A FAMILY OF CRITICALLY FINITE MAPS WITH SYMMETRY

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Abstract

The symmetric group $\mathcal{S}_n$ acts as a reflection group on $\mathbf{CP}^{n-2}$ (for $n \geq 3$). Associated with each of the $\binom{n}{2}$ transpositions in $\mathcal{S}_n$ is an involution on $\mathbf{CP}^{n-2}$ that pointwise fixes a hyperplane —the mirrors of the action. For each such action, there is a unique $\mathcal{S}_n$-symmetric holomorphic map of degree $n+1$ whose critical set is precisely the collection of hyperplanes. Since the map preserves each reflecting hyperplane, the members of this family are critically-finite in a very strong sense. Considerations of symmetry and critical-finiteness produce global dynamical results: each map's Fatou set consists of a special finite set of superattracting points whose basins are dense.

1. Overview

Complex dynamics in several dimensions has been the object of considerable recent study. Some specialized previous work in this field treats a variety of maps that share a common property: they respect the action of a finite group on a complex projective space. (See [**C1**], [**C2**], [**C3**].) The nature of these investigations leads to a consideration of issues pertaining to global dynamics. While the most significant dynamical claims possess experimental support, they remain theoretical conjectures. The current project stems from a desire to find symmetrical maps with interesting geometry and tractable dynamics. Its first fruit is an infinite family of special maps each of whose members respect the action of the symmetric group $\mathcal{S}_n$. In fact, for each $n \geq 3$, there is a unique holomorphic map g on $\mathbf{CP}^{n-2}$ whose critical set consists of an $\mathcal{S}_n$ orbit of $\binom{n}{2}$ hyperplanes that g preserves. This leads to a strong form of critical finiteness that yields several global dynamical results of the type that eluded earlier undertakings.

The treatment develops in three stages:

(1) Some background on special actions of $\mathcal{S}_n$ and their associated symmetrical maps.

(2) Proofs that the special family of critically-finite maps with $\mathcal{S}_n$ symmetry exists and that each member is unique and holomorphic.

(3) Proofs of claims concerning the dynamics of the maps (in the cases $n = 3, 4$). Specifically, each member has a certain attractor with dense basins. When $n > 4$, the claim concerning the attractor is conjectured.

Finally, some graphical results for low-dimensional cases appear.

2. $\mathcal{S}_n$ acts on $\mathbf{CP}^{n-2}$

The permutation action of the symmetric group $\mathcal{S}_n$ on $\mathbf{C}^n$ preserves the hyperplane

$$\mathcal{H} = \left\{ \sum_{k=1}^{n} x_k = 0 \right\} \simeq \mathbf{C}^{n-1}$$

and, thereby, restricts to a faithful $(n-1)$-dimensional irreducible representation. This action on $\mathbf{C}^{n-1}$ projects one-to-one to a group $\mathcal{G}_n$ on $\mathbf{H} := \mathbf{P}\mathcal{H} \simeq \mathbf{CP}^{n-2}$.

2.1. Special orbits and reflection hyperplanes. The smallest $\mathcal{G}_n$ orbit consists of the n points

$$[1 - n, 1, \ldots, 1], \ldots, [1, \ldots, 1, 1 - n].$$

(Square brackets indicate points in projective space.)

Corresponding to the $\binom{n}{2}$ transpositions (ij) in $\mathcal{S}_n$ are $\binom{n}{2}$ involutions

$$x_i \longleftrightarrow x_j$$

on $\mathbf{H}$ that generate $\mathcal{G}_n$ as a *complex reflection group*. Each generating involution fixes the point

$$[0, \ldots, 0, \overset{i}{\overbrace{1}}, 0, \ldots, 0, \overset{j}{\overbrace{-1}}, 0, \ldots, 0]$$

and pointwise fixes the companion hyperplane $\{x_i = x_j\}$. This point-hyperplane pair gives the only fixed points of the involution. They form $\mathcal{G}_n$ orbits of size $\binom{n}{2}$. For ease of reference, use the term "$\binom{n}{2}$-hyperplane".

2.2. Coordinates. The transformation $A\colon \mathbf{C}^n \to \mathbf{C}^{n-1}$ given by

$$u = Ax, \quad A = \begin{pmatrix} 1 & 0 & \ldots & 0 & -1 \\ 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -1 \end{pmatrix} = (a_{ij}), \quad a_{ij} = \begin{cases} 1 & i = j \\ -1 & j = n \\ 0 & \text{otherwise} \end{cases}$$

gives a special system of $n-1$ coordinates on $\mathbf{H}$ where the n-point orbit is

$$[1, 0, \ldots, 0], \ldots, [0, \ldots, 0, 1], [1, \ldots, 1].$$

Note that the null space of A is the euclidean orthogonal complement to $\mathbf{H}$. This change of coordinates has an "inverse"

$$x = Bu, \quad B = \begin{pmatrix} 1-n & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1-n & 1 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & 1 & 1-n \\ 1 & 1 & 1 & \ldots & 1 & 1 \end{pmatrix}$$

which gives

$$AB = -n\, I_{n-1}, \quad BA = \mathbf{1}_n - n\, I_n$$

where I_m is the $m \times m$ identity and $\mathbf{1}_n$ is the $n \times n$ matrix each entry of which is 1. Accordingly, A and B induce isomorphisms between $\mathbf{H}$ and $\mathbf{CP}^{n-2}$.

In u-coordinates, the $\binom{n}{2}$-hyperplanes are the $n-1$ coordinate hyperplanes $\{u_k = 0\}$ and the $\binom{n-1}{2}$ spaces $\{u_k = u_\ell\}$. The points determined by the intersections of the $\binom{n}{2}$-hyperplanes play a central role in subsequent developments. Their description is especially simple in u. (See Table 1.) With one exception, each orbit consists of points p_k and q_k with complementary coordinates.

Relative to the u space, $\mathcal{G}_n$ is generated over the permutation action $\mathcal{G}_{n-1}$ of $\mathcal{S}_{n-1}$ on the u_k by means of the involution

$$T = \begin{pmatrix} -1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \ldots & 1 & 0 \\ -1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

that transposes the pair $\{p_1, q_1\}$ and fixes the remaining members of the n-point orbit. Note that T is the u version of the transformation

$$x_1 \longleftrightarrow x_n.$$

n	Representative points on $n-2$ hyperplanes	Orbit size
$2m-1$	$p_k = [\overbrace{1,\ldots,1}^{k},\overbrace{0,\ldots,0}^{n-k-1}]$	$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$
	$q_k = [\overbrace{1,\ldots,1}^{n-k},\overbrace{0,\ldots,0}^{k-1}]$	$k = 1,\ldots,m-1$
$2m$	$p_k = [\overbrace{1,\ldots,1}^{k},\overbrace{0,\ldots,0}^{n-k-1}]$	$\begin{cases} \binom{n}{k} & k < m \\ \dfrac{1}{2}\binom{n}{k} = \binom{n-1}{k-1} & k = m \end{cases}$
	$q_k = [\overbrace{1,\ldots,1}^{n-k},\overbrace{0,\ldots,0}^{k-1}]$	$k = 1,\ldots,m$

TABLE 1. Points determined by intersections of $\binom{n}{2}$-hyperplanes.

3. $\mathcal{G}_n$ equivariants

Consider a map
$$f = [f_1,\ldots,f_{n-1}]$$
from **H** to itself given by homogeneous polynomials in
$$u = (u_1,\ldots,u_{n-1})$$
of degree r. In general, f can be meromorphic; that is, for some $p \in \mathbf{C}^{n-1}$, $f(p) = 0$ for every lift of f to $\mathbf{C}^{n-1}$. We say that f is $\mathcal{G}_n$-*equivariant* when it sends a group orbit to a group orbit. Algebraically, this means that f commutes with every element of $\mathcal{G}_n$. Obviously, f is $\mathcal{G}_{n-1}$-equivariant as well. It readily follows that each component f_k is invariant under the stabilizer Z_k of u_k. Thus, we can express a component by

$$f_k = \sum_{\ell=0}^{r} u_k^{r-\ell} A_{k,\ell}$$

where $A_{k,\ell}$ is a degree-ℓ Z_k invariant. Accordingly, each $A_{k,\ell}$ is taken to be a polynomial in the elementary symmetric functions in the complementary variables

$$\widehat{u}_k = (u_1,\ldots,u_{k-1},u_{k+1},\ldots,u_{n-1}).$$

Alternatively, we can employ the elementary symmetric functions in u when expressing $A_{k,\ell}$. This is a matter of expressing a polynomial in $\widehat{u}_k$

in terms of a polynomial in u and a polynomial in $\widehat{u}_k$ with lower degree. Specifically, let $\widehat{S}_m$ and S_m be the degree-m elementary symmetric functions in $\widehat{u}_k$ and u respectively. Taking $S_0 = 1$, the relations

$$\widehat{S}_m = S_m - u_k \, \widehat{S}_{m-1}$$

give a reductive scheme for the replacement process.

An immediate consequence of $\mathcal{G}_{n-1}$ equivariance is that

$$A_{j,\ell} = A_{k,\ell} := A_\ell \quad \text{for all } j, k, \ell.$$

We can say a bit more concerning the form that $\mathcal{G}_n$ equivariants take.

First, consider a point a that some element $M \in \mathcal{G}_n$ fixes and observe that

$$M f(a) = f(Ma) = f(a).$$

Thus, f either sends a to another fixed point of M or blows up at a —that is, for any lift $\widetilde{f}$ and $\widetilde{a}$ of f and a to $\mathbf{C}^{n-1}$, $\widetilde{f}(\widetilde{a}) = 0$. Applying this condition to the $\binom{n}{2}$-hyperplanes, provided that a is not a point of indeterminacy, each point on such a hyperplane must map to a point that is fixed by the involution that fixes the hyperplane pointwise. The only place for the image of such a point is on the hyperplane itself or its companion point. Under a holomorphic map, the image cannot be the companion point —this would force the entire hyperplane to collapse to the point. So, a holomorphic $\mathcal{G}_n$ equivariant f sends an $\binom{n}{2}$-hyperplane to itself. This circumstance forces f_k to be divisible by u_k and, thereby, requires the terms A_r to be a power of S_{n-1} or to vanish. In particular, when $r \leq n - 1$, $A_r = 0$ so that

$$f_k = u_k \sum_{\ell=0}^{r-1} u_k^{r-\ell-1} A_\ell.$$

By design, the map f has $\mathcal{G}_{n-1}$ symmetry. To be fully $\mathcal{G}_n$-equivariant, the map must commute with T as well. This condition places strong restrictions on the A_ℓ. The general form they take might be an interesting result, but not one taken up by the current investigation. Here, the quest is for a family of $\mathcal{G}_n$ equivariants with very special properties.

4. Reflection hyperplanes as critical sets: existence, uniqueness, and holomorphy

Explicit computation in low-degree cases reveals the existence of a unique holomorphic $\mathcal{G}_n$ equivariant whose critical set is precisely the

$\binom{n}{2}$-hyperplanes counted with multiplicity two. These maps conform to a general formula. Let

$$g = [g_1, \ldots, g_{n-1}]$$

where

$$g_\ell = u_\ell^3 \, G_\ell, \quad G_\ell = \sum_{k=0}^{n-2} (-1)^k \frac{k+1}{k+3} u_\ell^k \, S_{n,n-2-k},$$

and $S_{n,\ell}$ is the degree-ℓ elementary symmetric function in $u_1, \ldots, u_{n-1}$. In the degree-0 case, take $S_{n,0} = 1$. By construction, each g is equivariant under the group $\mathcal{G}_{n-1}$ that permutes the u_ℓ. In addition, the u_ℓ^3 factor in each coordinate implies that the maps are doubly critical on $n-1$ of the $\binom{n}{2}$-hyperplanes —namely, where $u_\ell = 0$. Were g to commute with the transformation T that generates $\mathcal{G}_n$ over $\mathcal{G}_{n-1}$, symmetry would provide for double criticality on the remaining $\binom{n-1}{2}$ of the $\binom{n}{2}$-hyperplanes —where $u_j = u_k$. Moreover, since a degree-$(n+1)$ map in $n-1$ variables has a critical set whose degree is

$$(n-1)\,n = 2\binom{n}{2},$$

g's critical set would consist exclusively of the $\binom{n}{2}$-hyperplanes.

This section develops rather technical arguments for three main results. According to Theorem 4.1, the $\binom{n}{2}$-hyperplanes form g's critical set with multiplicity two. Moreover, Theorem 4.2 informs us that there is only one such map for each $\mathcal{G}_n$ action. Theorem 4.3 states that each g is holomorphic on $\mathbf{H}$ which implies that g preserves each $\binom{n}{2}$-hyperplane $\mathcal{L}$ rather than collapse $\mathcal{L}$ to a lower-dimensional variety; a contraction would force the map to blow up.

Thus, g is a family of maps each member of which is holomorphic, doubly-critical on the $\binom{n}{2}$-hyperplanes, and critically-finite. As a standing assumption, let $n \geq 3$.

Theorem 4.1. *The respective g is T-equivariant, hence, $\mathcal{G}_n$-equivariant.*

Theorem 4.2. *Under the action of $\mathcal{G}_n$, g is the unique rational map of degree $n+1$ for which each $\binom{n}{2}$-hyperplane is doubly critical.*

Theorem 4.3. *Each member of the family g is holomorphic on $\mathbf{H}$.*

Proof of Theorem 4.1: Propositions 4.5 and 4.8 below establish that g is symmetric under T as well as under $\mathcal{G}_{n-1}$. Since T generates $\mathcal{G}_n$ over $\mathcal{G}_{n-1}$, g is $\mathcal{G}_n$-equivariant. $\qquad\square$

The proofs of the propositions rely on a formula that describes how the elementary symmetric functions transform under T. This result

was found by pattern detection in low-degree cases. For simplicity of appearance, express the functions $S_{n,k}(u)$ in the suppressed form $S_{n,k}$.

Lemma 4.4. *For $k \leq n$, the $\mathcal{G}_{n-1}$ invariants $S_{n,k}$ transform under T according to*

$$S_{n,k}(Tu) = \sum_{\ell=0}^{k} (-1)^{\ell} \binom{n-k+\ell}{n-k} u_1^{\ell} S_{n,k-\ell}.$$

Proof: Proofs of several technical lemmas appear at [**C4**].   □

The argument for the T-equivariance of g examines the coordinates individually.

Proposition 4.5. *The factor G_1 of g_1 is T-invariant (in a linear as well as projective sense).*

Proof: The proof amounts to manipulation of sums. Since n is fixed here, let $S_k = S_{n,k}$. Consider

$$G_1(Tu) = \sum_{k=0}^{n-2} (-1)^k \frac{k+1}{k+3} (-u_1)^k S_{n-2-k}(Tu) = \sum_{k=0}^{n-2} \frac{k+1}{k+3} u_1^k S_{n-2-k}(Tu).$$

By Lemma 4.4,

$$G_1(Tu) = \sum_{k=0}^{n-2} \frac{k+1}{k+3} u_1^k \left(\sum_{\ell=0}^{n-2-k} (-1)^{\ell} \binom{n-(n-2-k)+\ell}{n-(n-2-k)} u_1^{\ell} S_{n-2-k-\ell} \right)$$

$$= \sum_{k=0}^{n-2} \frac{k+1}{k+3} \left(\sum_{\ell=0}^{n-2-k} (-1)^{\ell} \binom{k+\ell+2}{k+2} u_1^{k+\ell} S_{n-2-(k+\ell)} \right).$$

Setting $m = k + \ell$,

$$G_1(Tu) = \sum_{k=0}^{n-2} \frac{k+1}{k+3} \left(\sum_{m=k}^{n-2} (-1)^{m-k} \binom{m+2}{k+2} u_1^m S_{n-2-m} \right).$$

Reversing the order of summation,

$$G_1(Tu) = \sum_{m=0}^{n-2} \left(\sum_{k=0}^{m} (-1)^{m-k} \frac{k+1}{k+3} \binom{m+2}{k+2} \right) u_1^m S_{n-2-m}$$

$$= \sum_{m=0}^{n-2} (-1)^m (m+2)! \left(\sum_{k=0}^{m} (-1)^k \frac{k+1}{(k+3)!} \frac{1}{(m-k)!} \right) u_1^m S_{n-2-m}.$$

Lemma 4.6 below gives the sum over k:

$$G_1(Tu) = \sum_{m=0}^{n-2} (-1)^m (m+2)! \frac{m+1}{(m+3)!} u_1^m S_{n-2-m}$$

$$= \sum_{m=0}^{n-2} (-1)^m \frac{m+1}{m+3} u_1^m S_{n-2-m}$$

$$= G_1(u). \qquad \square$$

Lemma 4.6.

$$\sum_{k=0}^{m} (-1)^k \frac{k+1}{(k+3)!(m-k)!} = \frac{m+1}{(m+3)!}.$$

Proof: See [**C4**]. $\square$

Corollary 4.7. *Each g is T-equivariant in the first coordinate.*

Proof: Let $[\cdot]_1$ specify a map's first coordinate. Then

$$g_1 \circ T = -u_1^3 G_1 \circ T = -u_1^3 G_1 = [T \circ g]_1. \qquad \square$$

To establish overall T-equivariance, it suffices to consider the behavior of g under T in just the second coordinate. This follows directly from the commutativity of T and the members $\tau_{2,m} \in \mathcal{G}_n$ that simply transpose the second and mth basis elements:

$$[0,1,0,\ldots,0] \overset{\tau_{2,m}}{\longleftrightarrow} [0,0,\ldots,0,\underbrace{1}_{m},0,\ldots,0]$$

provided that $m \neq 1, n$. Expressed in terms of $\mathcal{S}_n$, this amounts to the commutativity of the disjoint transpositions $(1n)$ and $(2m)$. So, noting that g is $\mathcal{G}_{n-1}$-equivariant, hence, $\tau_{2,m}$-equivariant, and given that g is T-equivariant in its second coordinate,

$$g_m \circ T = [g \circ T]_m$$

$$= [(\tau_{2,m} \circ g \circ \tau_{2,m}) \circ T]_m = [\tau_{2,m} \circ (g \circ T \circ \tau_{2,m})]_m = [g \circ T \circ \tau_{2,m}]_2$$

$$= [g \circ T]_2 \circ \tau_{2,m} = [T \circ g]_2 \circ \tau_{2,m} = [T \circ g \circ \tau_{2,m}]_2$$

$$= [\tau_{2,m} \circ T \circ g]_2 = [T \circ g]_m.$$

Proposition 4.8. *The second coordinate of g satisfies the equivariance condition*

$$g_2 \circ T = [T \circ g]_2.$$

Proof: First, express $g_2 \circ T$ in a way that's useful for comparison to $[T \circ g]_2$. Again, set $S_k = S_{n,k}$. Applying Lemma 4.4,

$$g_2(Tu) = (u_2 - u_1)^3 \sum_{k=0}^{n-2} (-1)^k \frac{k+1}{k+3} (u_2 - u_1)^k \, S_{n-2-k}(Tu)$$

$$= \sum_{k=0}^{n-2} \frac{k+1}{k+3} \left(\sum_{\ell=0}^{n-2-k} (-1)^{k+\ell} \binom{k+2+\ell}{k+2} u_1^\ell \, S_{n-2-k-\ell} \right) (u_2 - u_1)^{k+3}.$$

Setting $m = k + \ell$,

$$g_2(Tu) = \sum_{k=0}^{n-2} \frac{k+1}{k+3} \left(\sum_{m=k}^{n-2} (-1)^m \binom{m+2}{k+2} u_1^{m-k} \, S_{n-2-m} \right) (u_2 - u_1)^{k+3}.$$

Reversing the order of summation,

$$g_2(Tu) = \sum_{m=0}^{n-2} (-1)^m \left(\sum_{k=0}^{m} \frac{k+1}{k+3} \binom{m+2}{k+2} u_1^{m-k} (u_2 - u_1)^{k+3} \right) S_{n-2-m}.$$

Lemma 4.9 below establishes a useful identity for the sum over k so that

$$g_2(Tu) = u_2^3 \sum_{m=0}^{n-2} (-1)^m \frac{m+1}{m+3} u_2^m \, S_{n-2-m}$$

$$- u_1^3 \sum_{m=0}^{n-2} (-1)^m \frac{m+1}{m+3} u_1^m \, S_{n-2-m}$$

$$- u_1 u_2 \sum_{m=0}^{n-2} (-1)^m (u_2^{m+1} - u_1^{m+1}) \, S_{n-2-m}.$$

The first two terms are $g_2(u)$ and $g_1(u)$ respectively. Since their difference amounts to $[Tg(u)]_2$,

$$\left[Tg(u)\right]_2 - g_2(Tu) = u_1 u_2 \sum_{m=0}^{n-2} (-1)^m (u_2^{m+1} - u_1^{m+1}) \, S_{n-2-m}.$$

Adding and subtracting $-u_1 u_2 S_{n-1}$ on the right,

$$\left[Tg(u)\right]_2 - g_2(Tu) = u_1 u_2 \left(\left(-S_{n-1} + \sum_{m=0}^{n-2} (-1)^m u_2^{m+1} \, S_{n-2-m} \right) \right.$$

$$\left. - \left(-S_{n-1} + \sum_{m=0}^{n-2} (-1)^m u_1^{m+1} \, S_{n-2-m} \right) \right).$$

Let $m = p - 1$, while, for the apparent variables u_1 and u_2, set $x = u_2$ and $y = u_1$. The result is

$$\left[Tg(u)\right]_2 - g_2(Tu) = x\,y\left(-\sum_{p=0}^{n-1}(-1)^p\,x^p\,S_{n-1-p} + \sum_{p=0}^{n-1}(-1)^p\,y^p\,S_{n-1-p}\right)$$

$$= x\,y\left(-\prod_{k=1}^{n-1}(u_k - x) + \prod_{k=1}^{n-1}(u_k - y)\right).$$

Thus, when $x = u_2$ and $y = u_1$,

$$\left[Tg(u)\right]_2 - g_2(Tu) = 0. \qquad \square$$

Lemma 4.9.

$$\sum_{k=0}^{m}\frac{k+1}{k+3}\binom{m+2}{k+2}u_1^{m-k}\,(u_2 - u_1)^{k+3}$$

$$= \frac{m+1}{m+3}\left(u_2^{m+3} - u_1^{m+3}\right) - u_1\,u_2\left(u_2^{m+1} - u_1^{m+1}\right).$$

Proof: See [**C4**]. $\square$

Now we turn to the matter of uniqueness.

Proof of Theorem 4.2: Suppose that

$$h = [h_1, \ldots, h_{n-1}]$$

is a map of this type. The strategy is to compare g to h in terms of u coordinates. Since h is $\mathcal{G}_{n-1}$-equivariant and doubly critical on each $\{u_k = 0\}$, the components of h have the form

$$h_k = u_k^3\,H_k.$$

Furthermore, each H_k is a degree-$(n-2)$ invariant under an $\mathcal{S}_{n-2}$-isomorphic subgroup of $\mathcal{G}_{n-1}$, namely, the stabilizer of u_k. It follows that we can express these polynomials by

$$H_k = \sum_{\ell=0}^{n-2} u_k^{n-2-\ell}\,V_\ell$$

where V_ℓ is a $\mathcal{G}_{n-1}$ invariant of degree ℓ.

By $\mathcal{G}_{n-1}$ symmetry, we can examine a single component: h_1, say. Now, consider V_{n-2}. In the event that u_1 divides V_{n-2}, the associated component takes the form

$$h_1 = u_1^4\,\widehat{H_1}.$$

But this implies that $\{u_1 = 0\}$ is triply critical which is at odds with the assumption that h is doubly critical on the $\binom{n}{2}$-hyperplanes. By degree counting, the latter state of affairs completely accounts for the critical set.

Accordingly, assume that $V_{n-2} \not\equiv 0$ when $u_1 = 0$. We can now say that

$$V_{n-2} = u_1 X + Y$$

where no monomial in Y contains u_1. Hence, Y is invariant under the stabilizer in $\mathcal{G}_{n-1}$ of u_1. Lemma 4.10 below reveals that Y is divisible by each u_k except u_1, of course. Since the degree of Y is $n-2$, this result implies that

$$Y = \alpha \prod_{k=2}^{n-1} u_k$$

where $\alpha \in \mathbf{C} - \{0\}$. The $\mathcal{G}_{n-1}$ invariance of V_{n-2} requires that every element in the $\mathcal{G}_{n-1}$ orbit of Y appears in V_{n-2} and only these terms appear. Thus,

$$V_{n-2} = \alpha S_{n-2}.$$

Recalling the form of g, lift g and h to maps $\widetilde{g}$ and $\widetilde{h}$ on $\mathbf{C}^{n-1}$ so that

$$G_1\big|_{u_1=0} = H_1\big|_{u_1=0}.$$

Also, we can lift $\mathcal{G}_{n-1}$ trivially to a linear group $\widetilde{\mathcal{G}}$. Consequently, the $\widetilde{\mathcal{G}}$ equivariant $\widetilde{g} - \widetilde{h}$ is either the zero map or is both doubly critical along the $\binom{n}{2}$-hyperplanes and, as in the case considered above, has the contrary property that its first component is divisible by u_1^4. Hence, the former case is the only possibility so that $\widetilde{h} = \widetilde{g}$. $\square$

Evidently, g's uniqueness is due to its *full* $\mathcal{G}_n$ symmetry —that is, to its T-equivariance in addition to its symmetry under $\mathcal{G}_{n-1}$. The proof of the following lemma makes this explicit.

Lemma 4.10. *Define Y as above. For $k \neq 1$, $Y\big|_{u_k=0} = 0$.*

Proof: Let $k \neq 1$. Equivariance under T requires the components of h to satisfy the following identities:

(1) $$H_1 \circ T = H_1$$

(2) $$(u_k - u_1)^3 H_k \circ T = u_k^3 H_k - u_1^3 H_1.$$

(To lessen clutter, suppress explicit mention of the variable u, where possible.) By (1),

$$\sum_{\ell=0}^{n-2}(-u_1)^{n-2-\ell}\, V_\ell \circ T = H_1 \circ T = H_1 = \sum_{\ell=0}^{n-2} u_1^{n-2-\ell}\, V_\ell.$$

From this we obtain

$$V_{n-2}\circ T = V_{n-2} + \sum_{\ell=0}^{n-3}\left(V_\ell - (-1)^\ell\, V_\ell \circ T\right) u_1^{n-2-\ell}$$

which we can abbreviate to

(3) $$V_{n-2}\circ T = V_{n-2} + u_1\, W_{n-3}.$$

Turning to (2),

$$(u_k - u_1)^3 \sum_{\ell=0}^{n-2}(u_k - u_1)^{n-2-\ell}\, V_\ell \circ T = \sum_{\ell=0}^{n-2}\left(u_k^{n+1-\ell} - u_1^{n+1-\ell}\right) V_\ell$$

$$(u_k - u_1)^3\, V_{n-2}\circ T - (u_k^3 - u_1^3)\, V_{n-2} = \sum_{\ell=0}^{n-3}\Big(\left(u_k^{n+1-\ell} - u_1^{n+1-\ell}\right) V_\ell$$
$$- (u_k - u_1)^{n+1-\ell}\, V_\ell \circ T\Big).$$

Expanding the first binomial on the left, using (3), and rearranging gives

$$3\, u_1\, u_k\, (u_k - u_1)\, V_{n-2} = (u_k - u_1)^3\, u_1\, W_{n-3}$$

$$- \sum_{\ell=0}^{n-3}\Big(\left(u_k^{n+1-\ell} - u_1^{n+1-\ell}\right) V_\ell - (u_k - u_1)^{n+1-\ell}\, V_\ell \circ T\Big).$$

Dividing through by the common factor $u_k - u_1$,

$$3\, u_1\, u_k\, V_{n-2} = (u_k - u_1)^2\, u_1\, W_{n-3}$$

$$- \sum_{\ell=0}^{n-3}\left(\left(\sum_{m=0}^{n-\ell} u_1^m\, u_k^{n-\ell-m}\right) V_\ell - (u_k - u_1)^{n-\ell}\, V_\ell \circ T\right).$$

Restricting to $\{u_1 = u_k\}$,

$$3\, u_1^2 \left(V_{n-2}\big|_{u_1=u_k} \right) = \sum_{\ell=0}^{n-3} (n-\ell+1)\, u_1^{n-\ell} \left(V_\ell\big|_{u_1=u_k} \right)$$

$$= u_1^3 \sum_{\ell=0}^{n-3} (n-\ell+1)\, u_1^{n-\ell-3} \left(V_\ell\big|_{u_1=u_k} \right).$$

Note that this expression makes sense since $n \geq 3$. Thus,

$$3 \left(V_{n-2}\big|_{u_1=u_k} \right) = u_1 \sum_{\ell=0}^{n-3} (n-\ell+1)\, u_1^{n-3-\ell} \left(V_\ell\big|_{u_1=u_k} \right).$$

Finally, since $Y = V_{n-2}\big|_{u_1=0}$,

$$Y\big|_{u_k=0} = \left(V_{n-2}\big|_{u_1=0} \right)\big|_{u_1=u_k} = \left(V_{n-2}\big|_{u_1=u_k} \right)\big|_{u_1=0} = 0. \qquad \square$$

The upcoming proof of Theorem 4.3 exploits a dimension-reducing process of restricting g to intersections of $\binom{n}{2}$-hyperplanes. This cascade of intersections leads to the special point-orbits determined by the hyperplanes. At these points, the map's behavior is explicitly computable.

Proof of Theorem 4.3: When $n = 3$, g is one-dimensional and hence, holomorphic. As for the non-trivial cases $n > 3$, choose the "literal" lift of g to $\mathbf{C}^{n-1}$:

$$\widetilde{g} = (g_1, \ldots, g_{n-1})$$

where

$$g_\ell = u_\ell^3\, G_\ell \quad \text{and} \quad G_\ell = \sum_{k=0}^{n-2} (-1)^k \frac{k+1}{k+3} u_\ell^k\, S_{n,n-2-k}.$$

Let X denote the union of the $\binom{n}{2}$-hyperplanes lifted to hyperspaces through 0 in $\mathbf{C}^{n-1}$. Suppose there is a point $a \in \mathbf{C}^{n-1}$ where $\widetilde{g}(a) = 0$. By homogeneity,

$$(n+1)\, D\widetilde{g}(u) = D\widetilde{g}(u)\, u$$

where

$$D\widetilde{g}(u) = \left(\frac{\partial g_i(u)}{\partial g_j(u)} \right)$$

is the Jacobian matrix of $\widetilde{g}$. Thus, $\widetilde{g}$ is critical at a. That is, a is a zero eigenvector for $D\widetilde{g}(a)$. In this case, the map collapses in the "radial" direction defined by a. Since $\widetilde{g}$ is critical only on X, a lies on one of X's constituent hyperspaces; call this hyperspace $\mathcal{L}^{n-2}$ ($\simeq \mathbf{C}^{n-2}$) and consider the restriction $\widetilde{g}_{n-2}$ of $\widetilde{g}$ to $\mathcal{L}^{n-2}$. (Note that the action of $\mathcal{G}_n$

restricted to $\mathcal{L}^{n-2}$ is isomorphic to $\mathcal{S}_{n-2}$ so that $\widetilde{g}_{n-2}$ is *not* the member of the family g for dimension $n-2$ where the action is that of $\mathcal{S}_{n-1}$.)

Since
$$\widetilde{g}_{n-2}(a) = 0,$$
a is a zero eigenvector for $D\widetilde{g}_{n-2}(a)$; the critical set of $\widetilde{g}_{n-2}$ contains a. But, a zero eigenvector v for $D\widetilde{g}_{n-2}(a)$ corresponds to a radial collapse in the v direction so that v is also a zero eigenvector for $D\widetilde{g}(a)$. But, as Lemma 4.11 below describes, $\det D\widetilde{g}_{n-2}(a)$ does not vanish identically on $\mathcal{L}^{n-2}$ so that the critical set of $\widetilde{g}_{n-2}$ is a proper algebraic subset of X and $\mathcal{L}^{n-2}$. Hence, the only possible location for a is where some hyperspace in X different from $\mathcal{L}^{n-2}$ intersects $\mathcal{L}^{n-2}$. Denote this intersection by $\mathcal{L}^{n-3}$.

Further reducing the dimension, let
$$\widetilde{g}_{n-3} = \widetilde{g}\big|_{\mathcal{L}^{n-3}}$$
so that $\widetilde{g}_{n-3}(a) = 0$ and a is critical for $\widetilde{g}_{n-3}$. As above, a belongs to the intersection of $\mathcal{L}^{n-3}$ with a hyperspace in X that does not contain $\mathcal{L}^{n-3}$.

This reduction continues with the outcome at each stage that a belongs to the intersection of $\binom{n}{2}$-hyperplanes. When the procedure arrives at dimension three, a lies on two planes through 0 in $\mathbf{C}^{n-1}$ —that is, a point in $\mathbf{CP}^{n-2}$— that are intersections of $\binom{n}{2}$-hyperspaces. But, Lemma 4.12 below implies that $\widetilde{g} \neq 0$ at these points. $\square$

Lemma 4.11. *For the restriction $\widehat{g}$ of $\widetilde{g}$ to any space $\mathcal{L}^m$ of dimension $m \neq 0$ determined by the intersection of hyperspaces in X, $\det D\widehat{g} \not\equiv 0$.*

Proof: By the permutation action of $\mathcal{G}_{n-1}$ on the u_k, we can take
$$\mathcal{L}^m = \left(\bigcap_{k=1}^{p}\{u_k = 0\}\right) \bigcap \left(\bigcap_{\substack{i,j=1,\ldots,n-m-p-2\\p<\ell_i<\ell_j}} \{u_{\ell_i} = u_{\ell_j}\}\right).$$

Any $\mathcal{L}^m$ space that is partially determined by the intersection of p sets of the form $\{u_k = 0\}$ belongs to the $\mathcal{G}_{n-1}$-orbit of the set specified above. Relabel the coordinates on $\mathcal{L}^m$ so that the restriction is expressed
$$\widehat{g}(\widehat{u}) = \widetilde{g}_m\big|_{\mathcal{L}^m}(\widehat{u}) = \begin{pmatrix}\widehat{g}_1(\widehat{u})\\\vdots\\\widehat{g}_m(\widehat{u})\end{pmatrix} = \begin{pmatrix}u_1^3\,\widehat{G}_1(\widehat{u})\\\vdots\\u_m^3\,\widehat{G}_m(\widehat{u})\end{pmatrix} \quad \text{where } \widehat{u} = \begin{pmatrix}u_1\\\vdots\\u_m\end{pmatrix}.$$

Let $D\widehat{g}_i$ be the Jacobian of $\widehat{g}_i$ so that

$$D\widehat{g}(\widehat{u}) = \begin{pmatrix} D\widehat{g}_1(\widehat{u}) \\ \vdots \\ D\widehat{g}_m(\widehat{u}) \end{pmatrix}.$$

In order for $\det D\widehat{g} \equiv 0$, the set $\{D\widehat{g}_i,\ i = 1, \ldots, m\}$ must be linearly dependent in functional terms. To establish linear independence, consider the relation

$$\sum_{j}^{m} a_j\, D\widehat{g}_j = 0.$$

By homogeneity,

$$(n+1)\,\widehat{g}_j(\widehat{u}) = D\widehat{g}_j(\widehat{u})\,\widehat{u}$$

and

$$\lambda(\widehat{u}) := \sum_{j}^{m} a_j\, u_j^3\, \widehat{G}_j = \sum_{j}^{m} a_j\, \widehat{g}_j = 0.$$

But, on $\mathcal{L}^m$ there are m members of the $\mathcal{G}_{n-1}$ orbit of

$$p_1 = (1, 0, \ldots, 0),$$

namely,

$$\widehat{p}_k = (0, \ldots, 0, \overset{k}{\overbrace{1}}, 0, \ldots, 0).$$

Since

$$\lambda(\widehat{p}_k) = a_k\, \widehat{G}_k(\widehat{p}_k),$$

the proof of Lemma 4.12 yields $a_k = 0$.   □

Lemma 4.12. *For the points p_m that represent the orbits determined by the intersections of $\binom{n}{2}$-hyperplanes,*

$$\widetilde{g}(p_m) \neq 0.$$

Proof: Recall that

$$p_m = (\underbrace{1, \ldots, 1}_{m}, 0, \ldots, 0), \quad m = 1, \ldots, \left\lceil \frac{n-1}{2} \right\rceil.$$

It suffices to compute $G_1(p_m)$.

A straightforward calculation gives

$$S_{n,k}(p_m) = \begin{cases} 0 & k > m \\ \binom{m}{k} & k \leq m \end{cases}.$$

With this,

$$G_1(p_m) = \sum_{k=0}^{n-2}(-1)^k\frac{k+1}{k+3}S_{n,n-2-k}(p_m)$$

$$= \sum_{k=n-2-m}^{n-2}(-1)^k\frac{k+1}{k+3}\binom{m}{n-2-k}.$$

Setting $p = n - 2 - k$,

$$G_1(p_m) = \sum_{p=0}^{m}(-1)^{n-2-p}\frac{n-p-1}{n-p+1}\binom{m}{p}$$

$$= (-1)^n\sum_{p=0}^{m}(-1)^p\frac{n-p-1}{n-p+1}\binom{m}{p}.$$

From Lemma 4.13 below,

$$G_1(p_m) = (-1)^n\frac{2(-1)^{m-1}}{(n+1)\binom{n}{m}} \neq 0. \qquad\qquad \square$$

Lemma 4.13.

$$\sum_{p=0}^{m}(-1)^p\frac{n-p-1}{n-p+1}\binom{m}{p} = \frac{2(-1)^{m-1}}{(n+1)\binom{n}{m}}.$$

Proof: See [**C4**]. $\square$

5. Reflection hyperplanes as critical sets: global dynamics

Let $\mathcal{L}^{n-3}$ generically denote an $\binom{n}{2}$-hyperplane and let X refer to the union of the $\mathcal{L}^{n-3}$. Where m of the $\mathcal{L}^{n-3}$ intersect to form a $\mathbf{CP}^{n-2-m}$, call the resulting space $\mathcal{L}^{n-2-m}$. (Note that more than m of the $\mathcal{L}^{n-3}$ can pass through an $\mathcal{L}^{n-2-m}$.)

Not only is g critically-finite on $\mathbf{H} \simeq \mathbf{CP}^{n-2}$ with critical set consisting of the $\mathcal{L}^{n-3}$ hyperplanes, the restriction $g\big|_{\mathcal{L}^{n-2-m}}$ is also critically-finite, having a collection of the $\mathcal{L}^{n-3-m}$ for its critical set. In [**FS1**], such behavior is called *strict* critical finiteness (Section 7). In fact, *all* of the $\mathcal{L}^{n-3-m}$ on an $\mathcal{L}^{n-2-m}$ are critical for $g\big|_{\mathcal{L}^{n-2-m}}$ though not with the same multiplicity.

5.1. The Fatou set of g. Following standard practice, the *Fatou set* F_g is where the family of iterates $\{g^k\}$ is normal and the *Julia set* J_g is the complement of F_g.

The behavior of g on an $\mathcal{L}^{n-3}$ plays a central dynamical role. Again, lift g to $\mathbf{C}^{n-1}$:

$$\widetilde{g} = (g_1, \ldots, g_{n-1})$$

with

$$g_\ell = u_\ell^3 \, G_\ell \quad \text{and} \quad G_\ell = \sum_{k=0}^{n-2} (-1)^k \frac{k+1}{k+3} u_\ell^k \, S_{n,n-2-k}.$$

For a space $\mathcal{L}^m \subset \mathbf{CP}^k$ lifted to $\mathbf{C}^{k+1}$, call the lifted space $\widetilde{\mathcal{L}}^{m+1}$.

Proposition 5.1. *For any $a \in \mathcal{L}^{n-3}$, g is critical in the direction off of the hyperplane.*

Proof: By symmetry, consider the $\widetilde{\mathcal{L}}^{n-2}$ given by $\{u_1 = 0\}$. For any $a \in \{u_1 = 0\}$, the first row of $D\widetilde{g}(a)$ vanishes. Thus, the local behavior of $\widetilde{g}$ collapses points onto $\widetilde{\mathcal{L}}^{n-2}$. Explicit calculation reveals that the collapse occurs in the direction of $(2, 1, \ldots, 1)$. $\qquad\square$

Recall that the p_m represent the point sets of $\mathcal{G}_n$ orbits determined by intersecting the $\mathcal{L}^{n-3}$. Refer to these orbits as "p_m-points". First of all, each such point is superattracting in all directions.

Theorem 5.2. *Under g, the fixed p_m-points are superattracting in every direction. Conversely, the only points that are superattracting in every direction are the p_m-points.*

Proof: To establish that, at p_m, g is critical in every direction in $\mathbf{CP}^{n-1}$ show that the Jacobian $D\widetilde{g}$ at p_m has rank 1. Here, p_m is lifted in the literal way. It then follows that, since $\widetilde{g}(p_m) \neq 0$, there are $n-2$ non-radial directions through p_m that have zero eigenvalue.

The Jacobian has the form

$$D\widetilde{g} = \begin{pmatrix} (a_{ij}) & (b_{ij}) \\ 0 & 0 \end{pmatrix}$$

where

$$a_{ij} = \begin{cases} 3\, G_i(p_m) + \dfrac{\partial G_i}{\partial u_i}(p_m) & i = j \\[2mm] \dfrac{\partial G_i}{\partial u_j}(p_m) & i \neq j \end{cases}, \quad i, j \leq m$$

$$b_{ij} = \frac{\partial G_i}{\partial u_{m+j}}(p_m), \qquad\qquad\quad i \leq m < j.$$

With $S_k = S_{n,k}$ a straightforward calculation establishes that, for $\ell \leq m$,

$$\frac{\partial S_k}{\partial u_\ell}(p_m) = \begin{cases} 0 & k > m \\ \binom{m-1}{k-1} & k \leq m \end{cases}$$

so that $\frac{\partial G_i}{\partial u_j}(p_m)$ is the same value for $i, j \leq m$ with $i \neq j$. Similarly, $\frac{\partial G_i}{\partial u_\ell}(p_m)$ is the same value for $\ell > m$. It remains to show that

$$3\, G_i(p_m) + \frac{\partial G_i}{\partial u_i}(p_m) = \frac{\partial G_j}{\partial u_k}(p_m) \quad \text{for all } i, j, k \leq m.$$

By manipulation of sums,

$$\frac{\partial G_i}{\partial u_i}(p_m) = \sum_{k=0}^{n-2}(-1)^{n-k}\frac{n-k-1}{n-k+1}(n-2-k)S_k(p_m)$$

$$+ \sum_{k=0}^{n-2}(-1)^{n-k}\frac{n-k-1}{n-k+1}\frac{\partial S_p}{\partial u_i}(p_m).$$

The second sum is $\frac{\partial G_j}{\partial u_\ell}(p_m)$ for $j, \ell \leq m$ and $j \neq \ell$. To show that the first sum amounts to $-3\, G_i(p_m)$, notice that, from the proof of Theorem 4.3,

$$\sum_{k=0}^{n-2}(-1)^{n-k}\frac{n-k-1}{n-k+1}(n-2-k)S_k(p_m)$$

$$= (-1)^n \sum_{k=0}^{m}(-1)^k\frac{n-k-1}{n-k+2}(n-2-k)\binom{m}{k}$$

$$= (-1)^n \sum_{k=0}^{m}(-1)^k\frac{n-k-1}{n-k+1}((n-k+1)-3)\binom{m}{k}$$

$$= (-1)^n \sum_{k=0}^{m}(-1)^k (n-1-k)\binom{m}{k} - 3\, G_i(p_m).$$

Finally, the calculation reduces to showing that the first sum vanishes. This follows readily by splitting the sum into two terms each of which is

a binomial expansion of $1 - 1$. Specifically,

$$\sum_{k=0}^{m}(-1)^k\,(n-1-k)\binom{m}{k} = (n-1)\sum_{k=0}^{m}(-1)^k\binom{m}{k} - \sum_{k=0}^{m}(-1)^k\,k\binom{m}{k}$$

$$= (n-1)(1-1)^m + m\sum_{k=1}^{m}(-1)^k\binom{m-1}{k-1}$$

$$= m\,(1-1)^{m-1}.$$

Thus, the nonzero rows of $D\widetilde{g}(p_m)$ are identical and the matrix has rank 1.

For the converse claim, consider a point q that is critical in every direction. When g is restricted to any intersection $\mathcal{L}^k$ of hyperplanes each of which is an $\mathcal{L}^{n-2}$, q is again critical for the restriction $g\big|_{\mathcal{L}^k}$. Hence, q lies on some $\mathcal{L}^{n-2}$ that does not contain $\mathcal{L}^k$ and so, is determined by the intersection of $\mathcal{L}^{n-2}$ spaces. $\qquad\square$

Now for the issue of the Fatou set F_g. Is there a Fatou component of g that is not in the basin of a p_m point?

Theorem 5.3. *For $n = 3, 4$, F_g consists of the basins of attraction of the p_m-points.*

Proof: When $n = 3$, the one-dimensional map g has three fixed critical p_m-points. A basic result in one-dimensional dynamics states that the Fatou set of a rational map with periodic critical points consists only of superattracting basins; indeed, the basins have full measure in $\mathbf{CP}^1$.

In the two-dimensional case $n = 4$, the claim follows from Theorem 5.2 and [**FS1**, Theorem 7.7]. The latter implies that if a holomorphic map f on $\mathbf{CP}^2$ has a critical set C such that 1) C is periodic and 2) $\mathbf{CP}^2 - C$ is Kobayashi hyperbolic, then f has only superattracting basins in its Fatou set. See below for an explanation of the fact that condition 2) applies to g. $\qquad\square$

The general case remains open.

Conjecture 5.4. *For $n \geq 5$, F_g consists of the basins of attraction of the p_m-points.*

One approach to this claim adopts a technique from the proof of Theorem 4.3: reduction of dimension to the one-dimensional case where some things are understood. The argument for Theorem 5.6 employs the same idea. Assume an arbitrary choice of $n \geq 5$.

The question of whether the basins of the p_m-points exhaust F_g calls for some preparation. Following [**U**], let C_f be the critical set of a holomorphic map f on $\mathbf{CP}^m$,

$$D_f := \bigcup_{k=1}^{\infty} f^k(C_f) \quad \text{and} \quad E_f := \bigcap_{k=1}^{\infty} f^k(\overline{D_f})$$

be the *postcritical set* and the ω-limit set of C_f respectively. Also, the *Fatou limit set* Λ_f is where the forward orbits of Fatou components accumulate. In the case of g, $D_g = E_g = X$.

Let $p \in F_g$ and U be the Fatou component to which p belongs. For a critically-finite map f, $\Lambda_f \subset E_f$ [**U**, Theorem 5.1]. Accordingly, the forward orbit $\{g^k(p)\}$ of p accumulates on some $\mathcal{L}^{n-3}$ and, by Proposition 5.1, is attracted to that $\mathcal{L}^{n-3}$ —call it $\mathcal{L}^{n-3}$ as well. Accordingly,

$$g^n(U) \longrightarrow \mathcal{L}^{n-3}.$$

The claim also follows from [**M**, Theorem 2.36] —a result established by consideration of expansion in the Kobayashi metric on the complement of the postcritical set.

The task now is to show that

$$g^r(U) \cap \mathcal{L}^{n-3} \neq \emptyset$$

for some r. An argument might develop in two steps: 1) the orbit of a point that is Fatou for g accumulates at points that are Fatou for $\widetilde{g} := g|_{\mathcal{L}^{n-3}}$; 2) a point that is Fatou for $\widetilde{g}$ is also Fatou for g and, thereby, belongs to a Fatou component in $\mathbf{CP}^{n-2}$.

To treat the first claim, let $q \in \mathcal{L}^{n-3}$ be a limit point of $\{g^k(p)\}$ with $g^{n_k}\big|_K \to h$ where $h: K \to \mathcal{L}^{n-3}$, $K \subset U$ is a neighborhood of p, and $h(p) = q$.

Suppose that q belongs to the Julia set $J_{\widetilde{g}}$. By Proposition 5.1, g is superattracting at $g^k(q)$ in some direction away from $\mathcal{L}^{n-3}$ for all k. This equips q with a stable set

$$S_q = \{x \mid \operatorname{dist}(g^{n_k}(x), g^{n_k}(q)) \longrightarrow 0\}$$

transverse to $\mathcal{L}^{n-3}$. If $\widetilde{g}$ were hyperbolic—as in the case $n = 4$, one might expect that the Kobayashi expansion at q would produce saddle-like behavior and force U to contain Julia points for g.

To see claim 2) above, let $q \in F_{\widetilde{g}}$ with a neighborhood $\widetilde{N}$ on which $\{\widetilde{g}^k\}$ is normal. Take N to be the connected neighborhood of q that is absorbed by $\widetilde{N}$ and includes $\widetilde{N}$; that is, N is the connected component of the stable set of $\widetilde{N}$

$$S_{\widetilde{N}} = \bigcup_{x \in \widetilde{N}} S_x$$

where $\widetilde{N} \subset N$ and S_x is the stable set of x. Every point in N belongs to some S_x. Thus, if $\widetilde{g}^{n_k}$ converges to $\widetilde{h}$ on $\widetilde{N}$, then g^{n_k} converges on N to

$$h(y) = \widetilde{h}(x), \quad y \in S_x.$$

The claims 1) and 2) imply that some $g^r(U)$ intersects $\mathcal{L}^{n-3}$; indeed, $g^r(U) \cap \mathcal{L}^{n-3}$ is a Fatou component for $\widetilde{g}$. By the critical finiteness of $\widetilde{g}$, the forward orbit of $g^r(U) \cap \mathcal{L}^{n-3}$ meets some $\mathcal{L}^{n-4}$ in Fatou points for $g\big|_{\mathcal{L}^{n-4}}$.

This cascade continues until some $g^s(U)$ makes contact with a line $\mathcal{L}^1$, in particular, with the Fatou set of $g\big|_{\mathcal{L}^1}$. Since $g\big|_{\mathcal{L}^1}$ has fixed critical points, it has only superattracting basins. The only critical points on $\mathcal{L}^1$ are p_m-points. Hence, $g^s(U) \cap \mathcal{L}^1$ lies in the basin of attraction of some such point.

How "large" are the basins of the p_m-points? First of all, let

$$B_f := \bigcup_{k \geq 0} f^{-k}(C_f)$$

be the *precritical* set of f. The following basic result yields that the closure of B_g contains the Julia set J_g [**FS1**, Proposition 6.5].

Theorem 5.5. *If* $f : \mathbf{CP}^k \to \mathbf{CP}^k$ *is holomorphic and* $\mathbf{CP}^k - \overline{B_f}$ *is hyperbolically embedded,*

$$J_f \subset A_f := \bigcap_{n>0} \overline{\bigcup_{m>n} f^{-m}(C_f)}.$$

To apply this result to g, we must see that it satisfies the hypotheses. By Theorem 4.3, g is holomorphic on $\mathbf{CP}^{n-2}$. Two theorems of M. Green imply that $\mathcal{L}^{n-1-m} - \overline{B_g\big|_{\mathcal{L}^{n-1-m}}}$ is hyperbolically embedded in $\mathcal{L}^{n-1-m}$ (taking $\mathcal{L}^{n-2} = \mathbf{H}$). (For details on Green's results, consult [**FS1**, Section 5].) To see this, suppose that, for $n \geq 4$ and $m \geq 2$,

$$\phi : \mathbf{C} \longrightarrow \mathcal{L}^{n-1-m} - \overline{B_g\big|_{\mathcal{L}^{n-1-m}}}$$

is holomorphic. Then $\phi(\mathbf{C})$ omits at least $n - m + 1$ hypersurfaces in $\mathcal{L}^{n-1-m}$, namely, some $\mathcal{L}^{n-m-2}$ spaces and their preimages. By one

of Green's theorems (Theorem 5.6 in [**FS1**]), $\phi(\mathbf{C})$ is contained in a compact complex hypersurface. Since such a hypersurface intersects the omitted hypersurfaces, $\phi(\mathbf{C})$ omits at least three points and so, is constant. The statement concerning hyperbolic embedding follows from Green's other theorem (Theorem 5.5 in [**FS1**]).

One other preliminary: since $C_g \subset g^{-1}(C_g)$, $J_g \subset A_g = \overline{B_g}$. We can now establish a bit of F_g's global structure.

Theorem 5.6. *Under the assumption that Conjecture 5.4 holds, the Fatou set F_g is dense in* **H**.

Proof: Consider $j_0 \in J_g$ and let U_0 be a neighborhood of j_0. By Theorem 5.5, some precritical points meet U_0 so that, for some m,

$$g^m(U_0) \cap C_g \neq \emptyset.$$

If

$$U_1 := g^m(U_0) \cap \mathcal{L}^{n-3}$$

fails to contain Julia points, the case is made. Otherwise, take a Julia point $j_1 \in U_1$, a neighborhood of j_1.

The map $g\big|_{\mathcal{L}^{n-3}}$ is critically finite with critical set C_{n-3} in the intersection of $\mathcal{L}^{n-3}$ and the hyperplanes in X different from $\mathcal{L}^{n-3}$. Hence, C_{n-3} is a collection of $\mathcal{L}^{n-4}$ spaces. Implementing the argument given for j_0 and U_0 under g using j_1 and U_1 under $g\big|_{\mathcal{L}^{n-3}}$ produces a neighborhood of a Julia point j_2 on some $\mathcal{L}^{n-4}$. The descent continues until it reaches a Julia point j_{n-3} and neighborhood U_{n-3} on an $\mathcal{L}^1$. Thus, U_{n-3} meets the Fatou set of $g\big|_{\mathcal{L}^1}$. Since $g\big|_{\mathcal{L}^1}$ has fixed critical points that are p_m-points, its Fatou set consists of the superattracting basins of those p_m-points. Accordingly, U_{n-3} —hence, U_0— contains points in F_g. $\square$

5.2. A query on the structure of g's Julia set. For the restricted map $\widehat{g} = g\big|_{\mathcal{L}^{n-3}}$, the Julia set is given by

$$J_{\widehat{g}} = J_g \cap \mathcal{L}^{n-3}.$$

The inclusion $J_{\widehat{g}} \subset J_g \cap \mathcal{L}^{n-3}$ is clear. If $x \notin J_{\widehat{g}}$, then x belongs to a basin of a p_m-point so that $x \notin J_g$. At each point $p \in J_{\widehat{g}}$, the map is superattracting in the direction away from $\mathcal{L}^{n-3}$. Thus, there is a "stable set" S_p of points in J_g whose orbits are attracted to the orbit of p. Accordingly, there is a stable bundle over $J_{\widehat{g}}$

$$S_{J_{\widehat{g}}} := \bigcup_{p \in J_{\widehat{g}}} S_p \subset J_g.$$

Are the S_p one-dimensional manifolds? Are the preimages of the $S_{J_{\widehat{g}}}$ dense in J_g?

In the case $n = 4$, g restricts to a critically-finite map $\widehat{g}$ on an $\mathcal{L}^1$ that is one of the six lines of reflection for $\mathcal{G}_4$. Figure 4 displays the three basins of attraction for $\widehat{g}$. The Julia set $J_{\widehat{g}}$ consists of the boundaries of these basins. For each Julia point $p \in \mathcal{L}^1$, there is an S_p away from the line. What can be said about the structure of $S_{J_{\widehat{g}}}$?

What about the points

$$K := J_g - \bigcup_X S_{J_{\widehat{g}}}$$

that are not absorbed by X?

On an $\mathcal{L}^1$, each Julia point is non-wandering and has a contracting direction onto $\mathcal{L}^1$ and an expanding direction in $\mathcal{L}^1$. For a hyperbolic map on $\mathbf{CP}^2$, the literature describes a grading of the non-wandering set Ω by the expanding dimension [**FS2**]:

$$\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2.$$

The p_m-points comprise Ω_0 and $\cup_X J_{\widehat{g}} \subset \Omega_1$. The non-wandering points not on X belong to K. Since any neighborhood of such a point p contains an open set that is attracted to X, there is expansion at p. Does it happen that

$$\Omega \cap K \subset \Omega_2$$

so that g is hyperbolic?

6. Geometry and dynamics in low-dimension

To avoid confusion, let g_{n+1} represent the particular map g on the respective $\mathcal{G}_n$-symmetric $\mathbf{H}$.

6.1. The one-dimensional case: g_4 and Halley's method. When $n = 3$, the reflecting "hyperplanes" consist of a three-point orbit. With these points located at

$$\{1, \rho, \rho^2 \mid \rho = e^{2\pi i/3}\},$$

the map's inhomogeneous expression on $\{u_2 \neq 0\}$ is

$$z \longrightarrow \frac{z(z^3 - 2)}{2z^3 - 1}.$$

We can realize the $\mathcal{G}_3$ action on $\mathbf{CP}^1$ by the polyhedral configuration of a double triangular pyramid —two regular tetrahedra joined at a face. The two-point orbit resides at 0 and ∞ and defines two hemispheres in the usual way. Accordingly, the unit circle corresponds to the equatorial

boundary between hemispheres and the 3-points $\{1, \rho, \rho^2\}$ are vertices where four faces congregate.

Consider the degree-4 map that fixes the vertices of each face and sends one face F to four others: F itself and the three faces in the hemisphere not containing F. This symmetrical construction results in $\mathcal{G}_3$-equivariant behavior. At the three equatorial vertices, the map opens up a face's internal angle of $\pi/2$ to an angle of $3\pi/2$ so that the local behavior is cubing. This makes the 3-point orbit doubly-critical and, by degree counting, the entire critical set. Accordingly, this map must be g_4. Since g_4 has periodic critical points, the superattracting basins constitute its Fatou set and, moreover, have full measure in $\mathbf{CP}^1$. A portrait of the basins appears in Figure 1.

It turns out that g_4 is Halley's Method —a variation on Newton's Method— for a cubic polynomial. (See [**ST**] for a description of Halley's Method in real variables.) In the coordinates selected above, the polynomial to which we apply Halley's method is

$$z^3 - 1.$$

FIGURE 1. Dynamics of g_4 on the $\mathcal{S}_3$-symmetric $\mathbf{CP}^1$

6.2. The map in two dimensions. Since g_{n+1} has real coefficients, it preserves the $\mathbf{RP}^{n-2}$ of points whose coordinates can be expressed by real numbers. Call this space R. Under $\mathcal{G}_4$, R has the structure of a projective cube. We can view this as a hemisphere where one vertex is at the pole and the other three vertices lie along a circle whose center is the distinguished vertex. The 3-point orbit (i.e., the face-centers) lies on another circle centered at the north pole.

Figure 2 displays the basins of attraction of g_5 on R. In the affine plane of the picture, the vertices of the cube are

$$(0,0), (1,0), \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$$

while the three face-centers are the edge-midpoints

$$\left(-\frac{1}{2}, 0\right), \left(\frac{1}{4}, \pm\frac{\sqrt{3}}{4}\right)$$

of the equilateral triangle formed by the three vertices that are not $(0,0)$. The map is given by

$$(x, y) \longrightarrow$$

$$\left(\frac{3\left(15 x^4 + 12 x^5 - 30 x^2 y^2 - 5 y^4 + 20 x y^4\right)}{1 - 10 x^2 + 20 x^3 + 30 x^4 + 40 x^5 - 10 y^2 - 60 x y^2 + 60 x^2 y^2 - 80 x^3 y^2 + 30 y^4 - 120 x y^4}, \right.$$

$$\left. \frac{24\left(-5 x y^3 + 5 x^2 y^3 + y^5\right)}{1 - 10 x^2 + 20 x^3 + 30 x^4 + 40 x^5 - 10 y^2 - 60 x y^2 + 60 x^2 y^2 - 80 x^3 y^2 + 30 y^4 - 120 x y^4} \right).$$

The six lines of reflection run along the edges and a diagonal of a face. These lines carve the hemisphere into twelve triangles each of which is a fundamental domain for the reflection group action $\mathcal{G}_4$. Viewing the "hemi-cube" from above an edge, Figure 3 reveals the map's action on a fundamental triangle: one triangle stretches and twists onto five other associated triangles.

Returning to u coordinates, one of the six mirrors —say, $\{u_3 = 0\}$— is $\mathbf{Z}_2$-stable. Restricted to this line, g_5 has three superattracting points:

- A two-point $\mathbf{Z}_2$ orbit of type p_1 points $[1, 0, 0]$ and $[0, 1, 0]$ (where $\{u_2 = 0\}$ and $\{u_1 = 0\}$ intersect $\{u_3 = 0\}$).
- A one-point $\mathbf{Z}_2$ orbit of the point $p_2 = [1, 1, 0]$ (where $\{u_1 = u_2\}$ intersects $\{u_3 = 0\}$).
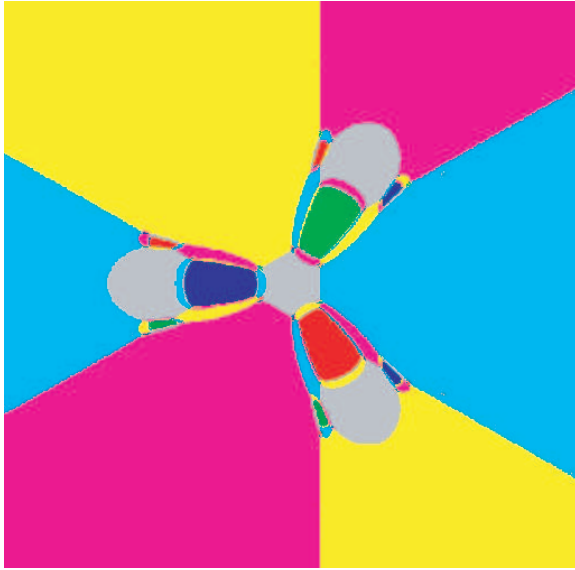
FIGURE 2. Dynamics of g_5 on the $\mathcal{S}_4$-symmetric $\mathbf{RP}^2$
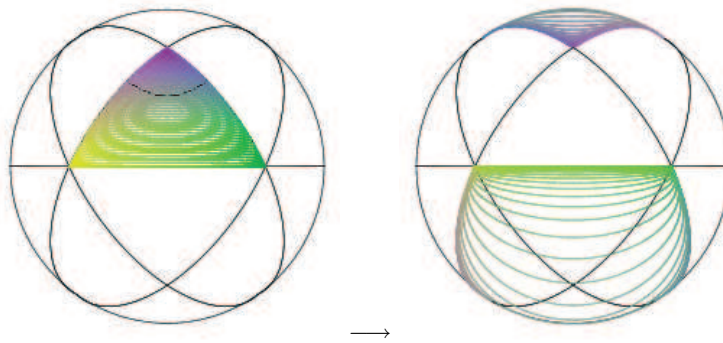

FIGURE 3. Geometry of g_5 on the $\mathcal{S}_4$-symmetric $\mathbf{RP}^2$

In coordinates where the two-point orbit is ± 1 and the one-point orbit is 0, the map takes the form

$$z \longrightarrow \frac{4\,z^3(z^2+5)}{15\,z^4 + 10\,z^2 - 1}.$$

Figure 4 shows their basins of attraction on the line. Notice that this $\mathbf{CP}^1$ intersects R in an $\mathbf{RP}^1$ that corresponds to a line of reflective symmetry in Figure 2 and the horizontal mirror in Figure 4 —for instance, the line that passes through the red, gray, and yellow basins.
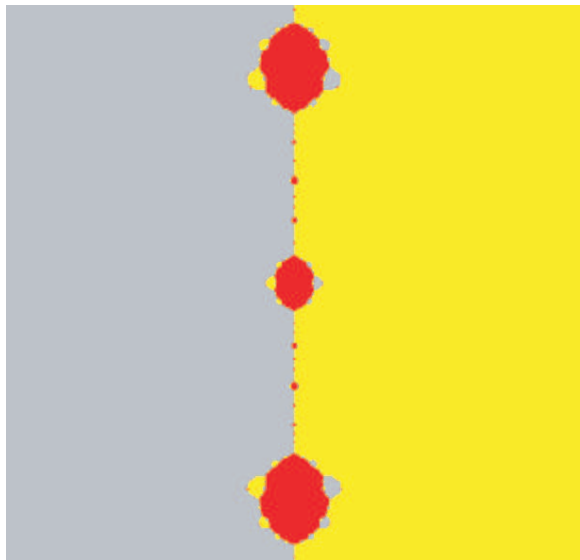
FIGURE 4. Dynamics of g_5 on the $\mathbf{Z}_2$-symmetric $\mathbf{CP}^1$

6.3. The three-dimensional map: a cascade of critical finiteness. A component of g_6's critical set is a $\mathbf{CP}^2$. On the $\mathcal{S}_3$-symmetric $\{u_4 = 0\}$ the map has three $\mathcal{S}_3$ orbits of superattracting points:

- Type p_1 points $[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0]$.
- Type p_2 points $[1, 1, 0, 0], [1, 0, 1, 0], [0, 1, 1, 0]$.
- $p_3 = [1, 1, 1, 0]$.

In the basin plot on the corresponding $\mathbf{RP}^2$ (Figure 5), the geometry is that of a projective double triangular pyramid and these points respectively occupy

$$(1, 0), \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) \quad \left(-\frac{1}{2}, 0\right), \left(\frac{1}{4}, \pm\frac{\sqrt{3}}{4}\right) \quad (0, 0).$$

The map is given by

$$(x, y) \longrightarrow$$

$$\Big(9 \left(15\, x^4 + 24\, x^5 + 15\, x^6 - 30\, x^2\, y^2 - 15\, x^4\, y^2 - 5\, y^4 + 40\, x\, y^4 - 35\, x^2\, y^4 - 5\, y^6 \right),$$

$$- 72\, y^3 \left(5\, x - 10\, x^2 + 5\, x^3 - 2\, y^2 + 5\, x\, y^2 \right) \Big) /$$

$$\Big(1 - 15\, x^2 + 40\, x^3 + 90\, x^4 + 240\, x^5 + 130\, x^6 - 15\, y^2 - 120\, x\, y^2 + 180\, x^2\, y^2$$

$$- 480\, x^3\, y^2 + 30\, x^4\, y^2 + 90\, y^4 - 720\, x\, y^4 + 630\, x^2\, y^4 + 90\, y^6 \Big).$$

This image makes for interesting comparison to the $\mathcal{S}_4$-symmetric Figure 2.
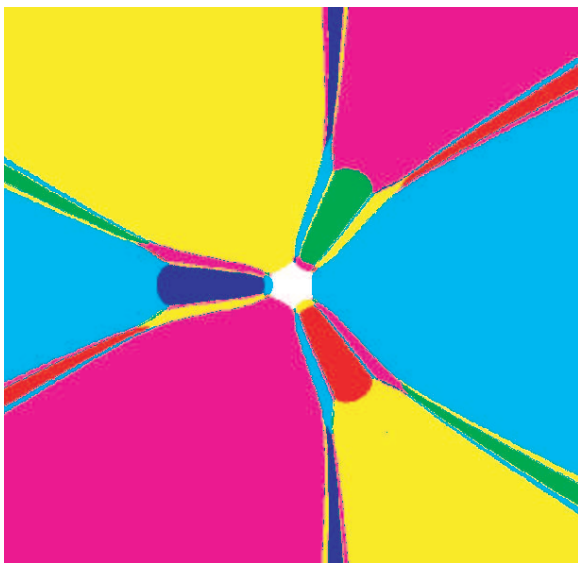
FIGURE 5. Dynamics of g_6 on the $\mathcal{S}_3$-symmetric $\mathbf{RP}^2$

On the critical component $\{u_4 = 0\}$, $g|_{\{u_4=0\}}$ has two types of critical line: $\{u_3 = 0\}$ and $\{u_2 = u_3\}$. The respective lines have $\mathbf{Z}_2$ and trivial symmetry. As for superattracting points, the former line contains $[1, 0, 0, 0]$, $[0, 1, 0, 0]$ (a two-point $\mathbf{Z}_2$ orbit) and $[1, 1, 0, 0]$ while on the latter line we find $[1, 0, 0, 0]$, $[0, 1, 1, 0]$, $[1, 1, 1, 0]$. In the respective basin plots for g_6 restricted to the lines (Figure 6 and Figure 7), these points

are $\pm 1, 0$, and $1, 0, -1$ while the maps are

$$z \longrightarrow \frac{8\,z^3(3\,z^2 + 5)}{5\,z^6 + 45\,z^4 + 15\,z^2 - 1}$$

and

$$z \longrightarrow \frac{8\,z^4(z^2 - 2\,z + 5)}{5\,z^6 + 30\,z^5 + 15\,z^4 + 20\,z^3 - 5\,z^2 - 2\,z + 1}.$$

As before, each $\mathbf{CP}^1$ intersects the $\mathbf{RP}^2$ of Figure 5 in an $\mathbf{RP}^1$: the three lines

$$\{u_k = 0 \mid k = 1, 2, 3\}$$

give the edges of the "triangle" whose vertices are

$$(1, 0), \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$$

and the three lines

$$\{u_k = u_\ell \mid k, \ell = 1, 2, 3\}$$

correspond to the lines of reflective symmetry through $(0, 0)$.
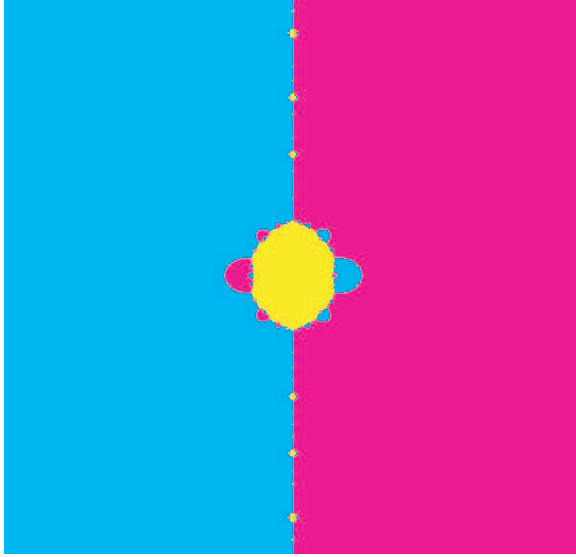
FIGURE 6. Dynamics of g_6 on the $\mathbf{Z}_2$-symmetric $\mathbf{CP}^1$
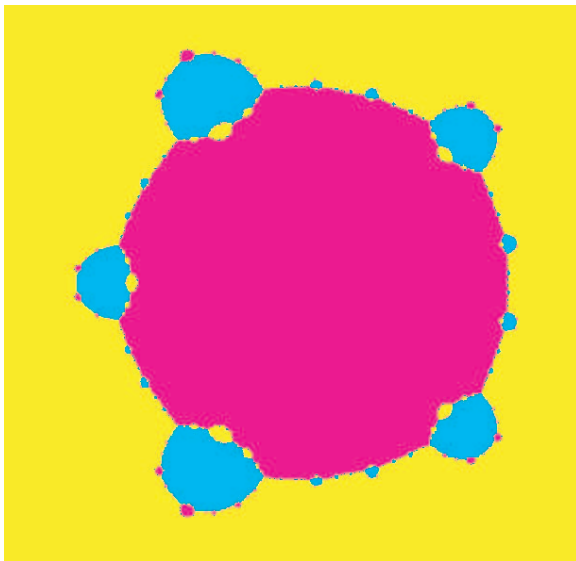
FIGURE 7. Dynamics of g_6 on the $\mathbf{Z}_1$-symmetric $\mathbf{CP}^1$

Acknowledgements

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