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WEIGHTED L^p ESTIMATES FOR THE $\overline{\partial}$-EQUATION ON CONVEX DOMAINS OF FINITE TYPE

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Abstract

We prove non-isotropic L^p $(1 \leq p \leq \infty)$ estimates with weights for solutions of the Cauchy-Riemann equation on bounded convex domains of finite type in $\mathbb{C}^n$ using the integral kernel method. We also give an example which guarantees the optimum of the estimates.

1. Introduction and statement of results

Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain of finite type m with a defining function ρ. In this paper we treat a certain weighted L^p estimates for the $\overline{\partial}$-equation on Ω. The notation $\delta(\zeta)$ will stand for the distance from ζ to the boundary of Ω, $b\Omega$, which is up to constants $|\rho(\zeta)|$. With solutions using the integral kernel introduced by Cumenge [**Cum01a**] we can prove the following theorem.

Theorem 1.1. *For the domain Ω as above the equation $\overline{\partial}u = f$ has a solution u in Ω such that for $1 \leq p < \infty$*

$$\int_\Omega \delta(\zeta)^{\alpha-1}|u(\zeta)|^p \, dV(\zeta) \leq C_{p,\alpha} \int_\Omega \delta(\zeta)^{\alpha-1+p}||f(\zeta)||^p \, dV(\zeta), \quad \alpha > 0 \tag{1}$$

and

$$\sup_{\zeta\in\Omega} \delta(\zeta)^{\alpha-1}|u(\zeta)| \leq C_\alpha \sup_{\zeta\in\Omega} \delta(\zeta)^{\alpha}||f(\zeta)||, \quad \alpha > 1, \tag{2}$$

if f is a smooth $(n,1)$-form with $\overline{\partial}f = 0$ and the right hand sides of (1) *and* (2) *are finite. Here the non-isotropic norm of forms, $||\cdot|| = ||\cdot||_\Omega$ is defined by $||f(\zeta)|| = \sup_{v\in\mathbb{C}^n\setminus\{0\}} |f(\zeta)(v)|/k(\zeta,v)$, where $k(\zeta,v)^{-1}$ is a weighted boundary distance of $\zeta \in \Omega$ in the direction v.*

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Remark. (i) The non-isotropic norm $||\cdot||$ was first introduced by Bruna, Charpentier and Dupain [**BCD98**]. The definition of the quantity $k(\zeta, v)$ is rather complicate even though $k(\zeta, v)^{-1}$ is a natural weighted boundary distance, so we postpone the precise definition in the next section. (ii) Theorem 1.1 implies that for given a smooth $(0,1)$-form f with $\overline{\partial} f = 0$ one can find a solution for the $\overline{\partial}$-equation on Ω and that solution satisfies the inequalities (1) and (2).

If Ω is strongly pseudoconvex, (1) and (2) were proved by Ahn-Cho [**AC02**] and Dautov-Henkin [**DH79**]. When the domain is convex of finite type, Cumenge [**Cum01a**] proved (1) in case $p = 1$. Therefore it seems natural to extend Cumenge's result to other L^p-norms, $1 \leq p \leq \infty$. There are a number of papers related to the $\overline{\partial}$-equation on convex domains of finite type m. We mention a few of them, which are closely related to our work; Diederich-Fischer-Fornæss [**DFF99**], Cumenge [**Cum01a**] independently proved $1/m$-Hölder estimates and Cumenge [**Cum01a**], Fischer [**Fis01**] obtained the best possible L^p estimates with respect to the isotropic norm. Diederich-Mazzilli [**DM01**] (resp. Cumenge [**Cum01b**]) obtained the characterization of the zero sets of functions in the Nevanlinna (resp. Nevanlinna-Djrbachian) classes using non-isotropic $L^1(b\Omega)$ (resp. weighted nonisotropic $L^1(\Omega)$) estimates for the solution of the $\overline{\partial}$-equation on Ω.

Here we briefly sketch the methods used to prove Theorem 1.1. First, to obtain the solution of $\overline{\partial}$ on Ω we use an integral representation introduced by Berndtsson-Andersson [**BA82**]. In the integral representation the key part is a kernel with a weight containing the Bergman kernel of Ω. Second, for the estimates of the kernel and integrals relevant to the kernel we use non-isotropic polydiscs and ε-extremal coordinates of McNeal on Ω [**McN94**]. Last, in order to pass into L^p estimates from L^1 estimates we employ a variation of Hölder's inequality used in [**AC02**].

2. Preliminaries

2.1. Integral kernels and solution operators. Let $B(z, \zeta)$ be the Bergman kernel for the domain Ω. Then $B(z, \zeta)$ is holomorphic in z and antiholomorphic in ζ. Moreover $B(z, \zeta) \in C^\infty(\overline{\Omega} \times \overline{\Omega} \setminus \{(\zeta, \zeta),\ \zeta \in b\Omega\})$ and the boundary behavior of $B(z, \zeta)$ is now well understood by [**McN94**]. We define

$$Q = Q(z, \zeta) = \frac{1}{B(\zeta, \zeta)} \sum_{j=1}^{n} \left(\int_0^1 \frac{\partial B}{\partial z_j}(z_t, \zeta)\, dt \right) dz_j, \quad z_t = \zeta + t(z - \zeta).$$

For sufficiently large integer N to be determined and fixed later we define the weighed kernel on $(z,\zeta)\in\overline{\Omega}\times\Omega\setminus\{(\zeta,\zeta),\ \zeta\in b\Omega\}$,

$$\begin{aligned}K(z,\zeta)&=\sum_{k=0}^{n-1}c_{n,k,N}\left(\frac{B(z,\zeta)}{B(\zeta,\zeta)}\right)^{N-k}\frac{\partial_z|z-\zeta|^2\wedge(\overline{\partial}_\zeta Q)^k\wedge(d\partial_z|\zeta-z|)^{n-k-1}}{|\zeta-z|^{2n-2k}}\\&=\sum_{k=0}^{n-1}c_{n,k,N}K^{(k)}(z,\zeta),\end{aligned}$$

where $c_{n,k,N}=-(-1)^{n(n-1)/2}\binom{N}{n}$ and $K^{(k)}(z,\zeta)$ is the k-th term in the summation. For $f\in C^1_{(n,1)}(\Omega)$ with $\overline{\partial}f=0$, if we define

$$u(z)=C\int_\Omega K(z,\zeta)\wedge f(\zeta),\quad z\in\Omega \tag{3}$$

then it is known that $\overline{\partial}u=f$ [**Cum01a**].

2.2. McNeal's result on the geometry of convex domains of finite type. We adapt to the notation of [**Cum01a**] and [**McN94**].

2.2.1. Weighted boundary distance. Define the radius of the largest complex disc centered at z in the direction v that fits in the domain $\{z:\rho(z)<\rho(\zeta)+\varepsilon\}$,

$$\tau(z,v,\varepsilon)=\sup\{r>0:|\rho(z+\lambda v)-\rho(z)|\le\varepsilon,\ |\lambda|\le r,\ \lambda\in\mathbb{C}\}.$$

Introduce a weighted boundary distance $k(z,v,\varepsilon)=\delta(z)/\tau(z,v,\varepsilon)$ and write $k(z,v)$ when $\varepsilon=\delta(z)/2$.

2.2.2. Non-isotropic polydisc. If $\{v_1,\dots,v_n\}$ is an ε-extremal basis of McNeal at z (see [**McN94**] and [**BCD98**] for the precise definition), then the non-isotropic polydisc at z with radius ε is defined by

$$P(z,\varepsilon)=\left\{w=z+\sum_{j=1}^n w_jv_j,\ |w_j|\le c\tau(z,v_j,\varepsilon)\right\},$$

where c is chosen so that $w\in P(z,\varepsilon)$ implies $|\rho(w)-\rho(z)|<\varepsilon$. The properties of τ which were proved by McNeal [**McN94**] are the following:

$$\tau(z,v_1,\varepsilon)\approx\varepsilon\lesssim\tau(z,v_n,\varepsilon)\le\cdots\le\tau(z,v_2,\varepsilon)\lesssim\varepsilon^{\frac{1}{m}} \tag{4}$$

and for $0<\varepsilon_1\le\varepsilon_2$

$$\left(\frac{\varepsilon_1}{\varepsilon_2}\right)\tau(z,v,\varepsilon_2)\lesssim\tau(z,v,\varepsilon_1)\lesssim\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{\frac{1}{m}}\tau(z,v,\varepsilon_2). \tag{5}$$

Proposition 2.1 ([**McN94**]).

(i) *For all* $C > 0$, $\operatorname{vol} P(z, C\varepsilon) \approx \operatorname{vol} P(z,\varepsilon)$ *uniformly in* z, ε *with constants depending on* C.
(ii) $\operatorname{vol} P(z,\varepsilon) \approx \operatorname{vol} P(\zeta,\varepsilon)$ *if* $P(z,\varepsilon) \cap P(\zeta,\varepsilon) \neq \emptyset$.
(iii) $\tau(\zeta, v, \varepsilon) \approx \tau(z, v, \varepsilon)$ *for* $\zeta \in P(z,\varepsilon)$.
(iv) *If* $\{v_1, \dots, v_n\}$ *is an* ε*-extremal basis at* z*, then we have* $\operatorname{vol} P(z,\varepsilon) \approx \prod_{i=1}^n \tau(z, v_i, \varepsilon)^2$.

2.2.3. Tent and quasi distance. For $z \in \overline{\Omega}$ close to the boundary and $\eta > 0$, $T(z,\eta) = P(\pi(z),\eta) \cap \Omega$ is called the tent at z of radius η, where $\pi(z)$ is the projection of z to the boundary. The quasi distance of McNeal is defined as follows:

$$\mathcal{M}(z,\zeta) \approx \mathcal{M}(\zeta,z) = \inf\{\eta : \zeta \in P(z,\eta)\}$$

for $|\zeta - z| \ll 1$ and z close to $b\Omega$. For $v \in \mathbb{C}^n$ with $|v| = 1$, $\varphi \in C^\infty(\Omega)$ let $D_v\varphi$ denote the directional derivatives of φ in the direction v. From now on set $\eta = \eta(z,\zeta) = |\rho(z)| + |\rho(\zeta)| + \mathcal{M}(z,\zeta)$. Then the following proposition is proved by McNeal [**McN94**].

Proposition 2.2. *For every* $p \in b\Omega$ *there exists a neighborhood* U *of* p *such that for* $\zeta, z \in U \cap \Omega$, $\mu, \nu \in \mathbb{N}$, $v, v' \in \mathbb{C}^n$ *with* $|v| = |v'| = 1$,

(i) $|D_v^\mu \overline{D}_{v'}^\nu B(z,\zeta)| \le C(\mu,\nu)\tau(\zeta,v,\eta)^{-\mu}\tau(\zeta,v',\eta)^{-\nu}(\operatorname{vol} T_{\zeta,z})^{-1}$. *Here* $\operatorname{vol} T_{\zeta,z}$ *is the volume of the smallest tent containing both* z, ζ.
(ii) *For* $\zeta \in U \cap \Omega$, $B(\zeta,\zeta) \ge (\operatorname{vol} P(\zeta,\delta))^{-1}$, $\delta = \delta(\zeta) = |\rho(\zeta)|/2$.

2.2.4. Coverings. There exists a constant $\beta > 1$ such that

$$\mathcal{M}(z,\zeta) < \varepsilon \Rightarrow \zeta \in P(z,\beta\varepsilon), \quad z \in U \cap \Omega,\ 0 < \varepsilon \ll 1,$$

where U is some neighborhood defined in Proposition 2.2. For the integral estimates we define the covering of $W \cap \Omega$

$$\mathcal{C}_0(z) = P(z, \beta\, d(z)) \cap W \cap \Omega,$$

$$\mathcal{C}_\ell(z) = \{\zeta \in \Omega \cap W : 2^{\ell-1}\, d(z) \le \mathcal{M}(z,\zeta) < 2^\ell\, d(z)\}, \quad \ell \ge 1,$$

where $W = 1/2U$.

3. Kernel estimates

3.1. Estimate of the term $K^{(k)}(z,\zeta)$, $0 \le k \le n-1$. Now we want to write all forms with respect to extremal coordinates of McNeal at ζ or z. Let $\{e_j^{(\ell)} = e_j^{(\ell)}(\zeta),\ 1 \le j \le n\}$ be a $\beta 2^\ell\delta$-extremal basis at ζ and $(e_j^\ell(z))_j$ a $\beta 2^\ell d$-extremal basis at z, respectively. If $v_j = v_j^{(\ell)}(\zeta)$ is the

j-th component of their coordinates, we denote $L_j^{(\ell)} = \partial/\partial v_j$ and $L_j^{(\ell)*}$ which is the dual of $L_j^{(\ell)}$. To simplify notations, in any ambiguous case, we write $L_j^{(z)}$, $\overline{L}_j^{(\zeta)}$, $L_j^{*(z)}$, $\overline{L}_j^{*(\zeta)}$ for $L_j^{(\ell)(z)}$, $\overline{L}_j^{(\ell)(\zeta)}$, $L_j^{(\ell)*(z)}$, $\overline{L}_j^{(\ell)*(\zeta)}$, $1 \le j \le n$, where the superscripts z, ζ mean the derivations act on the variables z, ζ, respectively. To save us from confusion, we denote $\operatorname{dist}(\zeta, \Omega)$ and $\operatorname{dist}(z, b\Omega)$ by $\delta = \delta(\zeta)$ and $d = d(z)$, respectively.

First we estimate $K^{(k)}(z,\zeta) \wedge f(\zeta)$, $1 \le k \le n-1$. Let $\mathcal{R} = \int_0^1 \partial_z B(z_t,\zeta)\, dt$. Then computing the k-th exterior product we obtain

$$\left(\overline{\partial}_\zeta Q\right)^k = c_k \frac{\overline{\partial}_\zeta B(\zeta,\zeta)}{B(\zeta,\zeta)^{k+1}} \wedge \mathcal{R} \wedge (\overline{\partial}_\zeta \mathcal{R})^{k-1} + \frac{\left(\overline{\partial}_\zeta \mathcal{R}\right)^k}{B(\zeta,\zeta)^k}.$$

Using this and expressing all forms in terms of $\overline{L}_j^{(\zeta)}$ and $L_j^{(z)}$'s, we have

$$K^{(k)}(z,\zeta) \wedge f(\zeta) = \frac{B(z,\zeta)^{N-k}}{B(\zeta,\zeta)^N} \frac{1}{|\zeta - z|^{2n-2k}} \left[H_1 + H_2\right],$$

where

$$\begin{aligned} H_1 = \frac{1}{B(\zeta,\zeta)} \sum_{I,J} L_{i_0}^{(z)} |\zeta - z|^2 \big(\overline{L}_{j_1}^{(\zeta)} B\big)(\zeta,\zeta) \int_0^1 \big(L_{i_1}^{(z)} B\big)(z_t,\zeta)\, dt \\ \times \prod_{\nu=2}^{k} \left(\int_0^1 \big(\overline{L}_{j_\nu}^{(\zeta)} L_{i_\nu}^{(z)} B\big)(z_t,\zeta)\, dt \right) \left[f(\zeta)(e_{j_n}^{(\ell)})\right] \\ \times A_{IJS}(z,\zeta)\, L_S^{*(\zeta)} \wedge \overline{L}_J^{*(\zeta)} \wedge L_I^{*(z)} \end{aligned}$$

$$\begin{aligned} H_2 = \sum_{I,J} L_{i_0}^{(z)} |\zeta - z|^2 \prod_{\nu=1}^{k} \left(\int_0^1 \big(\overline{L}_{j_\nu}^{(\zeta)} L_{i_\nu}^{(z)} B\big)(z_t,\zeta)\, dt \right) \left[f(\zeta)(e_{j_n}^{(\ell)})\right] \\ \times A_{IJS}(z,\zeta)\, L_S^{*(\zeta)} \wedge \overline{L}_J^{*(\zeta)} \wedge L_I^{*(z)}. \end{aligned}$$

Here $I = I(k) = I_k \cup I_k{}'$, $I_k = \{i_0, i_1, \dots, i_k\}$ and $I_k{}' = \{i_{k+1}, \dots, i_{n-1}\}$; $J = J(k) = J_k \cup J_k{}'$, $J_k = \{j_1, j_2, \dots, j_k\}$ and $J_k{}' = \{j_{k+1}, \dots, j_n\}$, $i_\nu, j_\nu \in S = \{1, 2, \dots, n\}$, $L_I^{*(z)} = L_{i_0}^{*(z)} \wedge \cdots \wedge L_{i_{n-1}}^{*(z)}$, etc. and A_{IJS} is uniformly bounded on $\Omega \times \Omega$. Note that if $k = 1$, then the product term $\prod_{\nu=2}^k (\dots)$ of H_1 does not appear.

Next we note the following inequality

$$|f(\zeta)(e_j^{(\ell)}(\zeta))| \le ||f(\zeta)|| \frac{\delta(\zeta)}{\tau(\zeta, e_j^{(\ell)}(\zeta), \delta)} \lesssim ||f(\zeta)||. \tag{6}$$

Then it is easy to see that

$$\begin{aligned}\left|K^{(0)}(z,\zeta)\wedge f(\zeta)\right| &\le ||f(\zeta)||\left|K^{(0)}(z,\zeta)\right|\\ &\le ||f(\zeta)||\left(\frac{|B(z,\zeta)|}{B(\zeta,\zeta)}\right)^N\frac{1}{|\zeta-z|^{2n-1}}.\end{aligned}$$

First we estimate $|B(z,\zeta)|/B(\zeta,\zeta)$, $\int_0^1 L_{i_1}^{(z)}B(z_t,\zeta)\,dt$ and $\int_0^1 \overline{L}_{j_n u}^{(\zeta)}L_{i_\nu}^{(z)}B(z_t,\zeta)\,dt$ on the neighborhood U of $p\in b\Omega$, where U is the neighborhood defined in Proposition 2.2. By Proposition 2.2 (ii), (4) and (5) the following estimates can be proved:

$$\frac{|B(z,\zeta)|}{B(\zeta,\zeta)}\le\frac{\delta(\zeta)}{\eta(z,\zeta)}$$

$$\left|\int_0^1 L_{i_1}^{(z)}B(z_t,\zeta)\,dt\right|\lesssim\left[\operatorname{vol}P(\zeta,\delta)\tau(\zeta,e_{i_1}^{(\ell)},\delta)\right]^{-1}$$

$$\left|\int_0^1 \overline{L}_{j_\nu}^{(\zeta)}L_{i_\nu}^{(z)}B(z_t,\zeta)\,dt\right|\lesssim\left[\operatorname{vol}P(\zeta,\delta)\tau(\zeta,e_{j_\nu}^{(\ell)},\delta)\tau(\zeta,e_{i_\nu}^{(\ell)},\delta)\right]^{-1}.$$

(For details see [**Cum01a**].) Combining above estimates and the inequality (6) we have for $1\le k\le n-1$

$$\begin{aligned}&\left|K^{(k)}(z,\zeta)\wedge f(\zeta)\right|\\ &\lesssim\sum_{I_k,J_k}\frac{\delta(\zeta)^{N-k}}{\eta(z,\zeta)^{N-k}}\frac{\delta(\zeta)||f(\zeta)||}{|\zeta-z|^{2n-2k-1}}\frac{1}{\prod_{\nu=1}^k\tau(\zeta,e_{j_\nu}^{(\ell)},\delta)\tau(\zeta,e_{i_\nu}^{(\ell)},\delta)}\frac{1}{\tau(\zeta,e_{j_n}^{(\ell)},\delta)}\end{aligned}$$

and

$$\left|K^{(0)}(z,\zeta)\wedge f(\zeta)\right|\lesssim\frac{\delta(\zeta)^{N-1}}{\eta(z,\zeta)^N}\frac{\delta(\zeta)||f(\zeta)||}{|\zeta-z|^{2n-1}}.$$

For the simplification of notation we write for $z,\zeta\in U$

$$\begin{aligned}K_+^{(0)}(z,\zeta)&=\frac{\delta(\zeta)^{N-1}}{\eta(z,\zeta)^N}\frac{1}{|\zeta-z|^{2n-1}}\\ K_+^{(k)}(z,\zeta)&=\frac{\delta(\zeta)^{N-k}}{\eta(z,\zeta)^{N-k}}\frac{1}{|\zeta-z|^{2n-2k-1}}\\ &\quad\times\frac{1}{\prod_{\nu=1}^k\tau(\zeta,e_{j_\nu}^{(\ell)},\delta)\tau(\zeta,e_{i_\nu}^{(\ell)},\delta)}\frac{1}{\tau(\zeta,e_{j_n}^{(\ell)},\delta)}.\end{aligned}$$

3.2. Estimates of $K_+^{(k)}(z,\zeta)$ on coverings $\{\mathcal{C}_\ell(z)\}$ and $\{\mathcal{C}_\ell(\zeta)\}$. We only consider the estimates of $K_+^{(k)}(z,\zeta)$ for $k=1,n-1$. In the integral estimates other cases can be reduced to cases $k=1,n-1$. If $k=0$, we can directly estimate integrals on the neighborhood W. First assume that $\zeta\in U$ is fixed and we will estimate $K_+^{(k)}(z,\zeta)$ on $\mathcal{C}_\ell(\zeta)$. By the definition of $\mathcal{C}_\ell(\zeta)$, $\mathcal{M}(z,\zeta)$ and (5), it is easy to see that

$$\eta(z,\zeta)\approx 2^\ell\delta(\zeta), \qquad\qquad z\in\mathcal{C}_\ell(\zeta)$$
$$\tau(\zeta,e_j^{(\ell)}(\zeta),\delta)\gtrsim 2^{-\ell}\tau(\zeta,e_j^{(\ell)}(\zeta),\beta 2^\ell\delta).$$

Note that $\tau_{j_n}^{(\ell)}(\zeta)\le\tau_2^{(\ell)}(\zeta)$. Therefore we have for $z\in\mathcal{C}_\ell(\zeta)$

$$K_+^{(n-1)}(z,\zeta)\lesssim\frac{\tau_2^{(\ell)}(\zeta)}{2^{(N-3n+2)\ell}|\zeta-z|\prod_{j=1}^n\tau_j^{(\ell)}(\zeta)^2} \tag{7}$$

$$K_+^{(1)}(z,\zeta)\lesssim\sum_{I_1',J_1'}\frac{\tau_{i_0}^{(\ell)}\prod_{\nu=2}^{n-1}\tau_{i_\nu}^{(\ell)}(\zeta)\tau_{j_\nu}^{(\ell)}(\zeta)}{2^{(N-4)\ell}\prod_{j=1}^n\tau_j^{(\ell)}(\zeta)^2|\zeta-z|^{2n-3}}, \tag{8}$$

where $\tau_j^{(\ell)}(\zeta)=\tau(\zeta,e_j^{(\ell)}(\zeta),\beta 2^\ell\delta(\zeta))$. Next assume that $z\in U$ is fixed and we will estimate $K^{(k)}(z,\zeta)$ on $\mathcal{C}_\ell(z)$. By Proposition 2.1 (ii) and (iii), we have

$$\tau(\zeta,e_j^{(\ell)},\delta)\gtrsim\left(\frac{\delta}{2^\ell d}\right)\tau(\zeta,e_j^{(\ell)},\beta 2^\ell\delta)\approx\left(\frac{\delta}{2^\ell d}\right)\tau(z,e_j^{(\ell)}(z),\beta 2^\ell d).$$

Since $\eta(z,\zeta)\approx 2^\ell\, d(z)$, $\zeta\in\mathcal{C}_\ell(z)$ we have

$$K_+^{(n-1)}(z,\zeta)\lesssim\left(\frac{\delta}{2^\ell d}\right)^{N-3n+2}\frac{1}{|\zeta-z|}\frac{\tau_2^{(\ell)}(z)}{\prod_{j=1}^n\tau_j^{(\ell)}(z)^2} \tag{9}$$

and

$$\begin{aligned}K_+^{(1)}(z,\zeta)&\lesssim\sum_{\substack{i,j,p=1\\ i\neq p}}^n\left(\frac{\delta}{2^\ell d}\right)^{N-4}\frac{1}{|\zeta-z|^{2n-3}}\frac{1}{\tau_j^{(\ell)}(z)\tau_i^{(\ell)}(z)\tau_p^{(\ell)}(z)}\\ &\lesssim\sum_{I_1',J_1'}\left(\frac{\delta}{2^\ell d}\right)^{N-4}\frac{1}{|\zeta-z|^{2n-3}}\frac{\tau_{i_0}^{(\ell)}(z)\prod_{\nu=2}^{n-1}\tau_{i_\nu}^{(\ell)}(z)\tau_{j_\nu}^{(\ell)}(z)}{\prod_{j=1}^n\tau_j^{(\ell)}(z)^2}.\end{aligned} \tag{10}$$

4. Integral estimates

In this section we verify preliminary integral estimates that are an essential step to prove our Theorem 1.1.

Lemma 4.1. *Let $\alpha > 0$ and $\varepsilon > 0$ with $\alpha - 1 - \varepsilon > -1$. Then for $k = 0, 1, \ldots, n-1$ we have*

$$\int_\Omega d(z)^{\alpha-1-\varepsilon} K_+^{(k)}(z,\zeta)\, dV(z) \le C_{\alpha,\varepsilon}\delta(\zeta)^{\alpha-\varepsilon-1} \tag{11}$$

$$\int_\Omega \delta(\zeta)^{-\varepsilon} K_+^{(k)}(z,\zeta)\, dV(\zeta) \le C_\varepsilon d(z)^{-\varepsilon}. \tag{12}$$

Proof: We prove (11) and (12) for $k = 0, 1, n-1$. The other cases can be reduced to the cases $k = 0, 1, n-1$. Since the only singularity is of the form $|\zeta - z|^{-j}$, we may assume that $z, \zeta \in W$. W can be covered by $\cup_\ell \mathcal{C}_\ell(z)$ and $\cup_\ell \mathcal{C}_\ell(\zeta)$ so basically we have to deal with the domain of the form $\mathcal{C}_\ell(z)$ or $\mathcal{C}_\ell(\zeta)$.

(i) By the estimate (7) we have for any integer $\ell \ge 0$

$$\begin{aligned} &\int_{\mathcal{C}_\ell(\zeta)} d(z)^{\alpha-1-\varepsilon} K_+^{(n-1)}(z,\zeta)\, dV(z) \\ &\qquad \lesssim \frac{\tau_2^{(\ell)}(\zeta)}{2^{(N-3n+2)\ell}\prod_{j=1}^n \tau_j^{(\ell)}(\zeta)^2} \int_{P(\zeta,\beta 2^\ell\delta)} \frac{d(z)^{\alpha-1-\varepsilon}}{|\zeta - z|}\, dV(z). \end{aligned} \tag{13}$$

To obtain a desired estimate, we use the system of coordinate associated to the basis $(e_1^{(\ell)}(\zeta), \ldots, e_n^{(\ell)}(\zeta))$. We set

$$w_k = \langle \zeta - z, e_k^{(\ell)} \rangle, \quad 1 \le k \le n, \quad t_1 = -\rho(z), \quad t_2 = \operatorname{Im} w_1 \tag{14}$$

and for $2 \le k \le n$,

$$\begin{aligned} t_{2k-1} &= \operatorname{Re} w_k, \\ t_{2k} &= \operatorname{Im} w_k. \end{aligned}$$

Since $\tau_1^{(\ell)}(\zeta) \approx 2^\ell \delta$ and $d(z) \lesssim 2^\ell d$ for $z \in P(\zeta, \beta 2^\ell d)$, by the coordinates change (14) we have

$$
\begin{aligned}
&\int_{P(\zeta,\beta 2^\ell \delta)} \frac{d(z)^{\alpha-1-\varepsilon}}{|\zeta - z|}\, dV(z) \\
&\qquad \lesssim \int_{\substack{|t_j|<2^\ell\delta,\, j=1,2\\ |w_j|<\tau_j^{(\ell)},\, j\geq 2}} \frac{t_1^{\alpha-1-\varepsilon}\, dt_1\, dt_2\, dV(w_2,\dots,w_n)}{|w_2|} \\
&\qquad \lesssim \frac{1}{\alpha-\varepsilon}(2^\ell\delta)^{\alpha-\varepsilon+1} \int_{|w_j|<\tau_j^{(\ell)},\, j\geq 2} \frac{dV(w_2,\dots,w_n)}{|w_2|} \\
&\qquad \lesssim \frac{1}{\alpha-\varepsilon}(2^\ell\delta)^{\alpha-\varepsilon-1} \prod_{j=1}^{n} \frac{\tau_j^{(\ell)}(\zeta)^2}{\tau_2^{(\ell)}(\zeta)}.
\end{aligned}
\tag{15}
$$

If we choose an integer N so that $N - 3n + 3 - \alpha > 0$, then (13) and (15) give

$$
\begin{aligned}
&\int_{W\cap\Omega} d(z)^{\alpha-1-\varepsilon} K_+^{(n-1)}(z,\zeta)\, dV(z) \\
&\qquad\qquad \lesssim \sum_\ell 2^{-(N-3n+3-\alpha+\varepsilon)\ell}\delta(\zeta)^{\alpha-\varepsilon-1} \lesssim \delta(\zeta)^{\alpha-\varepsilon-1}.
\end{aligned}
$$

To prove (11) for $k = 1$ we have to consider the integral

$$
\int_{P(\zeta,\beta 2^\ell\delta)} \frac{d(z)^{\alpha-1-\varepsilon}}{|\zeta - z|^{2n-3}}\, dV(z).
$$

To change coordinates we again use the coordinates (14). Here we may assume that $i_0 < i_1$, $\nu = \min(i_0, j_1)$, $\mu = \max(i_0, j_1)$ and $r_{i_1} = |w_{i_1}|$. We first consider t_1, $t_{2\nu}$, t_{2i_2-1}, t_{2i_2} variables and then we integrate with the remaining $(2n-4)$ variables, t'. Since $\tau_1^{(\ell)} \approx 2^\ell\delta$ and $2^\ell\delta \lesssim \tau_\mu^{(\ell)}$, we have

$$\begin{aligned}
&\int_{P(\zeta,\beta 2^\ell\delta)} \frac{d(z)^{\alpha-1-\varepsilon}}{|\zeta-z|^{2n-3}}\,dV(z)\\
&\qquad\lesssim \int\cdots\int_{\substack{|t_{2k-1}|+|t_{2k}|<\tau_k^{(\ell)}\\ k=1,\dots,n}} \frac{t_1^{\alpha-1-\varepsilon}\,dt_1\cdots dt_{2n}}{|t|^{2n-3}}\\
&\qquad\lesssim \int_{\substack{|t_1|<\tau_1^{(\ell)}\\ |t_{2\nu}|<\tau_\nu^{(\ell)}}} t_1^{\alpha-1-\varepsilon}\,dt_1\,dt_{2\nu}\int_{\substack{r_{i_1}<\tau_{i_1}^{(\ell)}\\ |t'|<1}} \frac{r_{i_1}\,dr_{i_1}\,dV(t')}{(r_{i_1}+|t'|)^{2n-3}} \qquad (16)\\
&\qquad\lesssim \frac{1}{\alpha-\varepsilon}(2^\ell\delta)^{\alpha-1-\varepsilon}\tau_\nu^{(\ell)}\tau_\mu^{(\ell)}\int_{|t'|<1}\int_0^{\tau_{i_1}^{(\ell)}} \frac{r_{i_1}\,dr_{i_1}\,dV(t')}{(r_{i_1}+|t'|)^{2n-3}}\\
&\qquad\lesssim \frac{1}{\alpha-\varepsilon}(2^\ell)^{\alpha-1-\varepsilon}(\delta)^{\alpha-1-\varepsilon}\tau_{i_0}^{(\ell)}\tau_{i_1}^{(\ell)}\tau_{j_1}^{(\ell)}.
\end{aligned}$$

From the estimates (8) and (16), since $N-3-\alpha>0$ we see that

$$\int_{W\cap\Omega} d(z)^{\alpha-1-\varepsilon}K_+^{(1)}(z,\zeta)\,dV(z) \lesssim \sum_\ell 2^{-(N-3-\alpha+\varepsilon)\ell}\delta(\zeta)^{\alpha-1-\varepsilon}\lesssim \delta(\zeta)^{\alpha-1-\varepsilon}.$$

To prove (11) for $k=0$ we note that

$$\begin{aligned}
\eta(z,\zeta) &\approx |\rho(\zeta)|+|\rho(z)|+\mathcal{M}(z,\zeta)\\
&\gtrsim \rho(\zeta)|+|\rho(z)|+|\zeta_1-z_1|, \quad z,\zeta\in U\cap\Omega.
\end{aligned}$$

Here the coordinates $(z_1,\dots,z_n)$ are ε-extremal coordinates of McNeal at ζ. Hence there is a system of coordinates $t=(t_1,\dots,t_{2n})$ on $U\cap\Omega$ such that $t_1=-\rho(z)$, $t_2=\operatorname{Im}(\zeta_1-z_1)$ and $t'(\zeta)=(t_3(\zeta),\dots,t_{2n}(\zeta))=0$. From the definition of $K_+^{(0)}(z,\zeta)$ we have

$$\begin{aligned}
&\int_{W\cap\Omega} d(z)^{\alpha-1-\varepsilon}K_+^{0}(z,\zeta)\,dV(z) \qquad (17)\\
&\lesssim \delta(\zeta)^{N-1}\int_{|t|<1}\frac{|t_1|^{\alpha-1-\varepsilon}\,dt_1\cdots dt_{2n}}{(|t_1|+|t_2|+|\rho(\zeta)|)^N(|t_1-\rho(\zeta)|+|t_2|+|t'|)^{2n-1}}.
\end{aligned}$$

Introducing polar coordinates with respect to the variables t' we have

$$I_1(\zeta) = \int_{|t|<1} \frac{|t_1|^{\alpha-1-\varepsilon}\, dt_1 \cdots dt_{2n}}{(|t_1|+|t_2|+|\rho(\zeta)|)^N(|t_1-\rho(\zeta)|+|t_2|+|t'|)^{2n-1}}$$
$$\lesssim \int_{|(t_1,t_2)|<1} \frac{|t_1|^{\alpha-1-\varepsilon}\, dt_1\, dt_2}{(|t_1|+|t_2|+|\rho(\zeta)|)^N(|t_1-\rho(\zeta)|+|t_2|)}.$$

If we make the change of variables $t_1 = |\rho(\zeta)|t_1'$ and $t_2 = |\rho(\zeta)|t_2'$ and omit the primes, then we obtain

$$\text{(18)}\quad I_1(\zeta) \lesssim |\rho(\zeta)|^{-N+\alpha-\varepsilon} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|t_1|^{\alpha-1-\varepsilon}\, dt_1\, dt_2}{(|t_1|+|t_2|+1)^N(|t_1-1|+|t_2|)}.$$

If $\alpha-1-\varepsilon \geq 0$ then

$$J_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|t_1|^{\alpha-1-\varepsilon}\, dt_1\, dt_2}{(|t_1|+|t_2|+1)^N(|t_1-1|+|t_2|)}$$
$$\lesssim \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dt_1\, dt_2}{(|t_1|+|t_2|+1)^{N-\alpha+1+\varepsilon}(|t_1-1|+|t_2|)}$$
$$\lesssim \int_{0}^{\infty}\int_{0}^{\infty} \frac{dt_1\, dt_2}{(|t_1|+|t_2|+2)^{N-\alpha+1+\varepsilon}(|t_1|+|t_2|)}$$
$$\lesssim \int_0^\infty \frac{s\, ds}{s(s+2)^{N-\alpha+1+\varepsilon}} \lesssim 1 \qquad \text{if } N-\alpha>0.$$

If $-1<\alpha-1-\varepsilon<0$, then we have for $0<\gamma<1$

$$J_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dt_1\, dt_2}{|t_1|^{1-\alpha+\varepsilon}(|t_1|+|t_2|+1)^N(|t_1-1|+|t_2|)}$$
$$\lesssim \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dt_1\, dt_2}{|t_1|^{1-\alpha+\varepsilon}(|t_1|+|t_2|+1)^N|t_1-1)|^{\gamma}|t_2|^{1-\gamma})}$$
$$\lesssim \int_{-\infty}^{\infty} \frac{dt_1}{|t_1|^{1-\alpha+\varepsilon}|t_1-1|^{\gamma}(|t_1|+1)^{\frac{N}{2}}} \int_{-\infty}^{\infty} \frac{dt_2}{|t_2|^{1-\gamma}(|t_2|+1)^{\frac{N}{2}}}.$$

Since $0<\gamma<1$ and $0<1-\alpha+\varepsilon<1$, it is easy to see that

$$\int_{-\infty}^{\infty} \frac{dt_1}{|t_1|^{1-\alpha+\varepsilon}|t_1-1|^{\gamma}(|t_1|+1)^{\frac{N}{2}}}$$
$$\lesssim \int_0^\infty \frac{dt_1}{t_1^{1-\alpha+\varepsilon}(t_1+1)^{\frac{N}{2}}} + \int_0^\infty \frac{dt_1}{t_1^{\gamma}(t_1+1)^{\frac{N}{2}}} < \infty.$$

Since J_1 is bounded, by the inequalities (17) and (18), we have

$$\int_{W\cap\Omega} d(z)^{\alpha-1-\varepsilon} K^0_+(z,\zeta)\, dV(z) \lesssim \delta(\zeta)^{\alpha-1-\varepsilon}.$$

(ii) By the estimate (9) we have

$$\int_{\mathcal{C}_\ell(z)} \delta(\zeta)^{-\varepsilon} K^{(n-1)}_+(z,\zeta)\, dV(\zeta)$$
$$\lesssim \frac{\tau_2^{(\ell)}(z)}{(2^\ell d)^{N-3n+2}} \frac{1}{\prod_{j=1}^n \tau_j^{(\ell)}(z)^2} \int_{P(z,\beta 2^\ell d)} \frac{\delta(\zeta)^{N-3n+2-\varepsilon}}{|\zeta - z|}\, dV(\zeta).$$

We introduce new coordinates associated to the basis $(e_1^{(\ell)}(z), \ldots, e_n^{(\ell)}(z))$ and set

$$u_k = \langle \zeta - z, e_k^{(\ell)}\rangle, \quad 1 \le k \le n, \quad t_1 = -\rho(\zeta), \quad t_2 = \operatorname{Im} u_2 \tag{19}$$

and for $2 \le k \le n$,

$$\begin{aligned} t_{2k-1} &= \operatorname{Re} u_k, \\ t_{2k} &= \operatorname{Im} u_k. \end{aligned}$$

The same calculation as (i) shows that

$$\int_{P(z,\beta 2^\ell d)} \frac{\delta(\zeta)^{N-3n+2-\varepsilon}}{|\zeta - z|}\, dV(\zeta) \lesssim (2^\ell d)^{N-3n+2-\varepsilon} \frac{\prod_{j=1}^n \tau_j^{(\ell)}(z)^2}{\tau_2^{(\ell)}(z)}, \quad \varepsilon > 0.$$

Hence we have for $\varepsilon > 0$

$$\int_{W\cap\Omega} \delta(\zeta)^{-\varepsilon} K^{(n-1)}_+(z,\zeta)\, dV(\zeta) \lesssim \sum_\ell (2^\ell)^{-\varepsilon} d^{-\varepsilon} \lesssim d(z)^{-\varepsilon}.$$

By the estimates (10) we have

$$\int_{\mathcal{C}_\ell(z)} \delta(\zeta)^{-\varepsilon} K^{(1)}_+(z,\zeta)\, dV(\zeta)$$
$$\lesssim \sum_{I,J} \frac{\tau_{i_0}^{(\ell)}(z) \prod_{\nu=2}^{n-1} \tau_{i_\nu}^{(\ell)}(z)\tau_{j_\nu}^{(\ell)}(z)}{(2^\ell d)^{N-4} \prod_{j=1}^n \tau_j^{(\ell)}(z)^2} \int_{P(z,\beta 2^\ell d)} \frac{\delta(\zeta)^{N-4-\varepsilon}}{|\zeta - z|^{2n-3}}\, dV(\zeta).$$

Using the coordinate system (19) and estimating the integral as (16) we obtain

$$\int_{P(z,\beta 2^\ell d)} \frac{\delta(\zeta)^{N-4-\varepsilon}}{|\zeta - z|^{2n-3}}\, dV(\zeta) \lesssim (2^\ell d)^{N-4-\varepsilon} \tau_{i_0}^{(\ell)}(z)\tau_{i_1}^{(\ell)}(z)\tau_{j_1}^{(\ell)}(z).$$

Thus as before we have for $\varepsilon > 0$

$$\int_{W\cap\Omega} \delta(\zeta)^{-\varepsilon} K_+^{(1)}(z,\zeta)\, dV(\zeta) \lesssim \sum_\ell (2^\ell)^{-\varepsilon} d^{-\varepsilon} \lesssim d(z)^{-\varepsilon}.$$

Finally if $k = 0$ then we have

$$\int_{U\cap\Omega} \delta(\zeta)^{-\varepsilon} K_+^{(0)}(z,\zeta)\, dV(\zeta) \lesssim \int_{U\cap\Omega} \frac{\delta(\zeta)^{N-1-\varepsilon}}{(\eta(z,\zeta))^N} \frac{1}{|\zeta - z|^{2n-1}}\, dV(\zeta).$$

Using a system of coordinates with respect to an ε-extremal basis at $z \in U$ we have

$$\eta(z,\zeta) \gtrsim |\rho(z)| + |\rho(\zeta)| + |\zeta_1 - z_1|.$$

Using the coordinates $t_1 = -\rho(\zeta)$, $t_2 = \mathrm{Im}(\zeta_1 - z_1)$ and $t' = (t_3, \ldots, t_{2n})$ satisfying $t'(z) = 0$ we have

$$\begin{aligned} I_2(z) &= \int_{U\cap\Omega} \frac{\delta(\zeta)^{N-\varepsilon-1}}{(\eta(z,\zeta))^N} \frac{1}{|\zeta - z|^{2n-1}}\, dV(\zeta) \\ &\lesssim \int_{|t|<1} \frac{|t_1|^{N-\varepsilon-1}\, dt_1\, dt_2\, dV(t')}{(|t_1 - |\rho(z)|| + |t_2| + |t'|)^{2n-1}(|t_1| + |t_2| + |\rho(z)|)^N}. \end{aligned}$$

The right hand side of above inequality has the same type integration as J_1. Thus the same method can be applied to obtain $I_2(z) \lesssim d(z)^{-\varepsilon}$. □

5. Proof of Main Theorem

Now we come to the final step in the proof of Main Theorem. Let $u(z)$ be a solution of $\overline{\partial} u = f$ in (3). Then by the definition of $K_+^{(k)}(z,\zeta)$, $k = 0, 1, \ldots, n-1$, we have for each $z \in \Omega$

$$|u(z)| \lesssim \sum_{k=0}^{n-1} \int_\Omega \delta(\zeta)||f(\zeta)||K_+^{(k)}(z,\zeta)\, dV(\zeta). \tag{20}$$

Therefore to complete the proof of Theorem 1.1, it suffices to show that

$$\int_\Omega d(z)^{\alpha-1}\, dV(z) \left[\int_\Omega \delta(\zeta)||f(\zeta)||K_+^{(k)}(z,\zeta)\, dV(\zeta)\right]^p \lesssim \int_\Omega \delta(\zeta)^\alpha ||f(\zeta)||^p\, dV(\zeta). \tag{21}$$

Cumenge [**Cum01a**] have already proved the inequality (21) in case $p = 1$, so we do not repeat the proof. First assume that $1 < p < \infty$ and fix $\alpha > 0$ and p. We choose a sufficiently large integer N so that

$N-3n+3-\alpha>0$. Let $q>1$ be a positive real such that $1/p+1/q=1$. Then by Hölder's inequality and Lemma 4.1 (12) we have

$$\begin{aligned}\int_\Omega \delta(\zeta)||f(\zeta)||K_+^{(k)}(z,\zeta)\,dV(\zeta) &= \int_\Omega \Big(\delta||f||\big(K_+^{(k)}\big)^{1/p}\delta^{\varepsilon}\Big)\Big(\big(K_+^{(k)}\big)^{1/q}\delta^{-\varepsilon}\Big)\,dV \\ &\le \left\{\int_\Omega \delta^{p+\varepsilon p}||f||^p K_+^{(k)}\right\}^{\frac{1}{p}}\left\{\int_\Omega K_+^{(k)}\delta^{-\varepsilon q}\right\}^{\frac{1}{q}} \\ &\lesssim d^{-\varepsilon}\left\{\int_\Omega \delta^{p+\varepsilon p}||f||^p K_+^{(k)}\right\}^{\frac{1}{p}}.\end{aligned}$$

We apply Fubini's theorem to the left hand sid of (21) to obtain

$$\begin{aligned}&\int_\Omega d(z)^{\alpha-1}\,dV(z)\left[\int_\Omega \delta(\zeta)||f(\zeta)||K_+^{(k)}(z,\zeta)\,dV(\zeta)\right]^p \\ &\qquad\le \int_\Omega ||f(\zeta)||^p\delta(\zeta)^{p+\varepsilon p}\,dV(\zeta)\left[\int_\Omega d(z)^{\alpha-1-\varepsilon p}K_+^{(k)}(z,\zeta)\,dV(z)\right] \\ &\qquad\le \int_\Omega ||f(\zeta)||^p\delta(\zeta)^{p+\varepsilon p}\cdot\delta(\zeta)^{\alpha-1-\varepsilon p}\,dV(\zeta) \\ &\qquad= \int_\Omega \delta(\zeta)^{\alpha-1+p}||f(\zeta)||^p\,dV(\zeta)\end{aligned}$$

by Lemma 4.1 (11). Here we choose $\varepsilon>0$ so small that $\alpha-1-\varepsilon p>-1$. Next we prove the inequality for $p=\infty$. Again by the relation (20), it suffices to show that for all $z\in\Omega$,

$$d(z)^{\alpha-1}\int_\Omega \delta(\zeta)||f(\zeta)||K_+^{(k)}(z,\zeta)\,dV(\zeta)\le \sup_{\zeta\in\Omega}\delta(\zeta)^\alpha||f(\zeta)||, \quad \alpha>1.$$

By the Lemma 4.1 (12) we see that

$$\begin{aligned}&d(z)^{\alpha-1}\int_\Omega \delta(\zeta)||f(\zeta)||K_+^{(k)}(z,\zeta)\,dV(\zeta) \\ &\lesssim \sup_{\zeta\in\Omega}\big(\delta(\zeta)^\alpha||f(\zeta)||\big)\left[d(z)^{\alpha-1}\int_\Omega \delta(\zeta)^{-\alpha+1}K_+^{(k)}(z,\zeta)\,dV(\zeta)\right] \\ &\lesssim \sup_{\zeta\in\Omega}\big(\delta(\zeta)^\alpha||f(\zeta)||\big)\big[d(z)^{\alpha-1}\cdot d(z)^{-\alpha+1}\big] \\ &\lesssim \sup_{\zeta\in\Omega}\delta(\zeta)^\alpha||f(\zeta)||, \qquad\qquad \text{for } \alpha>1.\end{aligned}$$

6. Example

In this section we give an example to show that the estimates in Main Theorem are sharp in some sense at the cases $2 \le p < \infty$. Let $E_m = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = |z_1|^2 + |z_2|^m - 1 < 0\}$, where m is an even number. Then E_m is a convex domain of finite type m. If $1 \le p < \infty$ and $0 < \alpha < \infty$ we define a non-isotropic L^p space with weight α, $L^p_\alpha(E_m, ||\cdot||)$ that consists of all $(0,1)$-form f satisfying

$$||f||^p_{p,\alpha,E_m} = \int_{E_m} ||f(z)||^p |\rho(z)|^{\alpha-1+p}\, dV < \infty$$

and L^p space with weight α, $L^p_\alpha(\Omega)$ that consists of all measurable functions g satisfying

$$||g||^p_{p,\alpha} = \int_{E_m} |g(z)|^p |\rho(z)|^{\alpha-1}\, dV < \infty.$$

Now we can prove the following theorem.

Theorem 6.1. *For each $p \ge 2$ and α, there exists a $\overline{\partial}$-closed $(0,1)$-form $f \in L^p_\gamma(E_m.||\cdot||)$, for all $\gamma > \alpha$, or $f \in L^r_\alpha(E_m.||\cdot||)$, for all $r < p$, such that no solution to $\overline{\partial}u = f$ belongs to $L^p_\alpha(E_m)$.*

Proof: Fix $p \ge 2$ and α. Put $f(z_1, z_2) = d\overline{z}_2/(1 - z_1)^d$, where $dp - \alpha - p/m - 2/m + 1 = 2$. Then f is a $\overline{\partial}$-closed $(0,1)$-form on E_m. For simplicity of notation, we let $b_{z_1} = (1 - |z_1|^2)^{1/m}$. Then by the definition of $||\cdot||$, we have $||f(z_1, z_2)|| \lesssim (1 - |z_1|^2 - |z_2|^m)^{1/m-1}/|1 - z_1|^d$. It follows that

$$\|f\|^p_{p,\gamma,E_m} \lesssim \int_{|z_1|<1} \frac{dA(z_1)}{|1 - z_1|^{dp}} \int_{|z_2|<b_{z_1}} (1 - |z_1|^2 - |z_2|^m)^{\gamma-1+p/m}\, dA(z_2). \tag{22}$$

Using polar coordinate change, we have

$$\begin{aligned} I(z_1) &= \int_{|z_2|<b_{z_1}} (1 - |z_1|^2 - |z_2|^m)^{\gamma+p/m-1}\, dA(z_2) \\ &= 2\pi \int_0^{b_{z_1}} (1 - |z_1|^2 - r^m)^{\gamma+p/m-1} r\, dr \\ &= 2\pi (1 - |z_1|^2)^{\gamma+p/m-1+2/m} \int_0^1 (1 - s^m)^{\gamma+p/m-1} s\, ds \\ &\lesssim (1 - |z_1|^2)^{\gamma+p/m-1+2/m}, \end{aligned} \tag{23}$$

where we set $s = r/(1-|z_1|^2)^{1/m}$. To calculate the upper bound of (22) we need the following lemma:

Lemma 6.2 ([**Rud80**]). *For $z \in B_n = \{z \in \mathbb{C}^n : |z| < 1\}$, c real, $\eta > -1$, define*

$$J_{c,\eta}(z) = \int_{B_n} \frac{(1-|\zeta|^2)^\eta}{|1-\overline{\zeta}\cdot z|^{n+1+\eta+c}}\, dV(\zeta).$$

When $c < 0$, then $J_{c,\eta}$ is bounded in B_n. When $c > 0$, then $J_{c,\eta}(z) \approx (1-|z|^2)^{-c}$. Finally, $J_{0,\eta} \approx -\log(1-|z|^2)$.

From (22), (23) and by Lemma 6.2 it follows that if $\gamma > \alpha$, then

$$\begin{aligned}\|f\|^p_{p,\gamma,E_m} &\lesssim \int_{|z_1|<1} \frac{dA(z_1)}{|1-z_1|^{dp-\gamma+1-p/m-2/m}} \\ &= \lim_{r\to 1^-} \int_{|z_1|<1} \frac{dA(z_1)}{|1-z_1 r|^{dp-\gamma+1-p/m-2/m}} \lesssim 1\end{aligned}$$

since $dp-\gamma+1-p/m-2/m < 2$. Let $v(z_1, z_2) = \overline{z}_2/(1-z_1)^d$. Then it is clear that $\overline{\partial} v = f$ on E_m. On the other hand, we have

$$\|v\|^p_{p,\alpha} = \int_{|z_1|<1} \frac{dA(z_1)}{|1-z_1|^{dp}} \int_{|z_2|<b_{z_1}} |z_2|^p(1-|z_1|^2-|z_2|^m)^{\alpha-1}\, dA(z_2). \tag{24}$$

By polar coordinate change, we have

$$\begin{aligned}J(z_1) &= \int_{|z_2|<b_{z_1}} |z_2|^p(1-|z_1|^2-|z_2|^m)^{\alpha-1}\, dA(z_2) \\ &= 2\pi(1-|z_1|^2)^{\alpha-1} \int_0^{b_{z_1}} r^{p+1}\left(1-\frac{r^m}{1-|z_1|^2}\right)^{\alpha-1} dr \\ &= 2\pi(1-|z_1|^2)^{\alpha-1+\frac{(p+2)}{m}} \int_0^1 (1-s^m)^{\alpha-1} s^{p+1}\, ds \\ &\gtrsim (1-|z_1|^2)^{\alpha-1+\frac{(p+2)}{m}}.\end{aligned} \tag{25}$$

From (24) and (25) we have

$$\begin{aligned}
\|v\|_{p,\alpha}^p &\gtrsim \int_{|z_1|<1} \frac{(1-|z_1|^2)^{\alpha-1+\frac{(p+2)}{m}}}{|1-z_1|^{dp}}\, dA(z_1) \\
&= \lim_{r\to 1^-} \int_{|z_1|<1} \frac{(1-|z_1|^2)^{\alpha-1+\frac{(p+2)}{m}}}{|1-z_1 r|^{dp}}\, dA(z_1) \qquad (26)\\
&\approx \lim_{r\to 1^-} \log\left(\frac{1}{1-r^2}\right) = \infty,
\end{aligned}$$

by Lemma 6.2. Thus, $v \notin L^p_\alpha(B_2)$.

Next we consider the inner product $\langle h, v\rangle_\alpha$ for every $h \in L^2_\alpha(B_2)\cap \mathcal{O}(B_2)$. By Fubini's theorem, we have

$$\begin{aligned}
\langle h, v\rangle_\alpha &= \int_{E_m} h(\zeta)\,\overline{v(\zeta)}\,|\rho(\zeta_1,\zeta_2)|^{\alpha-1}\, dV(\zeta) \\
&= \int_{|\zeta_1|<1} \frac{dA(\zeta_1)}{(1-\overline{\zeta}_1)^d} \int_{|\zeta_2|<b_{\zeta_1}} \zeta_2 h(\zeta_1,\zeta_2)(1-|\zeta_1|^2-|\zeta_2|^m)^{\alpha-1}\, dA(\zeta_2).
\end{aligned}$$

Putting $\zeta_2 = re^{i\theta}$, we see

$$\begin{aligned}
&\int_{|\zeta_2|<b_{\zeta_1}} \zeta_2 h(\zeta_1,\zeta_2)(1-|\zeta_1|^2-|\zeta_2|^m)^{\alpha-1}\, dA(\zeta_2) \\
&\qquad = \int_0^{b_{\zeta_1}} \int_0^{2\pi} r^2 e^{i\theta} h(\zeta_1, re^{i\theta})(1-|\zeta_1|^2-r^m)^{\alpha-1}\, d\theta\, dr \\
&\qquad = \int_0^{b_{\zeta_1}} r^2(1-|\zeta_1|^2-r^m)^{\alpha-1}\left(\int_0^{2\pi} e^{i\theta} h(\zeta_1, re^{i\theta})\, d\theta\right) dr \\
&\qquad = \int_0^{b_{\zeta_1}} r^2(1-|\zeta_1|^2-r^m)^{\alpha-1}\cdot 0\cdot dr = 0,
\end{aligned}$$

since $h(\zeta_1,\cdot)$ is holomorphic. Thus v is orthogonal to $L^2_\alpha(B_2)\cap\mathcal{O}(E_m)$, i.e., v is the canonical solution for $\overline{\partial}u = f$. To complete our theorem we need another well-known theorem on the boundedness of the weighted Bergman projections on E_m.

Proposition 6.3 ([**Cho**], [**LS92**])**.** *Let $\mathbb{B}_\alpha\colon L^2_\alpha(E_m)\to L^2_\alpha(E_m)\cap\mathcal{O}(E_m)$ be the orthogonal projection, $\alpha>0$. Then $\mathbb{B}_\alpha\colon L^p_\alpha(E_m)\to L^p_\alpha(E_m)\cap\mathcal{O}(E_m)$ is a bounded operator for every $1<p<\infty$.*

Assume that $\overline{\partial}u = f \in L^p_\gamma(E_m, ||\cdot||)$ and $u\in L^p_\alpha(E_m)$, $\gamma>\alpha$. Then by the Proposition 6.3, $v = u - \mathbb{B}_\alpha(u)$ and it would be in $L^p_\alpha(E_m)$.

By (26) this is impossible. Hence there is no solution u in $L^p_\alpha(E_m)$ to the equation $\overline{\partial}u = f$.

If $r < p$, i.e. $dr - \alpha - r/2 < 2$, then it also follows by a similar calculation to the above that $f \in L^r_\alpha(E_m, ||\cdot||)$ and no solution u to $\overline{\partial}u = f$ belongs to $L^p_\alpha(E_m)$. □

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