subset C of T *convex* if $p, q \in C$ implies that $S(p, q) \subseteq C$. Of course convexity in this sense implies connectedness, and in fact connectedness implies convexity because of the assumption that T is a tree. That is, $S(p, q)$ is contained in any connected set that contains p and q. Of course connected subsets of T have a simple structure, since a connected subset of an interval in the real line is also an interval.

Because the distance on T is defined in terms of lengths of paths, open and closed balls in T are connected, and hence convex. The intersection of two convex sets is also convex, by definition.

Suppose that C_1, C_2, and C_3 are convex subsets of T such that $C_1 \cap C_2$, $C_1 \cap C_3$, and $C_2 \cap C_3$ are all nonempty. Let us check that $C_1 \cap C_2 \cap C_3$ is nonempty as well. Let p_{12}, p_{13}, and p_{23} be elements of $C_1 \cap C_2$, $C_1 \cap C_3$, and $C_2 \cap C_3$, respectively. Observe that

$$S(p_{12}, p_{13}) \subseteq S(p_{12}, p_{23}) \cup S(p_{23}, p_{13}), \tag{4.17}$$

since the right side defines a connected subset of T that contains p_{12} and p_{13}. As before, there is an isometric equivalence between $S(p_{12}, p_{13})$ and a closed and bounded interval I in the real line, where p_{12}, p_{13} correspond to the endpoints of I. On the other hand, $S(p_{12}, p_{13}) \cap S(p_{12}, p_{23})$ and $S(p_{12}, p_{13}) \cap S(p_{23}, p_{13})$ are closed convex subsets of $S(p_{12}, p_{13})$, and hence correspond to closed subintervals J, K of I. From (4.17) we obtain that $I \subseteq J \cup K$, which implies that $J \cap K \neq \emptyset$, since J and K are closed. Any element of $J \cap K$ corresponds to a point in $S(p_{12}, p_{13})$ that also lies in $S(p_{12}, p_{23})$ and $S(p_{23}, p_{13})$. Because C_1, C_2, and C_3 are convex, $S(p_{12}, p_{13}) \subseteq C_1$, $S(p_{12}, p_{23}) \subseteq C_2$, and $S(p_{23}, p_{13}) \subseteq C_3$. In other words, we get an element of the intersection of C_1, C_2, and C_3, as desired.

Because the intersection of convex sets is convex, one can iterate this result to obtain that if $C_1, C_2, \ldots, C_\ell$ are convex sets in T such that the intersection of any two of them is nonempty, then $\bigcap_{i=1}^{\ell} C_i \neq \emptyset$. For closed convex sets, which are then compact since T is compact, one can get the same result for an infinite family of convex sets. This uses the well-known general result that the intersection of a family of compact sets is nonempty if the intersection of every finite subfamily is nonempty.

This shows that $(T, \sigma(p, q))$ satisfies the hypotheses of Lemma 4.7, since closed balls are closed convex sets. This completes the proof of Lemma 4.16.

Of course there are analogous results for more complicated trees or tree-like sets. Let us note that one might have the set sitting inside of a Euclidean space, but where the internal geodesic metric is not quite the same as the restriction of the ambient Euclidean metric. If the two are comparable, in the sense that each is bounded by a constant multiple of the other, then the Lipschitz extension property for one metric follows from the same property for the other metric, with a modestly different constant.

5. The Sierpinski gasket and carpet

The *Sierpinski gasket* is the compact set in $\mathbf{R}^2$ which is constructed as follows. One starts with the unit equilateral triangle, with bottom left vertex at the origin and bottom side along the x_1-axis. By "triangle" we mean the closed set which includes both the familiar polygonal curve and its interior. This triangle can be subdivided into four parts each with sidelength equal to half of the original. The vertices of the four new triangles are vertices of the original triangle or midpoints of its sides. One removes the interior of the middle triangle, and keeps the other three triangles in the first stage. One then repeats the process for each of those triangles, and so on. The Sierpinski gasket is the compact set without interior which occurs in the limit, and which is the intersection of the sets which are finite unions of triangles which occur at the finite stages of the construction.

Similarly, the *Sierpinski carpet* is the compact set in $\mathbf{R}^2$ defined in the following manner. One starts with the unit square, where "square" also means the familiar polygonal curve together with its interior. One decomposes the unit square into nine smaller squares, each with sidelength equal to one-third that of the original. One removes the interior of the middle square, and keeps the remaining eight squares for the first stage of the construction. One then repeats the construction for each of the smaller squares, and so on. The Sierpinski carpet is the compact set without interior which occurs in the limit and is the intersection of the sets which are the finite unions of squares from the finite stages of the construction.

The Sierpinski gasket and carpet provide well-known basic examples of happy fractals. The main point is that if x, y are two elements of one of these sets, then x and y can be connected by a curve in the set whose length is bounded by a constant times $|x - y|$. This is not too difficult to show, using the sides of the triangles and squares to move around in the sets.

For neither of these sets is there a continuous retraction (let alone a Lipschitz retraction) from $\mathbf{R}^2$ onto the set. There is not even a continuous retraction from a neighborhood of the set onto the set. This is because in both cases there are arbitrarily small topological loops, given by boundaries of triangles or squares, which cannot be contracted to a point in the set, but can easily be contracted to a point in $\mathbf{R}^2$, within the particular triangle or square. If there were a retraction whose domain included such a triangle or square, then the contraction of the loop could be pushed back into the Sierpinski gasket or carpet, where in fact it cannot exist.

However, one can retract the complement of a triangle or square onto its boundary. If one removes a hole from each open triangle or square in the complement of the Sierpinski gasket or carpet, then one can define a continuous retraction on the fatter sets that one obtains, i.e., as the complement of the union of the holes. The domain of the retraction is reasonably fat, but it still does not contain a neighborhood of the Sierpinski gasket or carpet. If one is careful to choose the holes so that they always contain a disk of radius which is greater than or equal to a fixed positive constant times the diameter of the corresponding triangle or square, then one can get a Lipschitz retraction.

There are also nice Lipschitz retractions from the Sierpinski gasket or carpet onto subsets of itself. For instance, one can start by pushing parts of the gasket or carpet in individual triangles or squares to all or parts of the boundaries of these triangles or squares. One can often move what remains into the rest of the gasket or carpet that is not being moved.

6. Heisenberg groups

Let n be a positive integer. Define H_n first as a set by taking $\mathbf{C}^n \times \mathbf{R}$, where $\mathbf{C}$ denotes the complex numbers. The group law is given by

$$(w, s) \circ (z, t) = \left(w + z, s + t + 2 \operatorname{Im} \sum_{j=1}^{n} w_j \overline{z_j} \right), \tag{6.1}$$

where $\operatorname{Im} a$ denotes the imaginary part of a complex number a, and w_j, z_j denote the jth components of $w, z \in \mathbf{C}^n$.

It is not difficult to verify that this does indeed define a group structure on H_n. In this regard, notice that the inverse of (w, s) in H_n is given by

$$(w, s)^{-1} = (-w, -s). \tag{6.2}$$

For each positive real number r, define the "dilation" δ_r on H_n by

$$\delta_r(w, s) = (r\, w, r^2\, s). \tag{6.3}$$

One can check that these dilations define group automorphisms of H_n, i.e.,

$$\delta_r\big((w,s)\circ(z,t)\big) = \delta_r(w,s)\circ\delta_r(z,t). \tag{6.4}$$

Also, for $r_1, r_2 > 0$ we have that

$$\delta_{r_1}(\delta_{r_2}(w,s)) = \delta_{r_1\,r_2}(w,s). \tag{6.5}$$

Let us note that the group law and the dilations are compatible with the standard Euclidean topology on H_n, i.e., they define continuous mappings.

Let us call a nonnegative real-valued function $N(\cdot)$ on H_n a *norm* if it satisfies the following conditions: (a) N is continuous; (b) N takes the value 0 at the origin and is strictly positive at other points in H_n; (c) $N\big((w,s)^{-1}\big) = N(w,s)$ for all $(w,s) \in H_n$; (d) $N(\delta_r(w,s)) = r\,N(w,s)$ for all $r > 0$ and $(w,s) \in H_n$; and (e) N satisfies the triangle inequality with respect to the group structure on H_n, which is to say that

$$N\big((w,s)\circ(z,t)\big) \le N(w,s) + N(z,t) \tag{6.6}$$

for all $(w,s), (z,t) \in H_n$.

In many situations it is sufficient to work with a weaker notion, in which (6.6) is replaced by the "quasitriangle inequality" which says that there is a positive constant $C > 0$ so that the left side is less than or equal to C times the right side. It is very easy to write down explicit formulae for "quasinorms" which satisfy conditions (a)–(d) and this weaker version of (e), and in fact this weaker version of (e) is implied by the other conditions. Also, any two quasinorms are comparable, which is to say that each is bounded by a constant multiple of the other. Indeed, because of the homogeneity condition (d), this statement can be reduced to one on a compact set not containing the origin, where it follows from the continuity and positivity of the quasinorms.

Actual norms can be written down explicitly through simple but carefully-chosen formulae, as in [**52**]. Another aspect of this will be mentioned in a moment, but first let us define the distance function associated to a norm or quasinorm.

If N is a norm or quasinorm on H_n, then we can define an associated distance function $d_N(\cdot,\cdot)$ on H_n by

$$d_N\big((w,s),(z,t)\big) = N\big((w,s)^{-1}\circ(z,t)\big). \tag{6.7}$$

By construction, this distance function is automatically invariant under left translations on H_n, i.e.,

$$d_N\big((y,u)\circ(w,s),(y,u)\circ(z,t)\big) = d_N\big((w,s),(z,t)\big) \tag{6.8}$$

for all $(y,u),(w,s),(z,t) \in H_n$, simply because

$$\big((y,u)\circ(w,s)\big)^{-1}\circ\big((y,u)\circ(z,t)\big) = (w,s)^{-1}\circ(z,t). \tag{6.9}$$

We also have that $d(\cdot,\cdot)$ is nonnegative, equal to 0 when the two points in H_n are the same, and is positive otherwise, because of the corresponding properties of N. Similarly,

$$d_N\big((w,s),(z,t)\big) = d_N\big((z,t),(w,s)\big), \tag{6.10}$$

because of the symmetry property $N\big((w,s)^{-1}\big) = N(w,s)$ of N, and

$$d_N\big(\delta_r(w,s),\delta_r(z,t)\big) = r\,d_N\big((w,s),(z,t)\big) \tag{6.11}$$

by the homogeneity property of N.

If N is a norm, then (6.6) implies that d_N satisfies the usual triangle inequality for metrics. If N is a quasinorm, then d_N satisfies the weaker version for quasimetrics, in which the right side is multiplied by a fixed positive constant. Just as different quasinorms on H_n are comparable, the corresponding distance functions are too, i.e., they are each bounded by a constant times the other.

A basic and remarkable feature of the Heisenberg groups with this geometry is that they are happy fractals. In fact one can define the distance between two points in terms of the infimum of the lengths of certain paths between the two points, where the family of paths and the notion of length enjoy left-invariance and homogeneity properties which lead to the same kind of properties for the distance function as above. This kind of distance function can also be shown to be compatible with the Euclidean topology on H_n. These features imply that this distance function is of the form d_N for some N as above. The triangle inequality for the distance function is a consequence of its definition, and this leads to the triangle inequality for the corresponding N. A key subtlety in this approach is that there is a sufficiently-ample supply of curves used in the definition of the distance to connect arbitrary points in H_n, because the curves are required to satisfy nontrivial conditions on the directions of their tangent vectors.

Let us return to the setting of an arbitrary norm N on H_n. The triangle inequality can be rewritten as

$$\begin{aligned} N(w,s) &\le N(z,t) + d_N\big((w,s),(z,t)\big), \\ N(z,t) &\le N(w,s) + d_N\big((w,s),(z,t)\big) \end{aligned} \tag{6.12}$$

for all $(w,s),(z,t) \in H_n$. Thus

$$|N(w,s) - N(z,t)| \le d_N\big((w,s),(z,t)\big) \tag{6.13}$$

for all $(w,s),(z,t) \in H_n$.

For $(w,s) \neq 0$, define $\phi(w,s)$ by

$$\phi(w,s) = \delta_{N(w,s)^{-1}}(w,s). \tag{6.14}$$

Thus

$$N\big(\phi(w,s)\big) = 1 \tag{6.15}$$

by definition.

If (w,s), (z,t) are both nonzero elements of H_n, then

$$\begin{aligned} &d_N\big(\phi(w,s),\phi(z,t)\big) \\ &\qquad \le d_N\big(\phi(w,s),\delta_{N(w,s)^{-1}}(z,t)\big) + d_N\big(\delta_{N(w,s)^{-1}}(z,t),\phi(z,t)\big). \end{aligned} \tag{6.16}$$

The first term on the right can be rewritten as

$$d_N\big(\delta_{N(w,s)^{-1}}(w,s),\delta_{N(w,s)^{-1}}(z,t)\big) = N(w,s)^{-1}\, d_N\big((w,s),(z,t)\big), \tag{6.17}$$

which is reasonable and nice for our purposes. The second term on the right can be rewritten as

$$d_N\big(\delta_{N(z,t)\,N(w,s)^{-1}}(\phi(z,t)),\phi(z,t)\big). \tag{6.18}$$

Let us think of this as being of the form

$$d_N\big(\delta_r(y,u),(y,u)\big), \tag{6.19}$$

where r is a positive real number and $(y,u) \in H_n$ satisfies $N(y,u) = 1$. Of course this expression is equal to 0 when $r = 1$, and one can be interested in getting a bound for it in terms of $r - 1$.

Unfortunately one does not get a bound for (6.19) like $O(|r-1|)$ in general, but more like $O(\sqrt{|r-1|})$ for r reasonably close to 1. The bottom line is that the retraction ϕ onto the unit sphere for N is not Lipschitz, even in a small neighborhood of the sphere.

To look at it another way, although the dilation mapping δ_r is Lipschitz with constant r with respect to d_N on H_n, it does not have good Lipschitz properties as a function of r, except on a small set. This is in contrast to the case of Euclidean geometry, where dilation by r is uniformly Lipschitz as a function of r on bounded subsets.

A closely related point is that while there are curves of finite length joining the origin in H_n to arbitrary elements of H_n, the trajectories of the dilations do not have this property.

Certainly one can expect that it is more difficult to have Lipschitz retractions in the Heisenberg group than in Euclidean spaces, and this indicates that this is so even for relatively simple cases.

Another basic mapping to consider is

$$\psi(w, s) = \delta_{N(w,s)^{-2}}(w, s), \tag{6.20}$$

which takes H_n minus the origin to itself. This mapping is a reflection about the unit sphere for N, i.e., $\psi(w, s) = (w, s)$ when $N(w, s) = 1$, $N\big(\psi(w, s)\big) = N(w, s)^{-1}$, and $\psi(\psi(w, s)) = (w, s)$. Unlike the Euclidean case, there is once again trouble with the Lipschitz condition even on a small neighborhood of the unit sphere for N.

7. Some happy fractals from Helsinki

There are clearly numerous variations for the type of construction about to be reviewed. We shall focus on a simple family with a lot of self-similarity.

Let N be an odd integer greater than or equal to 5, and let Σ_0 denote the boundary of the unit cube in $\mathbf{R}^3$. Thus Σ_0 consists of 6 two-dimensional squares, each with sidelength 1.

In the first stage of the construction, we subdivide each of these 6 squares into N^2 squares with sidelength $1/N$. For each of the original 6 squares, we make a modification with the square of size $1/N$ in the middle. The "middle" makes sense because N is odd. Specifically, we remove the middle squares, and replace each one with the union of the other 5 squares in the boundary of the cube with one face the middle square in question and which lies outside the unit cube with which we started. The surface that results from Σ_0 by making these modifications is denoted Σ_1.

This procedure can also be described as follows. Let R_0 denote the unit cube, so that $\Sigma_0 = \partial R_0$. Now define R_1 to be the union of R_0 and the 6 cubes with sidelength $1/N$ whose interiors are outside R_0 and which have a face which is a middle square of a face of R_0. The surface Σ_1 is the boundary of R_1.

Using the decomposition of the boundary described in the first step, we can think of Σ_1 as the union of a bunch of two-dimensional squares of sidelength $1/N$. Namely, there are $6 \cdot (N^2 - 1) + 6 \cdot 5$ such squares. For each of these squares, we apply the same procedure as before. That is, we divide each square into N^2 squares of sidelength $1/N$ times the sidelength of the squares that we have, so that the new squares have sidelength $1/N^2$ in this second step. For each of the squares from the first step, we make modifications only at the middle smaller squares just described, one middle small square for each square from the second step. Each of these middle small squares is removed and replaced with the union of 5 squares of the same sidelength which are in the boundary of the cube with interior outside R_1 and with one face being the small middle square in question. The result is a surface Σ_2 consisting of a bunch of squares of sidelength $1/N^2$. The condition $N \geq 5$ is helpful for keeping the modifications at different places from bumping into each other or getting too close to doing that.

One can also describe this in terms of adding a bunch of cubes of sidelength $1/N^2$ to R_1, each with a face which is a middle square of a square from the first step, to get a new region R_2. The surface Σ_2 is the boundary of R_2.

This process can be repeated indefinitely to get regions R_j and surfaces $\Sigma_j = \partial R_j$ for all nonnegative integers j. In the limit we can take R to be the union of the R_j's, and Σ to be the boundary of R, which is the same as the Hausdorff limit of the Σ_j's.

Of course this procedure is completely analogous to ones in the plane for producing snowflake curves. However, one does not get *snowballs* in the technical sense introduced by Pekka Koskela, because there are a lot of curves of finite length. Indeed, whenever a square is introduced in the construction, its four boundary segments are kept intact for all future stages, and hence in the limit. One can verify that Σ is a happy fractal.

8. More on Lipschitz functions

Let $(M, d(x,y))$ be a metric space. Suppose that $f(x)$ is a real or complex-valued function on M, and that L is a nonnegative real number. We say that f is *L-Lipschitz* if

$$|f(x) - f(y)| \leq L\, d(x,y) \tag{8.1}$$

for all $x, y \in M$. Thus f is Lipschitz if it is L-Lipschitz for some L. If f is Lipschitz, then we define $\|f\|_{\mathrm{Lip}}$ to be the supremum of

$$\frac{|f(x) - f(y)|}{d(x,y)} \tag{8.2}$$

over all $x, y \in M$, where this ratio is replaced with 0 when $x = y$. In other words, f is $\|f\|_{\mathrm{Lip}}$-Lipschitz when f is Lipschitz, and $\|f\|_{\mathrm{Lip}}$ is the smallest choice of L for which f is L-Lipschitz. Note that $\|\cdot\|_{\mathrm{Lip}}$ is a seminorm, so that

$$\|a\,f + b\,g\|_{\mathrm{Lip}} \le |a|\,\|f\|_{\mathrm{Lip}} + |b|\,\|g\|_{\mathrm{Lip}} \tag{8.3}$$

for all constants a, b and Lipschitz functions f, g on M. Also, $\|f\|_{\mathrm{Lip}} = 0$ if and only if f is a constant function on M.

If f and g are real-valued L-Lipschitz functions on M, then the maximum and minimum of f, g, which are denoted $\max(f, g)$ and $\min(f, g)$, are L-Lipschitz functions too. Let us check this for $\max(f, g)$. It is enough to show that

$$\max(f, g)(x) - \max(f, g)(y) \le L\,d(x, y) \tag{8.4}$$

for all $x, y \in M$, since one can interchange the roles of x and y to get a corresponding lower bound for $\max(f, g)(x) - \max(f, g)(y)$. Assume, for the sake of definiteness, that $\max(f, g)(x) = f(x)$. Then we have

$$\begin{aligned} \max(f, g)(x) = f(x) &\le f(y) + L\,d(x, y) \\ &\le \max(f, g)(y) + L\,d(x, y), \end{aligned} \tag{8.5}$$

which is what we wanted.

Here is a generalization of this fact.

Lemma 8.6. *Let $\{f_\sigma\}_{\sigma \in A}$ be a family of real-valued functions on M which are all L-Lipschitz for some $L \ge 0$. Assume also that there is point p in M such that the set of real numbers $\{f_\sigma(p) : \sigma \in A\}$ is bounded from above. Then the set $\{f_\sigma(x) : \sigma \in A\}$ is bounded from above for every x in M (but not uniformly in x in general), and* $\sup\{f_\sigma(x) : \sigma \in A\}$ *is an L-Lipschitz function on M.*

Indeed, because f_σ is L-Lipschitz for all σ in A, we have that

$$f_\sigma(x) \le f_\sigma(y) + L\,d(x, y) \tag{8.7}$$

for all x, y in M. Applying this to $y = p$, we see that $\{f_\sigma(x) : \sigma \in A\}$ is bounded from above for every x, because of the analogous property for p. If $F(x) = \sup\{f_\sigma(x) : \sigma \in A\}$, then

$$F(x) \le F(y) + L\,d(x, y) \tag{8.8}$$

for all x, y in M, so that F is L-Lipschitz on M.

For the record, let us write down the analogous statement for infima of L-Lipschitz functions.

Lemma 8.9. *Let $\{f_\sigma\}_{\sigma\in A}$ be a family of real-valued functions on M which are all L-Lipschitz for some $L \geq 0$. Assume also that there is point q in M such that the set of real numbers $\{f_\sigma(q) : \sigma \in A\}$ is bounded from below. Then the set $\{f_\sigma(x) : \sigma \in A\}$ is bounded from below for every x in M, and $\inf\{f_\sigma(x) : \sigma \in A\}$ is an L-Lipschitz function on M.*

For any point w in M, $d(x,w)$ defines a 1-Lipschitz function of x on M. This can be shown using the triangle inequality. Suppose now that $f(x)$ is an L-Lipschitz function on M. For each $w \in M$, define $f_w(x) = f(w) + L\,d(x,w)$. The fact that f is L-Lipschitz implies that

$$f(x) \leq f_w(x) \quad \text{for all } x, w \in M. \tag{8.10}$$

Of course $f_x(x) = x$, and hence

$$f(x) = \inf\{f_w(x) : w \in M\}. \tag{8.11}$$

Each function $f_w(x)$ is L-Lipschitz in x, since $d(x,w)$ is 1-Lipschitz in w.

Similarly, we can set $\widetilde{f}_w(x) = f(x) - L\,d(x,w)$, and then we have that

$$f(x) = \sup\{\widetilde{f}_w(x) : w \in M\}, \tag{8.12}$$

and that $\widetilde{f}_w(x)$ is an L-Lipschitz function of x for every w.

Here is a variant of these themes. Let E be a nonempty subset of M, and suppose that f is a real-valued function on E which is L-Lipschitz, so that

$$|f(x) - f(y)| \leq L\,d(x,y) \tag{8.13}$$

for all x, y in M. For each w in E, set $f_w(x) = f(x) + L\,d(x,w)$ and $\widetilde{f}_w(x) = f(x) - L\,d(x,w)$. Consider

$$\begin{aligned} F(x) &= \inf\{f_w(x) : w \in E\}, \\ \widetilde{F}(x) &= \sup\{\widetilde{f}_w(x) : w \in E\}, \end{aligned} \tag{8.14}$$

for x in M. For the same reasons as before, $F(x) = \widetilde{F}(x) = f(x)$ when x lies in E. Using Lemmas 8.6 and 8.9, one can check that F and $\widetilde{F}$ are L-Lipschitz real-valued functions on all of M, i.e., they are extensions of f from E to M with the same Lipschitz constant L.

If $H(x)$ is any other real-valued function on M which agrees with f on E and is L-Lipschitz, then

$$\widetilde{f}_w(x) \leq H(x) \leq f_w(x) \tag{8.15}$$

for all w in E and x in M, and hence

$$\widetilde{F}(x) \le H(x) \le F(x) \tag{8.16}$$

for all x in M.

Remark 8.17. If S is any nonempty subset of M, define $\operatorname{dist}(x, S)$ for x in M by

$$\operatorname{dist}(x, S) = \inf_{y \in S} d(x, y). \tag{8.18}$$

This function is always 1-Lipschitz in x, as in Lemma 8.9.

9. Lipschitz functions of order α

Let $(M, d(x, y))$ be a metric space, and let α be a positive real number. A real or complex-valued function f on M is said to be *Lipschitz of order* α if there is nonnegative real number L such that

$$|f(x) - f(y)| \le L\, d(x, y)^\alpha \tag{9.1}$$

for all $x, y \in M$. This reduces to the Lipschitz condition discussed in Section 8 when $\alpha = 1$. We shall sometimes write $\operatorname{Lip} \alpha$ for the collection of Lipschitz functions of order α, which might be real or complex valued, depending on the context. One also sometimes refers to these functions as being "Hölder continuous of order α".

If f is Lipschitz of order α, then we define $\|f\|_{\operatorname{Lip} \alpha}$ to be the supremum of

$$\frac{|f(x) - f(y)|}{d(x, y)^\alpha} \tag{9.2}$$

over all $x, y \in M$, where this quantity is replaced with 0 when $x = y$. In other words, $\|f\|_{\operatorname{Lip} \alpha}$ is the smallest choice of L so that (9.1) holds for all $x, y \in M$. This defines a seminorm on the space of Lipschitz functions of order α, as before, with $\|f\|_{\operatorname{Lip} \alpha} = 0$ if and only if f is constant. Of course $\|f\|_{\operatorname{Lip} 1}$ is the same as $\|f\|_{\operatorname{Lip}}$ from Section 8.

If f and g are real-valued functions on M which are Lipschitz of order α with constant L, then $\max(f, g)$ and $\min(f, g)$ are also Lipschitz of order α with constant L. This can be shown in the same manner as for $\alpha = 1$. Similarly, the analogues of Lemmas 8.6 and 8.9 for Lipschitz functions of order α hold for essentially the same reasons as before.

However, if $\alpha > 1$, it may be that the only functions that are Lipschitz of order α are the constant functions. This is the case when $M = \mathbf{R}^n$, for instance, equipped with the standard Euclidean metric, because a function in $\operatorname{Lip} \alpha$ with $\alpha > 1$ has first derivatives equal to 0 everywhere. Instead of using derivatives, it is not hard to show that the function has to be constant through more direct calculation too. On any metric space M, a function which is Lipschitz or order α with $\alpha > 1$ is constant on every path of finite length.

This problem does not occur when $\alpha < 1$.

Lemma 9.3. *If $0 < \alpha \leq 1$ and a, b are nonnegative real numbers, then $(a+b)^\alpha \leq a^\alpha + b^\alpha$.*

To see this, observe that

$$\max(a,b) \leq (a^\alpha + b^\alpha)^{1/\alpha}, \tag{9.4}$$

and hence

$$\begin{aligned} a + b &\leq \max(a,b)^{1-\alpha}(a^\alpha + b^\alpha) \\ &\leq (a^\alpha + b^\alpha)^{1+(1-\alpha)/\alpha} = (a^\alpha + b^\alpha)^{1/\alpha}. \end{aligned} \tag{9.5}$$

Corollary 9.6. *If $(M, d(x,y))$ is a metric space and α is a real number such that $0 < \alpha \leq 1$, then $d(x,y)^\alpha$ also defines a metric on M.*

This is easy to check. The main point is that $d(x,y)^\alpha$ satisfies the triangle inequality, because of Lemma 9.3 and the triangle inequality for $d(x,y)$.

A function f on M is Lipschitz of order α with respect to the original metric $d(x,y)$ if and only if it is Lipschitz of order 1 with respect to $d(x,y)^\alpha$, and with the same norm. In particular, for each w in M, $d(x,w)^\alpha$ satisfies (9.1) with $L = 1$ when $0 < \alpha \leq 1$, because of the triangle inequality for $d(u,v)^\alpha$.

10. Some functions on the real line

Fix α, $0 < \alpha \leq 1$. For each nonnegative integer n, consider the function

$$2^{-n\alpha} \exp(2^n\, i\, x) \tag{10.1}$$

on the real line $\mathbf{R}$, where $\exp u$ denotes the usual exponential e^u. Let us estimate the $\mathrm{Lip}\,\alpha$ norm of this function.

Recall that

$$|\exp(i\,u) - \exp(i\,v)| \leq |u - v| \tag{10.2}$$

for all $u, v \in \mathbf{R}$. Indeed, one can write $\exp(i\,u) - \exp(i\,v)$ as the integral between u and v of the derivative of $\exp(i\,t)$, and this derivative is $i\exp(i\,t)$, which has modulus equal to 1 at every point.

Thus, for any $x, y \in \mathbf{R}$, we have that

$$|2^{-n\alpha} \exp(2^n\, i\, x) - 2^{-n\alpha} \exp(2^n\, i\, y)| \leq 2^{n(1-\alpha)}|x - y|. \tag{10.3}$$

Of course

$$(10.4)\quad |2^{-n\alpha}\exp(2^n\, i\, x) - 2^{-n\alpha}\exp(2^n\, i\, y)| \\ \le 2^{-n\alpha}|\exp(2^n\, i\, x)| + 2^{-n\alpha}|\exp(2^n\, i\, y)| = 2^{-n\alpha+1}$$

as well. As a result,

$$(10.5)\quad |2^{-n\alpha}\exp(2^n\, i\, x) - 2^{-n\alpha}\exp(2^n\, i\, y)| \\ \le \Big(2^{n(1-\alpha)}|x-y|\Big)^{\alpha}\Big(2^{-n\alpha+1}\Big)^{1-\alpha} = 2^{1-\alpha}|x-y|^{\alpha}.$$

This shows that the function (10.1) has $\mathrm{Lip}\,\alpha$ norm (with respect to the standard Euclidean metric on $\mathbf{R}$) which is at most $2^{1-\alpha}$. In the opposite direction, if $2^n(x-y) = \pi$, then

$$(10.6)\quad \begin{aligned} &|2^{-n\alpha}\exp(2^n\, i\, x) - 2^{-n\alpha}\exp(2^n\, i\, y)| \\ &\quad = 2^{-n\alpha}|\exp(2^n\, i\, x)| + 2^{-n\alpha}|\exp(2^n\, i\, y)| \\ &\quad = 2^{-n\alpha+1} = 2\pi^{-\alpha}|x-y|^{\alpha}, \end{aligned}$$

so that the $\mathrm{Lip}\,\alpha$ norm is at least $2\pi^{-\alpha}$.

Now suppose that $f(x)$ is a complex-valued function on $\mathbf{R}$ of the form

$$(10.7)\quad f(x) = \sum_{n=0}^{\infty} a_n\, 2^{-n\alpha}\exp(2^n\, i\, x),$$

where the a_n's are complex numbers. We assume that the a_n's are bounded, which implies that the series defining $f(x)$ converges absolutely for each x. Set

$$(10.8)\quad A = \sup_{n\ge 0}|a_n|.$$

Let m be a nonnegative integer. For each x in $\mathbf{R}$ we have that

$$(10.9)\quad \Big|\sum_{n=m}^{\infty} a_n\, 2^{-n\alpha}\exp(2^n\, i\, x)\Big| \le \sum_{n=m}^{\infty} A\, 2^{-n\alpha} = A\,(1-2^{-\alpha})^{-1}\, 2^{-m\alpha}.$$

If $m \ge 1$ and $x, y \in \mathbf{R}$, then (10.2) yields

$$\begin{aligned}
&\left|\sum_{n=0}^{m-1} a_n\, 2^{-n\alpha} \exp(2^n\, i\, x) - \sum_{n=0}^{m-1} a_n\, 2^{-n\alpha} \exp(2^n\, i\, y)\right| \\
&\qquad \le \sum_{n=0}^{m-1} A\, 2^{n(1-\alpha)} |x-y| \\
&\qquad \le A\, 2^{(m-1)(1-\alpha)} \left(\sum_{j=0}^{\infty} 2^{-j(1-\alpha)}\right) |x-y| \\
&\qquad = A\, 2^{(m-1)(1-\alpha)} (1 - 2^{-(1-\alpha)})^{-1} |x-y|.
\end{aligned} \tag{10.10}$$

Here we should assume that $\alpha < 1$, to get the convergence of $\sum_{j=0}^{\infty} 2^{-j(1-\alpha)}$.

Fix $x, y \in \mathbf{R}$. If $|x-y| > 1/2$, then we apply (10.9) with $m = 0$ to both x and y to get that

$$\begin{aligned}
|f(x) - f(y)| \le |f(x)| + |f(y)| &\le 2\, A\, (1 - 2^{-\alpha})^{-1} \\
&\le 2^{1+\alpha}\, A\, (1 - 2^{-\alpha})^{-1} |x-y|^{\alpha}.
\end{aligned} \tag{10.11}$$

Assume now that $|x-y| \le 1/2$, and choose $m \in \mathbf{Z}_+$ so that

$$2^{-m-1} < |x-y| \le 2^{-m}. \tag{10.12}$$

Combining (10.9) and (10.10), with (10.9) applied to both x and y, we obtain that

$$\begin{aligned}
&|f(x) - f(y)| \\
&\le 2\, A\, (1-2^{-\alpha})^{-1}\, 2^{-m\alpha} + A\, 2^{(m-1)(1-\alpha)} (1-2^{-(1-\alpha)})^{-1} |x-y| \\
&\le 2^{1+\alpha} A\, (1-2^{-\alpha})^{-1} |x-y|^{\alpha} + A\, 2^{-(1-\alpha)} (1-2^{-(1-\alpha)})^{-1} |x-y|^{\alpha}.
\end{aligned} \tag{10.13}$$

Therefore, for all $x, y \in \mathbf{R}$, we have that

$$\begin{aligned}
&|f(x) - f(y)| \\
&\qquad \le A(2^{1+\alpha}(1 - 2^{-\alpha})^{-1} + 2^{-(1-\alpha)}(1 - 2^{-(1-\alpha)})^{-1}) |x-y|^{\alpha}
\end{aligned} \tag{10.14}$$

when $0 < \alpha < 1$. In other words, f is Lipschitz of order α, and

$$\|f\|_{\mathrm{Lip}\,\alpha} \le \Big(\sup_{n \ge 0} |a_n|\Big) (2^{1+\alpha}(1-2^{-\alpha})^{-1} + 2^{-(1-\alpha)}(1-2^{-(1-\alpha)})^{-1}). \tag{10.15}$$

To get an inequality going in the other direction we shall compute as follows. Let $\psi(x)$ be a function on $\mathbf{R}$ such that the Fourier transform $\widehat{\psi}(\xi)$ of ψ,

$$\widehat{\psi}(\xi) = \int_{\mathbf{R}} \exp(i\, \xi\, x)\, \psi(x)\, dx \tag{10.16}$$

is a smooth function which satisfies $\widehat{\psi}(1) = 1$ and $\widehat{\psi}(\xi) = 0$ when $0 \le \xi \le 1/2$ and when $\xi \ge 2$. One can do this with $\psi(x)$ in the Schwartz class of smooth functions such that $\psi(x)$ and all of its derivatives are bounded by constant multiples of $(1+|x|)^{-k}$ for every positive integer k.

For each nonnegative integer j, let us write $\psi_{2^j}(x)$ for the function $2^j\,\psi(2^j\,x)$. Thus

$$\widehat{\psi_{2^j}}(\xi) = \widehat{\psi}(2^{-j}\,\xi). \tag{10.17}$$

In particular, $\widehat{\psi_{2^j}}(2^j) = 1$, and $\widehat{\psi_{2^j}}(2^l) = 0$ when l is a nonnegative integer different from j. Hence

$$\int_{\mathbf{R}} f(x)\,\psi_{2^j}(x)\,dx = \sum_{n=0}^{\infty} a_n\,2^{-n\alpha}\,\widehat{\psi_{2^j}}(2^n) = a_j\,2^{-j\alpha}. \tag{10.18}$$

On the other hand,

$$\int_{\mathbf{R}} \psi_{2^j}(x)\,dx = \widehat{\psi_{2^j}}(0) = \widehat{\psi}(0) = 0, \tag{10.19}$$

so that

$$\int_{\mathbf{R}} f(x)\,\psi_{2^j}(x)\,dx = \int_{\mathbf{R}} (f(x) - f(0))\,\psi_{2^j}(x)\,dx. \tag{10.20}$$

Therefore

$$\begin{aligned} \left|\int_{\mathbf{R}} f(x)\,\psi_{2^j}(x)\,dx\right| &\le \int_{\mathbf{R}} |f(x) - f(0)|\,|\psi_{2^j}(x)|\,dx \\ &\le \|f\|_{\mathrm{Lip}\,\alpha} \int_{\mathbf{R}} |x|^\alpha\,|\psi_{2^j}(x)|\,dx \\ &= \|f\|_{\mathrm{Lip}\,\alpha}\,2^{-j\alpha} \int_{\mathbf{R}} |x|^\alpha\,|\psi(x)|\,dx. \end{aligned} \tag{10.21}$$

Combining this with (10.18), we obtain that

$$|a_j| \le \|f\|_{\mathrm{Lip}\,\alpha} \int_{\mathbf{R}} |x|^\alpha\,|\psi(x)|\,dx \tag{10.22}$$

for all nonnegative integers j. The integral on the right side converges, because of the decay property of ψ.

If $\alpha = 1$, then let us pass to the derivative and write

$$f'(x) = \sum_{n=0}^{\infty} a_n \, i \exp 2^n \, i \, x \tag{10.23}$$

(where one should be careful about the meaning of f' and of this series). This leads to

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(x)|^2 \, dx = \sum_{n=0}^{\infty} |a_n|^2. \tag{10.24}$$

The main idea is that

$$\sum_{n=0}^{\infty} |a_n|^2 \le \|f\|_{\mathrm{Lip}\,1}^2 \tag{10.25}$$

if f is Lipschitz. Conversely, if $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, then the derivative of f exists in an L^2 sense, and in fact one can show that f' has "vanishing mean oscillation".

11. Sums on general metric spaces

Let $(M, d(x,y))$ be a metric space. For each integer n, suppose that we have chosen a complex-valued Lipschitz function $\beta_n(x)$ such that

$$\sup_{x \in M} |\beta_n(x)| \le 1 \quad \text{and} \quad \|\beta\|_{\mathrm{Lip}} \le 2^n. \tag{11.1}$$

Fix a real number α, $0 < \alpha < 1$.

Let a_n, $n \in \mathbf{Z}$ be a family (or doubly-infinite sequence) of complex numbers which is bounded, and set

$$A = \sup_{n \in \mathbf{Z}} |a_n|. \tag{11.2}$$

Consider

$$f(x) = \sum_{n \in \mathbf{Z}} a_n \, 2^{-n\alpha} \, \beta_n(x). \tag{11.3}$$

The sum on the right side does not really converge in general, although it would if we restricted ourselves to n greater than any fixed number, because of the bound on $\beta_n(x)$. However, this sum does converge "modulo constants", in the sense that the sum in

$$f(x) - f(y) = \sum_{n \in \mathbf{Z}} a_n \, 2^{-n\alpha} \, (\beta_n(x) - \beta_n(y)), \tag{11.4}$$

converges absolutely for all x, y in M.

To see this, suppose that k is any integer. For $n \ge k$ we have that

$$\sum_{n=k}^{\infty} |a_n|\, 2^{-n\alpha}\, |\beta_n(x)| \le A\,(1 - 2^{-\alpha})^{-1}\, 2^{-k\alpha}, \tag{11.5}$$

and similarly for y instead of x. For $n \le k-1$ we have that

$$\begin{aligned} \sum_{n=-\infty}^{k-1} |a_n|\, 2^{-n\alpha}\, |\beta_n(x) - \beta_n(y)| &\le A \sum_{n=-\infty}^{k-1} 2^{n(1-\alpha)}\, d(x,y) \\ &= A\, 2^{(k-1)(1-\alpha)} (1 - 2^{-(1-\alpha)})^{-1}\, d(x,y). \end{aligned} \tag{11.6}$$

Thus

$$\begin{aligned} &\sum_{n \in \mathbf{Z}} |a_n|\, 2^{-n\alpha}\, |\beta_n(x) - \beta_n(y)| \\ &\le A\,(1 - 2^{-\alpha})^{-1}\, 2^{-k\alpha} + A\, 2^{(k-1)(1-\alpha)} (1 - 2^{-(1-\alpha)})^{-1}\, d(x,y) \end{aligned} \tag{11.7}$$

for all $x, y \in M$ and $k \in \mathbf{Z}$.

12. The Zygmund class on R

Let $f(x)$ be a real or complex-valued function on the real line. We say that f lies in the *Zygmund class* Z if f is continuous and there is a nonnegative real number L such that

$$|f(x+h) + f(x-h) - 2\, f(x)| \le L\, |h| \tag{12.1}$$

for all $x, y \in \mathbf{R}$. In this case, the seminorm $\|f\|_Z$ is defined to be the supremum of

$$\frac{|f(x+h) + f(x-h) - 2\, f(x)|}{|h|} \tag{12.2}$$

over all $x, h \in \mathbf{R}$ with $h \ne 0$. This is the same as the smallest L so that (12.1) holds. Clearly f is in the Zygmund class when f is Lipschitz (of order 1), with $\|f\|_Z \le 2\, \|f\|_{\mathrm{Lip}}$.

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a bounded sequence of complex numbers, and consider the function $f(x)$ on $\mathbf{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} a_n\, 2^{-n} \exp(2^n\, i\, x). \tag{12.3}$$

Let us check that f lies in the Zygmund class, with $\|f\|_Z$ bounded in terms of

$$A = \sup_{n \geq 0} |a_n|. \tag{12.4}$$

Note that f is continuous.

Observe that

$$\begin{aligned} &|\exp(i(u+v)) + \exp(i(u-v)) - 2\exp(i\,u)| \\ &\qquad\qquad = |\exp(i\,v) + \exp(-i\,v) - 2| \end{aligned} \tag{12.5}$$

for all real numbers u, v, and that

$$\exp(i\,v) + \exp(-i\,v) - 2 = \int_0^v i(\exp(i\,t) - \exp(-i\,t))\,dt \tag{12.6}$$

when $v \geq 0$. Since $|\exp(i\,t) - \exp(-i\,t)| \leq 2\,t$ for $t \geq 0$, we obtain that

$$|\exp(i\,v) + \exp(-i\,v) - 2| \leq \int_0^v 2\,t\,dt = v^2. \tag{12.7}$$

Hence

$$|\exp(i(u+v)) + \exp(i(u-v)) - 2\exp(i\,u)| \leq v^2, \tag{12.8}$$

and this works for all real numbers u, v, since there is no real difference between $v \geq 0$ and $v \leq 0$.

Let x and h be real numbers, and let m be a nonnegative integer. From (12.8) we get that

$$\begin{aligned} &\left|\sum_{n=0}^{m} a_n\,2^{-n}\,(\exp(2^n\,i(x+h)) + \exp(2^n\,i(x-h)) - 2\exp(2^n\,i\,x))\right| \\ &\qquad\qquad \leq A \sum_{n=0}^{m} 2^{-n}\,2^{2n}\,|h|^2 \leq A\,2^{m+1}\,|h|^2. \end{aligned} \tag{12.9}$$

If $|h| \geq 1/2$, then

$$\begin{aligned} &|f(x+h) + f(x-h) - 2\,f(x)| \\ &\qquad \leq |f(x+h)| + |f(x-h)| + 2\,|f(x)| \leq 4\,A \leq 8\,A\,|h|. \end{aligned} \tag{12.10}$$

If $|h| \leq 1/2$, then choose a positive integer m such that $2^{-m-1} \leq |h| \leq 2^{-m}$. We can write $f(x+h) + f(x-h) - 2\,f(x)$ as

$$\sum_{n=0}^{m} a_n\, 2^{-n}\, (\exp(2^n\, i(x+h)) + \exp(2^n\, i(x-h)) - 2\exp(2^n\, i\, x)) \tag{12.11}$$

$$+ \sum_{n=m+1}^{\infty} a_n\, 2^{-n}\, (\exp(2^n\, i(x+h)) + \exp(2^n\, i(x-h)) - 2\exp(2^n\, i\, x)).$$

This leads to

$$\begin{aligned} &|f(x+h) + f(x-h) - 2\,f(x)| \\ &\quad \leq |f(x+h)| + |f(x-h)| + 2\,|f(x)| \\ &\quad \leq A\, 2^{m+1} |h|^2 + 4\,A\, 2^{-m} \\ &\quad \leq A \cdot 2 \cdot |h| + 4 \cdot A \cdot 2 \cdot |h| = 10\, A\, |h|. \end{aligned} \tag{12.12}$$

This shows that f lies in the Zygmund class, with constant less than or equal to $10\,A$.

13. Approximation operators, 1

Let $(M, d(x,y))$ be a metric space. Fix a real number α, $0 < \alpha < 1$, and let f be a real-valued function on M which is Lipschitz of order α. For each positive real number L, define $A_L(f)$ by

$$A_L(f)(x) = \inf\{f(w) + L\, d(x,w) : w \in M\} \tag{13.1}$$

for all x in M.

For arbitrary x, w in M we have that

$$f(w) \geq f(x) - \|f\|_{\mathrm{Lip}\,\alpha}\, d(x,w)^\alpha. \tag{13.2}$$

As a result,

$$f(w) + L\, d(x,w) \geq f(x) \tag{13.3}$$

when $L\, d(x,w)^{1-\alpha} \geq \|f\|_{\mathrm{Lip}\,\alpha}$. Thus we can rewrite (13.1) as

$$\begin{aligned} &A_L(f)(x) \\ &\qquad = \inf\{f(w) + L\, d(x,w) : w \in M,\ L\, d(x,w)^{1-\alpha} \leq \|f\|_{\mathrm{Lip}\,\alpha}\}, \end{aligned} \tag{13.4}$$

i.e., one gets the same infimum over this smaller range of w's. In particular, the set of numbers whose infimum is under consideration is bounded from below, so that the infimum is finite.

Because we can take $w = x$ in the infimum, we automatically have that

$$A_L(f)(x) \le f(x) \tag{13.5}$$

for all x in M. In the other direction, (13.2) and (13.4) lead to

$$\begin{aligned} A_L(f)(x) &\ge f(x) - \|f\|_{\mathrm{Lip}\,\alpha} \left(\frac{\|f\|_{\mathrm{Lip}\,\alpha}}{L} \right)^{\alpha/(1-\alpha)} \\ &= f(x) - \|f\|_{\mathrm{Lip}\,\alpha}^{1/(1-\alpha)} L^{-\alpha/(1-\alpha)}. \end{aligned} \tag{13.6}$$

We also have that $A_L(f)$ is L-Lipschitz on M, as in Lemma 8.9.

Suppose that $h(x)$ is a real-valued function on M which is L-Lipschitz and satisfies $h(x) \le f(x)$ for all x in M. Then

$$h(x) \le h(w) + L\,d(x, w) \le f(w) + L\,d(x, w) \tag{13.7}$$

for all x, w in M. Hence

$$h(x) \le A_L(f)(x) \tag{13.8}$$

for all x in M.

Similarly, one can consider

$$B_L(f)(x) = \sup\{f(w) - L\,d(x, w) : w \in M\}, \tag{13.9}$$

and show that

$$\begin{aligned} &B_L(f)(x) \\ &\quad = \sup\{f(w) - L\,d(x, w) : w \in M,\ L\,d(x, w)^{1-\alpha} \le \|f\|_{\mathrm{Lip}\,\alpha}\}. \end{aligned} \tag{13.10}$$

This makes it clear that the supremum is finite. As before,

$$f(x) \le B_L(f)(x) \le f(x) + \|f\|_{\mathrm{Lip}\,\alpha}^{1/(1-\alpha)} L^{-\alpha/(1-\alpha)}, \tag{13.11}$$

and $B_L(f)$ is L-Lipschitz. If $h(x)$ is a real-valued function on M which is L-Lipschitz and satisfies $f(x) \le h(x)$ for all x in M, then

$$B_L(f)(x) \le h(x) \tag{13.12}$$

for all x in M.

14. Approximation operators, 2

Let $(M, d(x,y))$ be a metric space, and let μ be a positive Borel measure on M. We shall assume that μ is a *doubling* measure, which means that there is a positive real number C such that

$$\mu(B(x,2r)) \le C\,\mu(B(x,r)) \tag{14.1}$$

for all x in M and positive real numbers r, and that the μ-measure of any open ball is positive and finite.

Let t be a positive real number. Define a function $p_t(x,y)$ on $M \times M$ by

$$\begin{aligned} p_t(x,y) &= 1 - t^{-1} d(x,y) && \text{when } d(x,y) \le t \\ &= 0 && \text{when } d(x,y) > t, \end{aligned} \tag{14.2}$$

and put

$$\rho_t(x) = \int_M p_t(x,y)\,d\mu(y). \tag{14.3}$$

This is positive for every x in M, because of the properties of μ. Also put

$$\phi_t(x,y) = \rho_t(x)^{-1}\,p_t(x,y), \tag{14.4}$$

so that

$$\int_M \phi_t(x,y)\,d\mu(y) = 1 \tag{14.5}$$

for all x in M by construction.

Fix a real number α, $0 < \alpha \le 1$, and let f be a complex-valued function on M which is Lipschitz of order α. Define $P_t(f)$ on M by

$$P_t(f)(x) = \int_M \phi_t(x,y)\,f(y)\,d\mu(y). \tag{14.6}$$

Because of (14.5),

$$P_t(f)(x) - f(x) = \int_M \phi_t(x,y)\,(f(y) - f(x))\,d\mu(y), \tag{14.7}$$

and hence

$$\begin{aligned} |P_t(f)(x) - f(x)| &\le \int_M \phi_t(x,y)|f(y) - f(x)|\,d\mu(y) \\ &\le \int_M \phi_t(x,y)\|f\|_{\mathrm{Lip}\,\alpha}\,t^\alpha\,d\mu(y) = \|f\|_{\mathrm{Lip}\,\alpha}\,t^\alpha. \end{aligned} \tag{14.8}$$

In the second step we employ the fact that $\phi_t(x,y) = 0$ when $d(x,y) \ge t$.

Suppose that x and z are elements of M, and consider

$$|P_t(f)(x) - P_t(f)(z)|. \tag{14.9}$$

If $d(x, z) \ge t$, then

$$\begin{aligned} &|P_t(f)(x) - P_t(f)(z)| \\ &\le |P_t(f)(x) - f(x)| + |f(x) - f(z)| + |P_t(f)(z) - f(z)| \\ &\le \|f\|_{\mathrm{Lip}\,\alpha}(2\,t^\alpha + d(x,z)^\alpha) \le 3\,t^{\alpha-1}\|f\|_{\mathrm{Lip}\,\alpha}\,d(x,z). \end{aligned} \tag{14.10}$$

Assume instead that $d(x, z) \le t$. In this case we write $P_t(f)(x)-P_t(f)(z)$ as

$$\begin{aligned} &\int_M (\phi_t(x,y) - \phi_t(z,y))\,f(y)\,d\mu(y) \\ &\qquad = \int_M (\phi_t(x,y) - \phi_t(z,y))\,(f(y) - f(x))\,d\mu(y), \end{aligned} \tag{14.11}$$

using (14.5). This yields

$$\begin{aligned} &|P_t(f)(x) - P_t(f)(z)| \\ &\le \int_M |\phi_t(x,y) - \phi_t(z,y)||f(y) - f(x)|\,d\mu(y) \\ &\le (2t)^\alpha\,\|f\|_{\mathrm{Lip}\,\alpha} \int_{\overline{B}(x,2t)} |\phi_t(x,y) - \phi_t(z,y)|\,d\mu(y), \end{aligned} \tag{14.12}$$

where the second step relies on the observation that $\phi_t(x, y) - \phi_t(z, y)$ is supported, as a function of y, in the set

$$\overline{B}(x,t) \cup \overline{B}(z,t) \subseteq \overline{B}(x,2t). \tag{14.13}$$

Of course

$$\begin{aligned} &\phi_t(x,y) - \phi_t(z,y) \\ &\qquad = (\rho_t(x)^{-1} - \rho_t(z)^{-1})\,p_t(x,y) + \rho_t(z)^{-1}\,(p_t(x,y) - p_t(z,y)). \end{aligned} \tag{14.14}$$

Notice that

$$|p_t(x,y) - p_t(z,y)| \le t^{-1}\,d(x,z) \tag{14.15}$$

for all y in M. To see this, it is convenient to write $p_t(u, v)$ as $\lambda_t(d(u, v))$, where $\lambda_t(r)$ is defined for $r \ge 0$ by $\lambda_t(r) = 1 - t^{-1}\,r$ when $0 \le r \le t$, and $\lambda_t(r) = 0$ when $r \ge t$. It is easy to check that λ_t is t^{-1}-Lipschitz, and hence $\lambda_t(d(u, v))$ is t^{-1}-Lipschitz on M as a function of u for each fixed v, since $d(u, v)$ is 1-Lipschitz as a function of u for each fixed v.

These computations and the doubling condition for μ permit one to show that

$$\int_{\overline{B}(x,2t)} |\phi_t(x,y) - \phi_t(z,y)|\, d\mu(y) \le C_1\, t^{-1}\, d(x,z) \tag{14.16}$$

for some positive real number C_1 which does not depend on x, z, or t. (Exercise.) Altogether, we obtain that

$$\|P_t(f)\|_{\mathrm{Lip}\,1} \le \max(3, 2^\alpha\, C_1) t^{\alpha-1}\, \|f\|_{\mathrm{Lip}\,\alpha}. \tag{14.17}$$

15. A kind of Calderón-Zygmund decomposition related to Lipschitz functions

Let $(M, d(x,y))$ be a metric space, and let f be a real-valued function on M. Consider the associated maximal function

$$N(f)(x) = \sup_{\substack{y \in M \\ y \ne x}} \frac{|f(y) - f(x)|}{d(y,x)}, \tag{15.1}$$

where this supremum may be $+\infty$.

Let L be a positive real number, and put

$$F_L = \{x \in M : N(f)(x) \le L\}. \tag{15.2}$$

We shall assume for the rest of this section that

$$F_L \ne \emptyset. \tag{15.3}$$

As in Section 13, define $A_L(f)$ by

$$A_L(f)(x) = \inf\{f(w) + L\, d(x,w) : w \in M\}. \tag{15.4}$$

We shall address the finiteness of this infimum in a moment. As before,

$$A_L(f)(x) \le f(x) \tag{15.5}$$

for all x in M.

If u is any element of F_L, then

$$|f(y) - f(u)| \le L\, d(y,u) \tag{15.6}$$

for all y in M. Let x and w be arbitrary points in M. The preceding inequality implies that

$$f(u) \le f(w) + L\, d(u,w), \tag{15.7}$$

and hence

$$\begin{aligned} f(u) - L\, d(x,u) &\le f(w) + L\,(d(u,w) - d(x,u)) \\ &\le f(w) + L\, d(x,w), \end{aligned} \tag{15.8}$$

by the triangle inequality. This yields

$$f(u) - L\,d(x,u) \le A_L(f)(x), \tag{15.9}$$

which includes the finiteness of $A_L(f)(x)$. If we take $x = u$, then we get $f(u) \le A_L(f)(u)$, so that

$$f(u) = A_L(f)(u) \quad \text{for all } u \in F_L. \tag{15.10}$$

For $x \notin F_L$, we obtain

$$f(x) - 2L\,d(x,u) \le A_L(f)(x) \tag{15.11}$$

for all u in F_L, by combining (15.9) and (15.6) with $y = x$. In other words,

$$f(x) - A_L(f)(x) \le 2L\,\mathrm{dist}(x, F_L). \tag{15.12}$$

Note that $A_L(f)$ is L-Lipschitz on M, by Lemma 8.9.

In the same way, if

$$B_L(f)(x) = \sup\{f(w) - L\,d(x,w) : w \in M\}, \tag{15.13}$$

then

$$f(x) \le B_L(f)(x) \le f(x) + 2L\,\mathrm{dist}(x, F_L) \tag{15.14}$$

for all x in M, and $B_L(f)$ is L-Lipschitz.

16. A brief overview of "atoms"

Let $(M, d(x,y))$ be a metric space, and let s be a positive real number. We say that $(M, d(x,y))$ is *Ahlfors-regular of dimension* s if M is complete as a metric space, and if there is a positive Borel measure μ on M such that

$$C_1^{-1}\, r^s \le \mu(\overline{B}(x,r)) \le C_1\, r^s \tag{16.1}$$

for some positive real number C_1, all x in M, and all $r > 0$ such that $r \le \operatorname{diam} M$ if M is bounded.

As a basic example, if M is n-dimensional Euclidean space $\mathbf{R}^n$ with the standard metric, and if μ is Lebesgue measure, then in fact $\mu(\overline{B}(x,r))$ is equal to a constant times r^n, where the constant is simply the volume of the unit ball. More exotically, one can consider simply-connected nonabelian nilpotent Lie groups, such as the Heisenberg groups. For these spaces one still has natural dilations as on Euclidean spaces, and Lebesgue measure is compatible with both the group structure and the dilations, in such a way that the measure of a ball of radius r is equal to a constant times r^s, where s is now a geometric dimension that is larger than the topological dimension. Other examples include fractals such as the Sierpinski gasket and carpet.

Fix a metric space $(M, d(x, y))$ and a measure μ on M satisfying the conditions in the definition of Ahlfors-regularity, with dimension s. The following fact is sometimes useful: there is a constant $k_1 \geq 1$ so that if x is an element of M and r, R are positive numbers, with $r \leq R$, then the ball $\overline{B}(x, R)$ can be covered by a collection of at most $k_1 (R/r)^s$ closed balls of radius r. If M is bounded, then we may as well assume that $r < \operatorname{diam} M$ here, because M is automatically contained in a single ball with radius $\operatorname{diam} M$. We may also assume that $R \leq \operatorname{diam} M$, since we could simply replace R with $\operatorname{diam} M$ if R is initially chosen to be larger than that.

To establish the assertion in the preceding paragraph, let us begin with a preliminary observation. Suppose that A is a subset of $\overline{B}(x, R)$ such that $d(x, y) > r$ for all x, y in A. Then the number of elements of A is at most $k_1 (R/r)^s$, if we choose k_1 large enough (independently of x, R, and r). Indeed,

$$\sum_{a \in A} \mu(\overline{B}(a, r/2)) = \mu\left(\bigcup_{a \in A} \overline{B}(a, r/2)\right) \leq \mu(\overline{B}(x, 3R/2)), \tag{16.2}$$

where the first equality uses the disjointness of the balls $\overline{B}(a, r/2)$, $a \in A$. The Ahlfors-regularity property then applies to give a bound on the number of elements of A of the form $k_1 (R/r)^s$. Now that we have such a bound, suppose that A is also chosen so that the number of its elements is maximal. Then

$$\overline{B}(x, R) \subseteq \bigcup_{a \in A} \overline{B}(a, r). \tag{16.3}$$

In other words, if z is an element of $\overline{B}(x, R)$, then $d(z, a) \leq r$ for some a in A, because otherwise we could add z to A to get a set which satisfies the same separation condition as A, but which has 1 more element. This yields the original assertion.

In particular, closed and bounded subsets of M are compact. This uses the characterization of compactness in terms of completeness and total boundedness, where the latter holds for bounded subsets of M by the result just discussed.

Let us look at some special families of functions on M, called *atoms*, as in [**19**]. For the sake of definiteness, we make the convention that a "ball" in M means a closed ball (with some center and radius), if nothing else is specified. Suppose that p is a real number and r is an extended real number such that

$$0 < p \le 1,\ 1 \le r \le \infty, \quad p < r. \tag{16.4}$$

An integrable complex-valued function $a(x)$ on M will be called a (p, r)-*atom* if it satisfies the following three conditions: first, there is a ball B in M such that the support of a is contained in B, i.e., $a(x) = 0$ when $x \in M \backslash B$; second,

$$\int_M a(x)\, d\mu(x) = 0; \tag{16.5}$$

and third,

$$\left(\frac{1}{\mu(B)} \int_M |a(x)|^r \, d\mu(x) \right)^{1/r} \le \mu(B)^{-1/p}. \tag{16.6}$$

If $r = \infty$, then (16.6) is interpreted as meaning that the supremum (or essential supremum, if one prefers) of a is bounded by $\mu(B)^{-1/p}$.

The size condition (16.6) may seem a bit odd at first. A basic point is that it implies

$$\int_M |a(x)|^p \, d\mu(x) \le 1, \tag{16.7}$$

by Jensen's inequality. The index r reflects a kind of regularity of the atom, and notice that a (p, r_1)-atom is automatically a (p, r_2)-atom when $r_1 \ge r_2$. There are versions of this going in the other direction, from r_2 to r_1, and we shall say more about this soon.

Suppose that $a(x)$ is a (p, r)-atom on M and that $\phi(x)$ lies in $\mathrm{Lip}\,\alpha$ on M for some α. Consider the integral

$$\int_M a(x)\, \phi(x)\, d\mu(x). \tag{16.8}$$

Let $B = \overline{B}(z, t)$ be the ball associated to $a(x)$ as in the definition of an atom. The preceding integral can be written as

$$\int_{\overline{B}(z,t)} a(x)\, (\phi(x) - \phi(z))\, d\mu(x), \tag{16.9}$$

using also (16.5). Thus

$$\begin{aligned} \left| \int_M a(x)\, \phi(x)\, d\mu(x) \right| &\le \int_{\overline{B}(z,t)} |a(x)|\, |\phi(x) - \phi(z)|\, d\mu(x) \\ &\le \mu(\overline{B}(z,t))^{1-(1/p)}\, t^\alpha\, \|\phi\|_{\mathrm{Lip}\,\alpha}. \end{aligned} \tag{16.10}$$

Ahlfors-regularity implies that

$$\left|\int_M a(x)\,\phi(x)\,d\mu(x)\right| \le C_1^{1-(1/p)}\,t^{(1-(1/p))s+\alpha}\,\|\phi\|_{\mathrm{Lip}\,\alpha}. \tag{16.11}$$

In particular,

$$\left|\int_M a(x)\,\phi(x)\,d\mu(x)\right| \le C_1^{1-(1/p)}\,\|\phi\|_{\mathrm{Lip}\,\alpha} \tag{16.12}$$

when $\alpha = ((1/p)-1)\,s$.

If we want to be able to choose $\alpha = ((1/p)-1)\,s$ and have $\alpha \le 1$, then we are lead to the restriction

$$p \ge \frac{s}{s+1}. \tag{16.13}$$

Indeed, this condition does come up for some results, even if much of the theory works without it. There can also be some funny business at the endpoint, so that one might wish to assume a strict inequality in (16.13), or some statements would have to be modified when equality holds.

In some situations this type of restriction is not really necessary, perhaps with some adjustments. Let us mention two basic scenarios. First, suppose that our metric space M is something like a self-similar Cantor set, such as the classical "middle-thirds" Cantor set. In this case there are a lot of $\mathrm{Lip}\,\alpha$ functions for all $\alpha > 0$, and, for that matter, there are a lot of functions which are locally constant. The computation giving (16.12) still works when $\alpha > 1$, and this is true in general.

On the other hand, if $M = \mathbf{R}^n$ with the standard Euclidean metric, then there other ways to define classes of more smooth functions, through conditions on higher derivatives. In connection with this, one can strengthen (16.5) by asking that the integral of an atom times a polynomial of degree at most some number is equal to 0. If one does this, then there are natural extensions of (16.12) for $\alpha > 1$, obtained by subtracting a polynomial approximation to $\phi(x)$.

A basic manner in which atoms can be used is to test localization properties of linear operators. Suppose that T is a bounded linear operator on $L^2(M)$, and that a is a $(p,2)$-atom on M. Consider

$$T(a) \tag{16.14}$$

(as well as $T^*(a)$, for that matter). This is well-defined as an element of $L^2(M)$, since a lies in $L^2(M)$. If $B = \overline{B}(z,t)$ is the ball associated to a in the definition of an atom, then the estimate

$$\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_M |T(a)(x)|^2\,d\mu(x)\right)^{1/2} \\
&\qquad \le \|T\|_{2,2}\left(\frac{1}{\mu(B)}\int_M |a(x)|^2\,d\mu(x)\right)^{1/2} \\
&\qquad \le \|T\|_{2,2}\,\mu(B)^{-1/p}
\end{aligned} \tag{16.15}$$

provides about as much information about $T(a)$ around B, on $2B = \overline{B}(z,2t)$, say, as one might reasonably expect to have. However, in many situations one can expect to have decay of $T(a)$ away from B, in such a way that

$$\|T(a)\|_p \le k \tag{16.16}$$

for some constant k which does not depend on a.

In this argument it is natural to take $r = 2$, but a basic result in the theory is that one has some freedom to vary r. Specifically, if b is a (p,r)-atom on M, then it is possible to write b as

$$b = \sum_i \beta_i\, b_i, \tag{16.17}$$

where each b_i is a (p,∞)-atom, each β_i is a complex number, and $\sum_i |\beta_i|^p$ is bounded by a constant that does not depend on b (but which may depend on p or r). Let us give a few hints about how one can approach this. As an initial approximation, one can try to write b as

$$b = \beta'\, b' + \sum_j \gamma_j\, c_j, \tag{16.18}$$

where b' is a (p,∞)-atom, β' is a complex number such that $|\beta'|$ is bounded by a constant that does not depend on b, each c_j is a (p,r)-atom, and $\sum_j |\gamma_j|^p \le 1/2$, say. If one can do this, then one can repeat the process indefinitely to get a decomposition as in (16.17). In order to derive (16.18), the method of Calderón-Zygmund decompositions can be employed.

Recall that

$$\left(\sum_k \tau_k\right)^p \le \sum_k \tau_k^p \tag{16.19}$$

for nonnegative real numbers τ_k and $0 < p \le 1$. As a consequence, if $\{f_k\}$ is a family of measurable functions on M such that

$$\int_M |f_k(x)|^p \, d\mu(x) \le 1 \quad \text{for all } k, \tag{16.20}$$

and if $\{\theta_k\}$ is a family of constants, then

$$\int_M \left|\sum_k \theta_k \, f_k(x)\right|^p d\mu(x) \le \sum_k |\theta_k|^p. \tag{16.21}$$

Because of this, bounds on $\sum_l |\alpha_l|^p$ are natural when considering sums of the form $\sum_l \alpha_l \, a_l$, where the a_l's are (p, r)-atoms and the α_l's are constants.

A fundamental theorem concerning atoms is the following. Suppose that T is a bounded linear operator on $L^2(M)$ again. (One could start as well with a bounded linear operator on some other L^v space, with suitable adjustments.) Suppose also that there is a constant k so that (16.16) holds for all $(p, 2)$-atoms, where $0 < p \le 1$, as before, or even simply for all (p, ∞)-atoms. Then T determines a bounded linear operator on L^q for $1 < q < 2$. This indicates how atoms are sufficiently abundant to be useful.

The proof of this theorem relies on an argument like the one in Marcinkeiwicz interpolation. In the traditional setting, one of the main ingredients is to take a function f in L^q on M, and, for a given positive real number λ, write it as $f_1 + f_2$, where $f_1(x) = f(x)$ when $|f(x)| \le \lambda$, $f_1(x) = 0$ when $|f(x)| > 0$, $f_2(x) = f(x)$ when $|f(x)| > \lambda$, and $f_2(x) = 0$ when $|f(x)| \le \lambda$. Notice in particular that f_1 lies in L^w for all $w \ge q$, and that f_2 lies in L^u for all $u \le q$. For the present purposes, the idea is to use decompositions which are better behaved, with f_2 having a more precise form as a sum of multiples of atoms. The Calderón-Zygmund method is again applicable, although it should be mentioned that one first works with (p, r)-atoms with one choice of r, and then afterwards makes a conversion to a larger r using the results described before.

In addition to considering the effect of T on atoms, one can consider the effect of T^* on atoms, and this leads to conclusions about T on L^q for $q > 2$, by duality.

References

[1] F. J. Almgren, Jr., The homotopy groups of the integral cycle groups, *Topology* **1** (1962), 257–299.

[2] L. Ambrosio and F. Serra Cassano, eds., *"Lecture Notes on Analysis in Metric Spaces"*, Scuola Normale Superiore, Pisa, 2000.

[3] L. Ambrosio and P. Tilli, *"Selected Topics on 'Analysis on Metric Spaces'"*, Scuola Normale Superiore, Pisa, 2000.

[4] P. Assouad, Espaces métriques, plongements, facteurs, Thèse de doctorat, Publications Mathématiques d'Orsay **223-7769**, U.E.R. Mathématique, Université Paris XI, Orsay (1977).

[5] P. Assouad, Étude d'une dimension métrique liée à la possibilité de plongements dans $\mathbf{R}^n$, *C. R. Acad. Sci. Paris Sér. A-B* **288(15)** (1979), A731–A734.

[6] P. Assouad, Plongements lipschitziens dans $\mathbf{R}^n$, *Bull. Soc. Math. France* **111(4)** (1983), 429–448.

[7] M. Barlow, T. Coulhon and A. Grigor'yan, Manifolds and graphs with slow heat kernel decay, *Invent. Math.* **144(3)** (2001), 609–649.

[8] A. Bellaïche and J.-J. Risler, eds., *"Sub-Riemannian geometry"*, Progress in Mathematics **144**, Birkhäuser Verlag, Basel, 1996.

[9] V. N. Berestovskiĭ and A. M. Vershik, Manifolds with intrinsic metric, and nonholonomic spaces, in: *"Representation theory and dynamical systems"*, Adv. Soviet Math. **9**, Amer. Math. Soc., Providence, RI, 1992, pp. 253–267.

[10] R. H. Bing, *"The geometric topology of* 3*-manifolds"*, American Mathematical Society Colloquium Publications **40**, American Mathematical Society, Providence, RI, 1983.

[11] M. Bonk, J. Heinonen and P. Koskela, Uniformizing Gromov hyperbolic spaces, *Astérisque* **270** (2001), 99 pp.

[12] M. Bonk, J. Heinonen and S. Rohde, Doubling conformal densities, *J. Reine Angew. Math.* **541** (2001), 117–141.

[13] M. Bonk and B. Kleiner, Quasisymmetric parametrizations of two-dimensional metric spheres, *Invent. Math.* **150(1)** (2002), 127–183.

[14] A. Borel, Compact Clifford-Klein forms of symmetric spaces, *Topology* **2** (1963), 111–122.

[15] A. Borel, *"Semisimple groups and Riemannian symmetric spaces"*, Texts and Readings in Mathematics **16**, Hindustan Book Agency, New Delhi, 1998.

[16] M. Bourdon, Immeubles hyperboliques, dimension conforme et rigidité de Mostow, *Geom. Funct. Anal.* **7(2)** (1997), 245–268.

[17] M. Bourdon and H. Pajot, Poincaré inequalities and quasiconformal structure on the boundary of some hyperbolic buildings, *Proc. Amer. Math. Soc.* **127(8)** (1999), 2315–2324.

[18] R. R. Coifman and G. Weiss, *"Analyse harmonique non-commutative sur certains espaces homogènes"*, Lecture Notes in Mathematics **242**, Springer-Verlag, Berlin-New York, 1971.

[19] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83(4)** (1977), 569–645.

[20] M. Coornaert, Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov, *Pacific J. Math.* **159(2)** (1993), 241–270.

[21] M. Coornaert, T. Delzant and A. Papadopoulos, *"Géométrie et théorie des groupes. Les groupes hyperboliques de Gromov"*, Lecture Notes in Mathematics **1441**, Springer-Verlag, Berlin, 1990.

[22] M. Coornaert and A. Papadopoulos, *"Symbolic dynamics and hyperbolic groups"*, Lecture Notes in Mathematics **1539**, Springer-Verlag, Berlin, 1993.

[23] T. Coulhon and A. Grigor'yan, Pointwise estimates for transition probabilities of random walks on infinite graphs, in: *"Fractals in Graz 2001"*, Analysis-Dynamics-Geometry-Stochastics, W. Woess and P. Grabner, eds., Trends in Mathematics, Birkhäuser Boston, Boston, MA, 2003, pp. 119–134.

[24] B. E. J. Dahlberg and C. E. Kenig, Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains, *Ann. of Math. (2)* **125(3)** (1987), 437–465.

[25] D. Danielli, N. Garofalo and D.-M. Nhieu, Notions of convexity in Carnot groups, Preprint (2002).

[26] G. David and S. Semmes, *"Fractured fractals and broken dreams. Self-similar geometry through metric and measure"*, Oxford Lecture Series in Mathematics and its Applications **7**, The Clarendon Press, Oxford University Press, New York, 1997.

[27] R. A. DeVore and R. C. Sharpley, Maximal functions measuring smoothness, *Mem. Amer. Math. Soc.* **47(293)** (1984), 115 pp.

[28] S. K. Donaldson and D. P. Sullivan, Quasiconformal 4-manifolds, *Acta Math.* **163(3–4)** (1989), 181–252.

[29] J. Duoandikoetxea, *"Fourier analysis"*, Translated and revised from the 1995 Spanish original by David Cruz-Uribe, Graduate Studies in Mathematics **29**, American Mathematical Society, Providence, RI, 2001.

[30] P. L. Duren, *"Theory of H^p spaces"*, Pure and Applied Mathematics **38**, Academic Press, New York-London, 1970.

[31] K. J. Falconer, *"The geometry of fractal sets"*, Cambridge Tracts in Mathematics **85**, Cambridge University Press, Cambridge, 1986.

[32] H. Federer, *"Geometric measure theory"*, Die Grundlehren der mathematischen Wissenschaften **153**, Springer-Verlag New York Inc., New York, 1969.

[33] J. L. Fernández and J. M. Rodríguez, Area growth and Green's function of Riemann surfaces, *Ark. Mat.* **30(1)** (1992), 83–92.

[34] G. B. Folland and E. M. Stein, *"Hardy spaces on homogeneous groups"*, Mathematical Notes **28**, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982.

[35] M. H. Freedman and F. Quinn, *"Topology of 4-manifolds"*, Princeton Mathematical Series **39**, Princeton University Press, Princeton, NJ, 1990.

[36] J. García-Cuerva, Weighted Hardy spaces, in: *"Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1"*, Proc. Sympos. Pure Math. XXXV, Part, Amer. Math. Soc., Providence, R.I., 1979, pp. 253–261.

[37] J. García-Cuerva and J. L. Rubio de Francia, *"Weighted norm inequalities and related topics"*, North-Holland Mathematics Studies **116**, Notas de Matemática **104**, North-Holland Publishing Co., Amsterdam, 1985.

[38] É. Ghys and P. de la Harpe, eds., *"Sur les groupes hyperboliques d'après Mikhael Gromov"*, Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988, Progress in Mathematics **83**, Birkhäuser Boston, Inc., Boston, MA, 1990.

[39] R. Goldblatt, *"Lectures on the hyperreals"*, An introduction to nonstandard analysis, Graduate Texts in Mathematics **188**, Springer-Verlag, New York, 1998.

[40] A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, *Bull. Amer. Math. Soc. (N.S.)* **36(2)** (1999), 135–249.

[41] A. GRIGOR'YAN, J. HU AND K.-S. LAU, Heat kernels on metric measure spaces and an application to semilinear elliptic equations, *Trans. Amer. Math. Soc.* **355(5)** (2003), 2065–2095 (electronic).

[42] M. GROMOV, Hyperbolic manifolds, groups and actions, in: *"Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)"*, Ann. of Math. Stud. **97**, Princeton Univ. Press, Princeton, N.J., 1981, pp. 183–213.

[43] M. GROMOV, Groups of polynomial growth and expanding maps, *Inst. Hautes Études Sci. Publ. Math.* **53** (1981), 53–73.

[44] M. GROMOV, Hyperbolic groups, in: *"Essays in group theory"*, Math. Sci. Res. Inst. Publ. **8**, Springer, New York, 1987, pp. 75–263.

[45] M. GROMOV AND P. PANSU, Rigidity of lattices: an introduction, in: *"Geometric topology: recent developments"* (Montecatini Terme, 1990), Lecture Notes in Math. **1504**, Springer, Berlin, 1991, pp. 39–137.

[46] P. HAJŁASZ AND P. KOSKELA, Sobolev met Poincaré, *Mem. Amer. Math. Soc.* **145(688)** (2000), 101 pp.

[47] J. HEINONEN, Calculus on Carnot groups, in: *"Fall School in Analysis"* (Jyväskylä, 1994), Report **68**, Univ. Jyväskylä, Jyväskylä, 1995, pp. 1–31.

[48] J. HEINONEN, *"Lectures on analysis on metric spaces"*, Universitext, Springer-Verlag, New York, 2001.

[49] J. HEINONEN AND P. KOSKELA, Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.* **181(1)** (1998), 1–61.

[50] J.-L. JOURNÉ, *"Calderón-Zygmund operators, pseudodifferential operators and the Cauchy integral of Calderón"*, Lecture Notes in Mathematics **994**, Springer-Verlag, Berlin, 1983.

[51] J. KIGAMI, *"Analysis on fractals"*, Cambridge Tracts in Mathematics **143**, Cambridge University Press, Cambridge, 2001.

[52] A. KORÁNYI AND H. M. REIMANN, Quasiconformal mappings on the Heisenberg group, *Invent. Math.* **80(2)** (1985), 309–338.

[53] A. KORÁNYI AND H. M. REIMANN, The complex cross ratio on the Heisenberg group, *Enseign. Math. (2)* **33(3–4)** (1987), 291–300.

[54] A. KORÁNYI AND H. M. REIMANN, Quasiconformal mappings on CR manifolds, in: *"Complex geometry and analysis"* (Pisa, 1988), Lecture Notes in Math. **1422**, Springer, Berlin, 1990, pp. 59–75.

[55] A. KORÁNYI AND H. M. REIMANN, Foundations for the theory of quasiconformal mappings on the Heisenberg group, *Adv. Math.* **111(1)** (1995), 1–87.

[56] T. W. Körner, *“Fourier analysis”*, Second edition, Cambridge University Press, Cambridge, 1989.

[57] S. G. Krantz, Lipschitz spaces, smoothness of functions, and approximation theory, *Exposition. Math.* **1(3)** (1983), 193–260.

[58] S. G. Krantz, *“Geometric analysis and function spaces”*, CBMS Regional Conference Series in Mathematics **81**, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993.

[59] S. G. Krantz, *“A panorama of harmonic analysis”*, Carus Mathematical Monographs **27**, Mathematical Association of America, Washington, DC, 1999.

[60] S. G. Krantz, *“Function theory of several complex variables”*, Reprint of the 1992 edition, AMS Chelsea Publishing, Providence, RI, 2001.

[61] S. G. Krantz and H. R. Parks, *“The geometry of domains in space”*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Boston, Inc., Boston, MA, 1999.

[62] T. J. Laakso, Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality, *Geom. Funct. Anal.* **10(1)** (2000), 111–123; Erratum, **12(3)** (2002), 650.

[63] T. J. Laakso, Plane with A_∞-weighted metric not bi-Lipschitz embeddable to $\mathbb{R}^N$, *Bull. London Math. Soc.* **34(6)** (2002), 667–676.

[64] G. Lu, J. J. Manfredi and B. Stroffolini, Convex functions on the Heisenberg group, Preprint (2002).

[65] J. Luukkainen, Assouad dimension: antifractal metrization, porous sets, and homogeneous measures, *J. Korean Math. Soc.* **35(1)** (1998), 23–76.

[66] R. A. Macías and C. Segovia, Lipschitz functions on spaces of homogeneous type, *Adv. in Math.* **33(3)** (1979), 257–270.

[67] R. A. Macías and C. Segovia, A decomposition into atoms of distributions on spaces of homogeneous type, *Adv. in Math.* **33(3)** (1979), 271–309.

[68] V. Magnani, Elements of geometric measure theory on sub-Riemannian groups, Thesis, Scuola Normale Superiore, Pisa (2002).

[69] P. Mattila, *“Geometry of sets and measures in Euclidean spaces”*, Fractals and rectifiability, Cambridge Studies in Advanced Mathematics **44**, Cambridge University Press, Cambridge, 1995.

[70] J. Mitchell, On Carnot-Carathéodory metrics, *J. Differential Geom.* **21(1)** (1985), 35–45.

[71] E. E. Moise, *"Geometric topology in dimensions* 2 *and* 3*"*, Graduate Texts in Mathematics **47**, Springer-Verlag, New York-Heidelberg, 1977.

[72] R. Montgomery, *"A tour of subriemannian geometries, their geodesics and applications"*, Mathematical Surveys and Monographs **91**, American Mathematical Society, Providence, RI, 2002.

[73] R. Monti, Distances, boundaries, and surface measures in Carnot-Carathéodory spaces, Thesis, Università di Trento (2001).

[74] F. Morgan, *"Geometric measure theory. A beginner's guide"*, Academic Press, Inc., Boston, MA, 1988.

[75] P. Pansu, Dimension conforme et sphère à l'infini des variétés à courbure négative, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **14(2)** (1989), 177–212.

[76] P. Pansu, Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un, *Ann. of Math. (2)* **129(1)** (1989), 1–60.

[77] M. Picardello and W. Woess, eds., *"Random walks and discrete potential theory"*, Proceedings of the conference held in Cortona, June 1997, Symposia Mathematica **XXXIX**, Cambridge University Press, Cambridge, 1999.

[78] M. S. Raghunathan, *"Discrete subgroups of Lie groups"*, Ergebnisse der Mathematik und ihrer Grenzgebiete **68**, Springer-Verlag, New York-Heidelberg, 1972.

[79] W. Rudin, *"Principles of mathematical analysis"*, Third edition, International Series in Pure and Applied Mathematics, McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976.

[80] W. Rudin, *"Function theory in the unit ball of* $\mathbf{C}^n$*"*, Grundlehren der Mathematischen Wissenschaften **241**, Springer-Verlag, New York-Berlin, 1980.

[81] D. Sarason, *"Function theory on the unit circle"*, Notes for lectures given at a Conference at Virginia Polytechnic Institute and State University, Blacksburg, Va., June 19–23, 1978, Virginia Polytechnic Institute and State University, Department of Mathematics, Blacksburg, Va., 1978.

[82] S. Semmes, Noyaux de la chaleur, marches aléatoires, analyse sur les variétés et les graphes, *Gaz. Math.* **95** (2003), 9–28.

[83] S. Semmes, An introduction to analysis on metric spaces, *Notices Amer. Math. Soc.* **50(4)** (2003), 438–443.

[84] S. Semmes, An introduction to Heisenberg groups in Analysis and Geometry, *Notices Amer. Math. Soc.* **50(6)** (2003), 640–646.

[85] E. M. STEIN, *"Singular integrals and differentiability properties of functions"*, Princeton Mathematical Series **30**, Princeton University Press, Princeton, N.J., 1970.

[86] E. M. STEIN, *"Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals"*, Princeton Mathematical Series **43**, Monographs in Harmonic Analysis III, Princeton University Press, Princeton, NJ, 1993.

[87] E. M. STEIN AND G. WEISS, *"Introduction to Fourier analysis on Euclidean spaces"*, Princeton Mathematical Series **32**, Princeton University Press, Princeton, N.J., 1971.

[88] R. S. STRICHARTZ, Sub-Riemannian geometry, *J. Differential Geom.* **24(2)** (1986), 221–263; Corrections, **30(2)** (1989), 595–596.

[89] R. S. STRICHARTZ, Analysis on fractals, *Notices Amer. Math. Soc.* **46(10)** (1999), 1199–1208.

[90] J.-O. STRÖMBERG AND A. TORCHINSKY, *"Weighted Hardy spaces"*, Lecture Notes in Mathematics **1381**, Springer-Verlag, Berlin, 1989.

[91] D. P. SULLIVAN, Hyperbolic geometry and homeomorphisms, in: *"Geometric topology"* (Proc. Georgia Topology Conf., Athens, Ga., 1977), Academic Press, New York-London, 1979, pp. 543–555.

[92] M. H. TAIBLESON AND G. WEISS, The molecular characterization of certain Hardy spaces, in: *"Representation theorems for Hardy spaces"*, Astérisque **77**, Soc. Math. France, Paris, 1980, pp. 67–149.

[93] F. J. THAYER, Nonstandard analysis of graphs, *Houston J. Math.* (to appear).

[94] A. TORCHINSKY, *"Real-variable methods in harmonic analysis"*, Pure and Applied Mathematics **123**, Academic Press, Inc., Orlando, FL, 1986.

[95] N. TH. VAROPOULOS, L. SALOFF-COSTE AND T. COULHON, *"Analysis and geometry on groups"*, Cambridge Tracts in Mathematics **100**, Cambridge University Press, Cambridge, 1992.

[96] W. WOESS, *"Random walks on infinite graphs and groups"*, Cambridge Tracts in Mathematics **138**, Cambridge University Press, Cambridge, 2000.

[97] A. H. ZEMANIAN, Nonstandard versions of conventionally infinite networks, *IEEE Trans. Circuits Systems I Fund. Theory Appl.* **48(10)** (2001), 1261–1265.

[98] A. H. ZEMANIAN, Hyperreal transients in transfinite RLC networks, *Internat. J. Circuit Theory Appl.* **29(6)** (2001), 591–605.

[99] A. H. ZEMANIAN, Nonstandard graphs, Preprint (2002).

[100] A. Zygmund, *“Trigonometric series. Vol. I, II”*, Reprinting of the 1968 version of the second edition with Volumes I and II bound together, Cambridge University Press, Cambridge-New York-Melbourne, 1977.

Department of Mathematics
Rice University
Box 1892
Houston TX 77251-1892
U.S.A.
E-mail address: semmes@math.rice.edu

Primera versió rebuda el 25 de novembre de 2002,
darrera versió rebuda el 28 de març de 2003.