Publ. Mat. **49** (2005), 111–125

# COMMUTATORS OF SINGULAR INTEGRALS ON GENERALIZED $L^p$ SPACES WITH VARIABLE EXPONENT

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*Abstract*

A classical theorem of Coifman, Rochberg, and Weiss on commutators of singular integrals is extended to the case of generalized $L^p$ spaces with variable exponent.

## 1. Introduction

Let $T$ be a Calderón-Zygmund singular integral operator

$$Tf(x) := \text{P.V.} \int_{\mathbb{R}^n} K(x-y)f(y)\,dy$$

with kernel $K(x) = \Omega(x)/|x|^n$, where $\Omega$ is homogeneous of degree zero, infinitely differentiable on the unit sphere $S^{n-1}$, and $\int_{S^{n-1}} \Omega = 0$.

All functions in the present paper are assumed to be real valued. By $L_c^\infty$ we denote the class of all bounded functions on $\mathbb{R}^n$ with compact support. Let $b$ be a locally integrable function on $\mathbb{R}^n$. Consider the commutator $[b,T]$ defined initially for any $f \in L_c^\infty$ by

$$[b,T]f := bT(f) - T(bf).$$

Recall that the space $BMO(\mathbb{R}^n)$ consists of all locally integrable functions $f$ such that

$$\|f\|_* := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|\,dx < \infty,$$

where $f_Q := |Q|^{-1}\int_Q f(y)\,dy$, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes, and $|Q|$ denotes the Lebesgue measure of $Q$.

2000 *Mathematics Subject Classification.* Primary: 42B20; Secondary: 46E30.
*Key words.* Commutator, Calderón-Zygmund singular integral, *BMO*, generalized $L^p$ space with variable exponent, local sharp maximal function.

A classical result of Coifman, Rochberg, and Weiss [**4**] states that if $b \in BMO(\mathbb{R}^n)$, then $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$; conversely, if $[b,R_i]$ is bounded on $L^p(\mathbb{R}^n)$ for every Riesz transform $R_i$, then $b \in BMO(\mathbb{R}^n)$. Janson [**12**] observed that actually for any singular integral $T$ (with kernel satisfying the above-mentioned conditions) the boundedness of $[b,T]$ on $L^p(\mathbb{R}^n)$ implies $b \in BMO(\mathbb{R}^n)$.

An important role in proving the latter implication is played by a translation invariant argument, that is, by an obvious fact that the translation operator is bounded on $L^p(\mathbb{R}^n)$. Therefore, it is natural to ask whether an analogous result holds if we replace $L^p(\mathbb{R}^n)$ by a more general function space for which the continuity of translations may fail to hold. We will consider the problem in *generalized $L^p$ spaces with variable exponent.*

Function spaces $L^{p(\cdot)}$ of Lebesgue type with variable exponent $p$ were studied for the first time by Orlicz [**22**]. Then Nakano considered spaces $L^{p(\cdot)}$ as an example of his modular spaces [**20**]. The theory of modular spaces and, in particular, generalized Orlicz spaces generated by Young functions with a parameter (Musielak-Orlicz spaces) is documented in [**19**]. The generalized $L^p$ spaces with variable exponent are a special case of Musielak-Orlicz spaces.

Let $p\colon \mathbb{R}^n \to [1,\infty)$ be a measurable function. Consider the convex modular (see [**19**, Chapter 1] for definitions and properties)

$$m(f,p) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx.$$

Denote by $L^{p(\cdot)}(\mathbb{R}^n)$ the set of all Lebesgue measurable functions $f$ on $\mathbb{R}^n$ such that $m(\lambda f, p) < \infty$ for some $\lambda = \lambda(f) > 0$. This set becomes a Banach space with respect to the *Luxemburg-Nakano norm*

$$\|f\|_{L^{p(\cdot)}} := \inf\Big\{\lambda > 0 : \ m(f/\lambda, p) \le 1\Big\}$$

(see, e.g., [**19**, Chapter 2]). Clearly, if $p(\cdot) = p$ is constant, then the space $L^{p(\cdot)}(\mathbb{R}^n)$ is isometrically isomorphic to the Lebesgue space $L^p(\mathbb{R}^n)$.

Observe, however, that spaces $L^{p(\cdot)}(\mathbb{R}^n)$ have attracted a great attention only several years ago in connection with problems of the boundedness of classical operators on $L^{p(\cdot)}(\mathbb{R}^n)$, which in turn were motivated by some questions in fluid dynamics. We mention here [**6**], [**8**], [**10**], [**14**], [**21**], [**24**] (see also the references therein).

It is easy to see that $L^{p(\cdot)}(\mathbb{R}^n)$ fail to be rearrangement-invariant, in general (see, e.g., [**2**, Chapter 2] for the definition and properties of rearrangement-invariant spaces). This means that neither good-$\lambda$ technique nor rearrangement inequalities may be applied for a generalization of some well-known results in harmonic analysis to the case of $L^{p(\cdot)}(\mathbb{R}^n)$. Also $L^{p(\cdot)}(\mathbb{R}^n)$ fail to be translation invariant, in general (see [**15**, Theorem 2.10]).

If a measurable function $p\colon \mathbb{R}^n \to [1,\infty)$ satisfies

$$1 < p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) =: p_+ < \infty, \tag{1.1}$$

then the function

$$p'(x) := p(x)/(p(x)-1)$$

is well defined and satisfies (1.1) itself.

Denote by $\mathcal{M}(\mathbb{R}^n)$ the set of all measurable functions $p\colon \mathbb{R}^n \to [1,\infty)$ such that (1.1) holds and the Hardy-Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Sufficient conditions guaranteeing $p \in \mathcal{M}(\mathbb{R}^n)$ are given in [**6**, Theorem 1.5], [**8**, Theorem 3.5], [**21**, Theorem 2.14] (see also [**14**] for weighted analogs).

Let $\mathcal{B}(X)$ be the class of all bounded sublinear operators on a Banach lattice $X$ and let $\|A\|_{\mathcal{B}(X)}$ denote the operator norm of $A \in \mathcal{B}(X)$.

Our main result is the following.

**Theorem 1.1.** *Suppose $p$ and $p'$ belong to $\mathcal{M}(\mathbb{R}^n)$.*

(a) *If $b \in BMO(\mathbb{R}^n)$, then $[b,T]$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ and*

$$\|[b,T]\|_{\mathcal{B}(L^{p(\cdot)})} \le C_p \|b\|_*.$$

(b) *If $\Omega$ is odd, $b$ belongs to the Zygmund space $L\log L(Q)$ for every cube $Q \subset \mathbb{R}^n$ and $[b,T]$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then $b \in BMO(\mathbb{R}^n)$ and*

$$\|b\|_* \le C'_p \|[b,T]\|_{\mathcal{B}(L^{p(\cdot)})}.$$

Our proof of Part (a) is motivated by an analog of the Fefferman-Stein theorem on the sharp maximal function for $L^{p(\cdot)}(\mathbb{R}^n)$ proved recently by Diening and Růžička [**10**, Theorem 3.6]. To prove Part (a), we combine a little bit more elaborate version of the latter result, based on the so-called local sharp maximal function and on a duality inequality due to the second author [**16**, Theorem 1], with a sharp function inequality for commutators due to Strömberg (see [**12**]) and Pérez [**23**, Lemma 3.1].

To prove Part (b), we first deduce that $[b,T]$ is also bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$. To make this step, we have to pay by stronger requirements of oddness of the kernel and of the local $L\log L$ integrability of $b$. Next, using an interpolation argument, we conclude that $[b,T]$ is bounded on $L^2(\mathbb{R}^n)$. This reduces the problem to the classical situation.

We do not know whether assumptions on the kernel $K$ and on $b$ in Part (b) of Theorem 1.1 can be relaxed.

The paper is organized as follows. Section 2 contains some auxiliary results. In Section 3 we prove Theorem 1.1. Section 4 contains some concluding remarks.

## 2. Auxiliary results

**2.1. Duality and density in spaces $L^{p(\cdot)}(\mathbb{R}^n)$.** For $p$ satisfying (1.1) the function $p'$ is well defined and one can equip the space $L^{p(\cdot)}(\mathbb{R}^n)$ with the *Orlicz type norm*

$$\|f\|^0_{L^{p(\cdot)}} := \sup\left\{\int_{\mathbb{R}^n} |f(x)g(x)|\,dx : g \in L^{p'(\cdot)}(\mathbb{R}^n),\ \|g\|_{L^{p'(\cdot)}} \le 1\right\}.$$

This norm is equivalent to the Luxemburg-Nakano norm (see [**15**, Theorem 2.3]):

$$\|f\|_{L^{p(\cdot)}} \le \|f\|^0_{L^{p(\cdot)}} \le r_p\|f\|_{L^{p(\cdot)}} \quad \Big(f \in L^{p(\cdot)}(\mathbb{R}^n)\Big), \tag{2.1}$$

where

$$r_p := 1 + 1/p_- - 1/p_+.$$

**Lemma 2.1** (see [**15**, Theorem 2.1])**.** *Let $p\colon \mathbb{R}^n \to [1,\infty)$ be a measurable function satisfying* (1.1)*. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then $fg$ is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le r_p\|f\|_{L^{p(\cdot)}}\|g\|_{L^{p'(\cdot)}}.$$

From [**15**, Theorem 2.11] we get the following.

**Lemma 2.2.** *Let $p\colon \mathbb{R}^n \to [1,\infty)$ be a measurable function satisfying* (1.1)*. Then $L^\infty_c$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$ and in $L^{p'(\cdot)}(\mathbb{R}^n)$.*

**2.2. Pointwise estimates for sharp maximal functions.** Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function is defined by

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|\,dy.$$

For $\delta > 0$ and $f \in L^\delta_{\mathrm{loc}}(\mathbb{R}^n)$, set also

$$f^{\#}_\delta(x) := \sup_{Q\ni x}\inf_{c\in\mathbb{R}}\left(\frac{1}{|Q|}\int_Q |f(y)-c|^\delta\,dy\right)^{1/\delta}.$$

The non-increasing rearrangement (see, e.g., [**2**, Chapter 2, Section 1]) of a measurable function $f$ on $\mathbb{R}^n$ is defined by

$$f^*(t) := \inf\Big\{\lambda > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \le t\Big\} \quad (0 < t < \infty).$$

Set also $f^{**}(t) := t^{-1}\int_0^t f^*(\tau)\,d\tau$.

For a fixed $\lambda \in (0,1)$ and a given measurable function $f$ on $\mathbb{R}^n$, consider the local sharp maximal function $M^{\#}_{\lambda} f$ defined by

$$M^{\#}_{\lambda} f(x) := \sup_{Q \ni x} \inf_{c\in\mathbb{R}} \big((f-c)\chi_Q\big)^* (\lambda|Q|)\,.$$

In all above definitions the supremums are taken over all cubes $Q \subset \mathbb{R}^n$ containing $x$.

**Proposition 2.3.** *If $\delta > 0$, $\lambda \in (0,1)$, and $f \in L^{\delta}_{\mathrm{loc}}(\mathbb{R}^n)$, then*

$$M^{\#}_{\lambda} f(x) \le (1/\lambda)^{1/\delta} f^{\#}_{\delta}(x) \quad (x \in \mathbb{R}^n). \tag{2.2}$$

*Proof:* Let $\varphi \in L^{\delta}_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. For every cube $Q$ containing $x \in \mathbb{R}^n$, by Chebyshev's inequality,

$$\big(|\varphi|^{\delta}\chi_Q\big)^*(\lambda|Q|) \le \frac{1}{\lambda|Q|}\int_Q |\varphi(y)|^{\delta}\,dy. \tag{2.3}$$

On the other hand, in view of [**2**, Chapter 2, Proposition 1.7],

$$\big(|\varphi|^{\delta}\chi_Q\big)^* = \big[(\varphi\chi_Q)^*\big]^{\delta}. \tag{2.4}$$

Take $\varphi = f - c$ with $c \in \mathbb{R}$. Then from (2.3) and (2.4) we get

$$\big((f-c)\chi_Q\big)^*(\lambda|Q|) \le (1/\lambda)^{1/\delta}\left(\frac{1}{|Q|}\int_Q |f(y)-c|^{\delta}\,dy\right)^{1/\delta}.$$

Taking the infimum over $c \in \mathbb{R}$ and then the supremum over all cubes $Q \subset \mathbb{R}^n$ containing $x$, we obtain (2.2). $\square$

**Theorem 2.4** (see [**1**, Theorem 2.1])**.** *If $0 < \delta < 1$, then for every $f \in L^{\infty}_c$,*

$$(Tf)^{\#}_{\delta}(x) \le c_{\delta,n} Mf(x) \quad (x \in \mathbb{R}^n).$$

A sharp function inequality for the commutator $[b,T]f$ was proved by Strömberg (the proof is contained in [**12**]). Afterwards, a more precise version of this result was given by Pérez [**23**]. We will need the following corollary from [**23**, Lemma 3.1].

**Theorem 2.5.** *If $0 < \delta < 1$, then for every $b \in BMO$ and $f \in L^{\infty}_c$,*

$$([b,T]f)^{\#}_{\delta}(x) \le c_{\delta,n}\|b\|_*\big(M(Tf)(x) + MMf(x)\big) \quad (x \in \mathbb{R}^n).$$

Actually, Theorems 2.4 and 2.5 were proved in [**1**] and [**23**], respectively, for smooth functions, but exactly the same proofs work for $L^{\infty}_c$ functions as well.

**Theorem 2.6** (see [**16**, Theorem 1]). *For a function $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a measurable function $\varphi$ satisfying*

$$|\{x : |\varphi(x)| > \alpha\}| < \infty \quad \textit{for all} \quad \alpha > 0, \tag{2.5}$$

*one has*

$$\int_{\mathbb{R}^n} |\varphi(x)g(x)|\,dx \le c_n \int_{\mathbb{R}^n} M^{\#}_{\lambda_n}\varphi(x)Mg(x)\,dx.$$

**2.3. On the boundedness of singular integral operators.**

**Theorem 2.7.** *If $p, p' \in \mathcal{M}(\mathbb{R}^n)$, then there exists a constant $c_p$ such that for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|Tf\|_{L^{p(\cdot)}} \le c_p\|f\|_{L^{p(\cdot)}}.$$

*Remark* 2.8. This result was proved by Diening and Růžička [**10**, Theorem 4.8] under the assumptions $p \in \mathcal{M}(\mathbb{R}^n)$ and $(p/s)' \in \mathcal{M}(\mathbb{R}^n)$ for some $s \in (0,1)$. Their proof is based on an analog of the Fefferman-Stein theorem on the sharp maximal function proved in the same paper [**10**, Theorem 3.6]. We shall give a little bit different proof for the sake of completeness. See also Remark 4.1 below.

*Proof:* Let $f \in L^\infty_c$. For any $g \in L^{p'(\cdot)}(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |(Tf)(x)g(x)|\,dx \le c_n \int_{\mathbb{R}^n} Mf(x)Mg(x)\,dx. \tag{2.6}$$

This inequality was proved in [**16**, Theorem 3]. It follows easily by putting $Tf$ in place of $\varphi$ in Theorem 2.6 and by using Proposition 2.3 along with Theorem 2.4. Notice that the application of Theorem 2.6 is justified due to the weak type $(1,1)$ of the operator $T$.

From (2.6), Lemma 2.1, and the condition $p, p' \in \mathcal{M}(\mathbb{R}^n)$ it follows that

$$\int_{\mathbb{R}^n} |(Tf)(x)g(x)|\,dx \le c_n r_p\|Mf\|_{L^{p(\cdot)}}\|Mg\|_{L^{p'(\cdot)}} \le c_p\|f\|_{L^{p(\cdot)}}\|g\|_{L^{p'(\cdot)}},$$

where $c_p := c_n r_p\|M\|_{\mathcal{B}(L^{p(\cdot)})}\|M\|_{\mathcal{B}(L^{p'(\cdot)})}$. Then, by (2.1),

$$\|Tf\|_{L^{p(\cdot)}} \le \|Tf\|^0_{L^{p(\cdot)}} \le c_p\|f\|_{L^{p(\cdot)}}.$$

From the latter inequality and Lemma 2.2 we get the theorem. $\square$

**2.4. Zygmund spaces $L\log L(Q)$ and $L_{\exp}(Q)$.** Let $Q$ be a cube in $\mathbb{R}^n$. The space $L\log L(Q)$ consists of all measurable functions $f$ on $Q$ for which

$$\int_Q |f(x)|\log^+|f(x)|\,dx < \infty$$

(here $\log^+ t := \max\{\log t, 0\}$). The space $L_{\exp}(Q)$ consists of all measurable functions $f$ on $Q$ for which there exists a $\lambda = \lambda(f)$ such that

$$\int_Q \exp(\lambda|f(x)|)\,dx < \infty.$$

It is well known (see, e.g., [**2**, Chapter 4, Section 6]) that $L\log L(Q)$ and $L_{\exp}(Q)$ can be equipped with the norms

$$\|f\|_{L\log L(Q)} := \int_0^{|Q|} f^*(t)\log(|Q|/t)\,dt = \int_0^{|Q|} f^{**}(t)\,dt,$$

$$\|f\|_{L_{\exp}(Q)} := \sup_{t\in(0,|Q|)} \frac{f^{**}(t)}{1+\log(|Q|/t)},$$

respectively. Easy manipulations with $f^*$ and $f^{**}$ lead us to the following well known Hölder inequality for Zygmund spaces. If $f \in L\log L(Q)$ and $g \in L_{\exp}(Q)$, then $fg \in L^1(Q)$ and

$$\int_Q |f(x)g(x)|\,dx \le 2\|f\|_{L\log L(Q)}\|g\|_{L_{\exp}(Q)}. \tag{2.7}$$

**2.5. A commuting relation for singular integrals.** The following statement represents one of the numerous variations on the theme of adjoint operators (cf. [**25**, Chapter 2, Section 5.3]), and it seems to be known. We shall give its proof for the sake of completeness.

**Proposition 2.9.** *Suppose $f$ and $\varphi$ are supported in a cube $Q \subset \mathbb{R}^n$. If $\varphi \in L^\infty(Q)$ and $f \in L\log L(Q)$, then*

$$\int_{\mathbb{R}^n} Tf(x)\varphi(x)\,dx = -\int_{\mathbb{R}^n} f(x)T\varphi(x)\,dx. \tag{2.8}$$

*Proof*: Set $K_\varepsilon(y) := K(y)\chi_{|y|>\varepsilon}$, $T_\varepsilon f := f * K_\varepsilon$, and

$$T^*f(x) := \sup_{\varepsilon>0} |T_\varepsilon f(x)|.$$

Since $K_\varepsilon$ is odd (because $K$ does), and the double integral

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_\varepsilon(x-y)f(y)\varphi(x)\,dy\,dx$$

converges absolutely, we clearly have

$$\int_{\mathbb{R}^n} T_\varepsilon f(x)\varphi(x)\,dx = -\int_{\mathbb{R}^n} f(x)T_\varepsilon\varphi(x)\,dx \quad (\varepsilon > 0). \tag{2.9}$$

Next, $|(T_\varepsilon f)\varphi| \le |\varphi|(T^*f)$ and $|f(T_\varepsilon\varphi)| \le |f|(T^*\varphi)$.

The conditions on $f$ and $\varphi$ imply $T^*f \in L^1(Q)$ and $T^*\varphi \in L_{\exp}(Q)$, respectively (see, e.g., [**25**, Chapter 2, Section 6.2]). Since $\varphi$ is supported in $Q$ and $\varphi \in L^\infty(Q)$, we have $\varphi(T^*f) \in L^1(\mathbb{R}^n)$. On the other hand, by the generalized Hölder inequality (2.7), $f(T^*\varphi) \in L^1(\mathbb{R}^n)$.

Hence, letting $\varepsilon \to 0$ in (2.9) and using the dominated convergence theorem, we get (2.8). □

**2.6. Interpolation in Banach lattices.** We fix here some terminology (cf. [**7**, Chapter 1] and [**3**]). Let $(\mathcal{R}, \Sigma, \mu)$ be a measure space and $X$ be a Banach space of (equivalence classes of a.e. equal) real valued measurable functions on $\mathcal{R}$ such that if $|g| \le |f|$ a.e., where $f \in X$ and $g$ is measurable, then $g \in X$ and $\|g\|_X \le \|f\|_X$. The space $X$ is called a *Banach lattice* on $(\mathcal{R}, \Sigma, \mu)$. The *Köthe dual* or *associate space* $X'$ of any Banach lattice $X$ on $(\mathcal{R}, \Sigma, \mu)$ is defined to be the space of real valued measurable functions $g$ on $\mathcal{R}$ for which $fg \in L^1(\mathcal{R}, \Sigma, \mu)$ for each $f \in X$. For every $g \in X'$, put

$$\|g\|_{X'} := \sup\left\{ \int_{\mathcal{R}} |fg|\, d\mu : f \in X,\ \|f\|_X \le 1 \right\}.$$

To ensure that this is a norm rather than a seminorm we must assume that $X$ is *saturated*, that is, every $E \in \Sigma$ with $\mu(E) > 0$ has a measurable subset $F$ of finite positive measure for which $\chi_F \in X$.

Let $X_0$ and $X_1$ be Banach lattices on $(\mathcal{R}, \Sigma, \mu)$ and $0 < \theta < 1$. The *Calderón product* (see [**3**, p. 123]) consists of all real valued measurable functions $f$ such that a.e. pointwise inequality $|f| \le \lambda |f_0|^{1-\theta} |f_1|^\theta$ holds for some $\lambda > 0$ and elements $f_j$ in $X_j$ with $\|f_j\|_{X_j} \le 1$ for $j = 0, 1$. The norm of $f$ in $X_0^{1-\theta} X_1^\theta$ is defined to be the infimum of all values $\lambda$ appearing in the above inequality. From results of [**3**, Sections 6, 7, and 13.6] one can extract the following interpolation theorem.

**Theorem 2.10.** *Let $X_0$ and $X_1$ be real Banach lattices, one of which is reflexive. Let $A$ be a linear operator bounded on $X_0$ and $X_1$. Then $A$ is bounded on $X_0^{1-\theta} X_1^\theta$ and*

$$\|A\|_{\mathcal{B}(X_0^{1-\theta} X_1^\theta)} \le 2 \|A\|_{\mathcal{B}(X_0)}^{1-\theta} \|A\|_{\mathcal{B}(X_1)}^\theta.$$

The following remarkable formula was proved by Lozanovskiĭ [**17**, Theorem 2] under some additional assumptions. Cwikel and Nilsson relaxed assumptions on $X_0$ and $X_1$ and proved the following (see [**7**, Theorem 7.2]).

**Theorem 2.11.** *For arbitrary Banach lattices $X_0$ and $X_1$,*

$$(X_0^{1-\theta} X_1^\theta)' = (X_0')^{1-\theta} (X_1')^\theta, \quad 0 < \theta < 1,$$

*with equality of the norms.*

We refer to [**18**, Chapter 15] for generalizations of Theorems 2.10 and 2.11 to the case of so-called *Calderón-Lozanovskiĭ spaces.*

A Banach lattice $X$ is said to have the Fatou property if for every a.e. pointwise increasing sequence $f_n$ of non-negative functions in $X$ with $\sup_n \|f_n\|_X < \infty$, the function $f$, defined by $f(x) := \lim_{n\to\infty} f_n(x)$, is in $X$ and $\|f\|_X = \lim_{n\to\infty} \|f_n\|_X$. It is well known (see, e.g., [**26**, p. 452]) that if $X$ is a saturated Banach lattice, then $X = X''$ isometrically if and only if $X$ has the Fatou property.

**Corollary 2.12.** *If $X$ is a saturated Banach lattice with the Fatou property and $X'$ is its associate space, then*

$$X^{1/2}(X')^{1/2} = L^2 \tag{2.10}$$

*with equality of the norms.*

*Proof:* This result is contained in [**17**, Theorem 5] in a slightly different form. For the convenience of the readers we reproduce here its proof from [**18**, p. 185].

Since $X$ is saturated, so is $X'$. Clearly, $X'$ has the Fatou property. By the hypothesis, $X$ has the Fatou property too. Then $X = X''$ and $X' = X'''$ with equalities of the norms. Put $Z = X^{1/2}(X')^{1/2}$. Applying Theorem 2.11 with $\theta = 1/2$, we get

$$\begin{aligned} Z' &= (X')^{1/2}(X'')^{1/2} = (X')^{1/2}X^{1/2} = Z, \\ Z'' &= (X'')^{1/2}(X''')^{1/2} = X^{1/2}(X')^{1/2} = Z \end{aligned} \tag{2.11}$$

with equalities of the norms.

If $f \in Z = Z' = Z''$ is a non-zero function, then

$$\|f\|_{Z'} = \|f\|_{Z''} = \sup_{\|g\|_{Z'}\le 1} \int_{\mathcal{R}} |fg|\, d\mu \ge \int_{\mathcal{R}} \frac{f^2}{\|f\|_{Z'}}\, d\mu = \frac{\|f\|_{L^2}^2}{\|f\|_{Z'}}.$$

Hence,

$$\|f\|_Z = \|f\|_{Z'} = \|f\|_{Z''} \ge \|f\|_{L^2}. \tag{2.12}$$

By duality, from the latter inequality we get

$$\|g\|_{L^2} \ge \|g\|_{Z'} \quad \text{for all} \quad g \in Z'. \tag{2.13}$$

From (2.12) and (2.13) it follows that $L^2 = Z'$ isometrically. Combining the latter equality with (2.11), we arrive at (2.10). □

We shall apply the results of this section in the following form.

**Theorem 2.13.** *Let* $p\colon \mathbb{R}^n \to [1,\infty)$ *be a measurable function satisfying* (1.1). *Let* $A$ *be a linear operator bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$ *and* $L^{p'(\cdot)}(\mathbb{R}^n)$. *Then* $A$ *is bounded on* $L^2(\mathbb{R}^n)$ *and*

$$\|A\|_{\mathcal{B}(L^2)} \le 2\sqrt{r_p}\|A\|^{1/2}_{\mathcal{B}(L^{p(\cdot)})}\|A\|^{1/2}_{\mathcal{B}(L^{p'(\cdot)})}. \tag{2.14}$$

*Proof:* It is easy to see that $L^{p(\cdot)}(\mathbb{R}^n)$ is a saturated Banach lattice on $\mathbb{R}^n$ equipped with the Lebesgue measure. By [**15**], (1.1) is equivalent to the reflexivity of $L^{p(\cdot)}(\mathbb{R}^n)$. Moreover, $L^{p(\cdot)}(\mathbb{R}^n)$ has the Fatou property (see, e.g., [**11**, Proposition 1.3]). So, we can apply Theorem 2.10 and Corollary 2.12.

From the obvious equality $r_p = r_{p'}$ and (2.1) we get

$$\|A\|_{\mathcal{B}([L^{p(\cdot)}]')} \le r_p\|A\|_{\mathcal{B}(L^{p'(\cdot)})}. \tag{2.15}$$

Applying Theorem 2.10 with $\theta = 1/2$ and Corollary 2.12 to $X_0 = X = L^{p(\cdot)}(\mathbb{R}^n)$ and $X_1 = X' = [L^{p(\cdot)}(\mathbb{R}^n)]'$, we get

$$\|A\|_{\mathcal{B}(L^2)} = \|A\|_{\mathcal{B}((L^{p(\cdot)})^{1/2}([L^{p(\cdot)}]')^{1/2})} \le 2\|A\|^{1/2}_{\mathcal{B}(L^{p(\cdot)})}\|A\|^{1/2}_{\mathcal{B}([L^{p(\cdot)}]')}. \tag{2.16}$$

Combining (2.15) and (2.16), we arrive at (2.14). □

## 3. Proof of Theorem 1.1

We start with Part (a). The proof of this part is similar to the proof of Theorem 2.7.

Let $f \in L^\infty_c$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$. In view of [**23**, Theorem 1.1], the function $[b,T]f$ satisfies (2.5). Thus, putting $[b,T]f$ in place of $\varphi$ in Theorem 2.6 and then applying Proposition 2.3 and Theorem 2.5, we get

$$\begin{aligned}\int_{\mathbb{R}^n}\Big|\big([b,T]f\big)(x)g(x)\Big|\,dx &\le c_n(1/\lambda_n)^{1/\delta}\int_{\mathbb{R}^n}([b,T]f)^\#_\delta(x)Mg(x)\,dx\\ &\le c'\|b\|_*\int_{\mathbb{R}^n}M(Tf)(x)Mg(x)\,dx\\ &\quad + c'\|b\|_*\int_{\mathbb{R}^n}MMf(x)Mg(x)\,dx\end{aligned}$$

with $c' := c_n c_{\delta,n}(1/\lambda_n)^{1/\delta}$. From the latter inequality, Lemma 2.1, Theorem 2.7, and the condition $p, p' \in \mathcal{M}(\mathbb{R}^n)$ it follows that

$$\begin{aligned} &\int_{\mathbb{R}^n} \Big| \big([b,T]f\big)(x) g(x) \Big| \, dx \\ &\qquad \le c' r_p \|b\|_* \Big( \|M(Tf)\|_{L^{p(\cdot)}} + \|MMf\|_{L^{p(\cdot)}} \Big) \|Mg\|_{L^{p'(\cdot)}} \\ &\qquad \le C_p \|b\|_* \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}}, \end{aligned} \tag{3.1}$$

where

$$C_p := c' r_p \|M\|_{\mathcal{B}(L^{p(\cdot)})} \|M\|_{\mathcal{B}(L^{p'(\cdot)})} \Big( \|T\|_{\mathcal{B}(L^{p(\cdot)})} + \|M\|_{\mathcal{B}(L^{p(\cdot)})} \Big).$$

From (3.1) and (2.1) we obtain for all $f \in L_c^\infty$,

$$\|[b,T]f\|_{L^{p(\cdot)}} \le \|[b,T]f\|^0_{L^{p(\cdot)}} \le C_p \|b\|_* \|f\|_{L^{p(\cdot)}}.$$

By Lemma 2.2, $[b,T]$ can be extended by continuity to a bounded linear operator on $L^{p(\cdot)}(\mathbb{R}^n)$, and $\|[b,T]\|_{\mathcal{B}(L^{p(\cdot)})} \le C_p \|b\|_*$. Part (a) is proved.

We turn now to the proof of Part (b). Suppose $b \in L \log L(Q)$ for any cube $Q$ and $[b,T]$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Let $f \in L_c^\infty$ and $\varphi \in L^{p(\cdot)}(\mathbb{R}^n)$. For natural $k$ set $\varphi_k := \min\{|\varphi|, k\} \chi_{B(0,k)}$, where $B(0,k)$ is the ball of radius $k$ centered at the origin. Let also $\varphi'_k = \varphi_k \operatorname{sgn}([b,T]f)$.

Clearly, $bf$ and $b\varphi'_k$ belong to $L \log L(Q)$ for any cube $Q \subset \mathbb{R}^n$ and any $k$. Applying Proposition 2.9 yields

$$\begin{aligned} \int_{\mathbb{R}^n} |\big([b,T]f\big)(x) \varphi_k(x)| \, dx &= \int_{\mathbb{R}^n} \big([b,T]f\big)(x) \varphi'_k(x) \, dx \\ &= \int_{\mathbb{R}^n} \big([b,T]\varphi'_k\big)(x) f(x) \, dx. \end{aligned}$$

Hence, by Lemma 2.1,

$$\begin{aligned} \int_{\mathbb{R}^n} |\big([b,T]f\big)(x) \varphi_k(x)| \, dx &\le r_p \|[b,T]\|_{\mathcal{B}(L^{p(\cdot)})} \|\varphi'_k\|_{L^{p(\cdot)}} \|f\|_{L^{p'(\cdot)}} \\ &\le r_p \|[b,T]\|_{\mathcal{B}(L^{p(\cdot)})} \|\varphi\|_{L^{p(\cdot)}} \|f\|_{L^{p'(\cdot)}}. \end{aligned}$$

By the Fatou convergence theorem,

$$\int_{\mathbb{R}^n} |\big([b,T]f\big)(x) \varphi(x)| \, dx \le r_p \|[b,T]\|_{\mathcal{B}(L^{p(\cdot)})} \|\varphi\|_{L^{p(\cdot)}} \|f\|_{L^{p'(\cdot)}}.$$

Hence, taking into account (2.1), for every $f \in L_c^\infty$,

$$\|[b,T]f\|_{L^{p'(\cdot)}} \le \|[b,T]f\|^0_{L^{p'(\cdot)}} \le r_p \|[b,T]\|_{\mathcal{B}(L^{p(\cdot)})} \|f\|_{L^{p'(\cdot)}}.$$

From the latter inequality and Lemma 2.2 we deduce that $[b,T]$ is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$ and $\|[b,T]\|_{\mathcal{B}(L^{p'(\cdot)})} \le r_p \|[b,T]\|_{\mathcal{B}(L^{p(\cdot)})}$. In view of Theorem 2.13, it follows that $[b,T]$ is bounded on $L^2(\mathbb{R}^n)$ and

$$\|[b,T]\|_{\mathcal{B}(L^2)} \le 2r_p \|[b,T]\|_{\mathcal{B}(L^{p(\cdot)})}. \tag{3.2}$$

On the other hand, by Janson's theorem [**12**], if $[b,T]$ is bounded on $L^2(\mathbb{R}^n)$, then $b \in BMO(\mathbb{R}^n)$. Moreover, from the proof in [**12**] one can see that there exists a positive constant $c_2(K)$ such that

$$\|b\|_* \le c_2(K) \|[b,T]\|_{\mathcal{B}(L^2)}. \tag{3.3}$$

Combining (3.2) and (3.3), we arrive at Part (b) with $C'_p := 2r_p c_2(K)$. Theorem 1.1 is proved.

## 4. Concluding remarks

*Remark* 4.1. We have learned recently that Diening has obtained a new characterization of the class $\mathcal{M}(\mathbb{R}^n)$. In particular, $p \in \mathcal{M}(\mathbb{R}^n)$ if and only if $p' \in \mathcal{M}(\mathbb{R}^n)$ [**9**, Theorem 8.1] and $p \in \mathcal{M}(\mathbb{R}^n)$ implies $(p/s)' \in \mathcal{M}(\mathbb{R}^n)$ for some $s \in (0,1)$ [**9**, Corollary 8.8]. So, the condition $p' \in \mathcal{M}(\mathbb{R}^n)$ in Theorems 1.1 and 2.7 can be removed. For singular integral operators this is noticed already in [**9**, Theorem 8.14].

*Remark* 4.2. Part (a) of Theorem 1.1 holds for a more general class of Calderón-Zygmund operators [**13**].

*Remark* 4.3. All results of this paper can be extended to the context of Banach function spaces in the sense of Luxemburg (see [**2**, Chapter 1, Definition 1.1]). From [**2**, Chapter 1, Theorem 3.11 and Corollary 4.4] it follows that if $X(\mathbb{R}^n)$ is a reflexive Banach function space, then $L^\infty_c$ is dense in $X(\mathbb{R}^n)$ and in its associate space $X'(\mathbb{R}^n)$. Theorems 1.1 and 2.7 remain valid for reflexive Banach function spaces under the assumption that the Hardy-Littlewood maximal function is bounded on $X(\mathbb{R}^n)$ and on $X'(\mathbb{R}^n)$. The proofs of these statements are minor modifications of those for Theorems 1.1 and 2.7.

*Remark* 4.4. In the recent preprint by Cruz-Uribe, Fiorenza, Martell, and Pérez [**5**] the authors use extrapolation theory to deduce the boundedness of a wide variety of operators on generalized $L^p$ spaces with variable exponent. In particular, they give a different proof of Theorem 1.1(a).

## Acknowledgments

The first author is supported by F.C.T. (Portugal) grant SFRH/BPD/11619/2002. The second author is grateful for the support given by "Centro de Matemática e Aplicações", Instituto Superior Técnico, Lisbon, Portugal, during his short stay in Lisbon in January-February of 2004.

We would like to thank Lars Diening and Aleš Nekvinda for sharing with us their preprints [**8**], [**9**], [**21**]. We also thank the referee who kindly informed us about the work [**5**] appeared after the submission of our paper for publication.


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Primera versió rebuda el 26 de gener de 2004,
darrera versió rebuda el 31 de juliol de 2004.