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SUMMATION OF COEFFICIENTS OF POLYNOMIALS ON ℓ_p SPACES

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Abstract: We investigate the summability of the coefficients of m-homogeneous polynomials and m-linear mappings defined on ℓ_p spaces. In our research we obtain results on the summability of the coefficients of m-linear mappings defined on $\ell_{p_1} \times \cdots \times \ell_{p_m}$. The first results in this respect go back to Littlewood [**17**] and Bohnenblust and Hille [**6**] for bilinear and m-linear forms on c_0, and Hardy and Littlewood [**15**] and Praciano-Pereira [**20**] for bilinear and m-linear forms on arbitrary ℓ_p spaces. Our results recover and in some case complete these old results through a general approach on vector valued m-linear mappings.

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1. Introduction

Every m-homogeneous polynomial P defined on ℓ_p with values on some Banach space X defines a family of coefficients $\big(c_\alpha(P)\big)_{\alpha\in\Lambda_m}$ (here Λ_m denotes the set of multi-indices that eventually become 0 such that $|\alpha| = \sum_j \alpha_j = m$) in the following way: consider T the unique symmetric m-linear form associated to P then, for $\alpha = (\alpha_1, \dots, \alpha_n, 0, \dots)$ with $\alpha_1 + \cdots + \alpha_n = m$ we have

$$c_\alpha(P) = \frac{m!}{\alpha_1! \cdots \alpha_n!} T(e_1, \overset{\alpha_1}{\dots}, e_1, \dots, e_n, \overset{\alpha_n}{\dots}, e_n).$$

Our interest is to investigate the summability properties of these coefficients. As consequences of results due to Aron and Globevnik for polynomials on c_0 [**3**, Corollary 1.4] and of Zalduendo for general ℓ_p spaces [**22**, Corollary 1] we have that there exists a constant $C > 0$ such that

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for every m-homogeneous polynomial $P\colon \ell_p \to \mathbb{C}$ (with $m < p < \infty$) we have

$$\left(\sum_{i=1}^{\infty} |P(e_i)|^{\frac{p}{p-m}}\right)^{\frac{p-m}{p}} \leq C\|P\|,$$

and the exponent is optimal (if the polynomial is defined on c_0 then the exponent is 1). This can be seen as summing the coefficients over the family of indices $\alpha = (0, \dots, 0, m, 0, \dots)$. If we sum over all coefficients the situation is pretty well understood for polynomials on c_0 (or ℓ_∞) by the results by Bohnenblust and Hille [**6**] for scalar-valued polynomials and by Defant and Sevilla-Peris [**12**] in the vector-valued setting. Following the spirit of [**12**] we focus on the coefficients of polynomials defined on some ℓ_p space with values on some other ℓ_u, computing the norm of the coefficients on a bigger ℓ_q. Then the main result of the paper is the following.

Theorem 1.1. *Let $1 \leq p \leq \infty$ and $1 \leq u \leq q \leq \infty$. Then there is $C > 0$ such that, for every continuous m-homogeneous polynomial $P\colon \ell_p \to \ell_u$ with coefficients $\big(c_\alpha(P)\big)$ we have*

$$\left(\sum_{\alpha \in \Lambda_m} \|c_\alpha(P)\|_{\ell_q}^{\rho}\right)^{1/\rho} \leq C\|P\|,$$

where ρ is given by

- (i) *If $1 \leq u \leq q \leq 2$, and*
  - (a) *if $\frac{mqu}{q-u} < p \leq \infty$, then $\rho = \frac{2m}{m+2(1/u-1/q-m/p)}$;*
  - (b) *if $\frac{2muq}{uq+2q-2u} < p \leq \frac{mqu}{q-u}$, then $\rho = \frac{2}{1+2(1/u-1/q-m/p)}$.*
- (ii) *If $1 \leq u \leq 2 \leq q$, and*
  - (a) *if $\frac{2mu}{2-u} < p \leq \infty$, then $\rho = \frac{2m}{m+2(1/u-1/2-m/p)}$;*
  - (b) *if $mu < p \leq \frac{2mu}{2-u}$, then $\rho = \frac{1}{1/u-m/p}$.*
- (iii) *If $2 \leq u \leq q \leq \infty$ and $mu < p \leq \infty$, then $\rho = \frac{1}{1/u-m/p}$.*

Moreover, the exponents in the cases (ia), (iib), *and* (iii) *are optimal. Also, the exponent in* (ib) *is optimal for $p > 2m$.*

We will approach the problem through multilinear mappings. Given an m-homogeneous polynomial P we take T the associated symmetric m-linear and denote $a_{i_1,\dots,i_m} = T(e_{i_1}, \dots, e_{i_m})$. Since $\|T\| \leq e^m\|P\|$ (see e.g. [**14**, Corollary 1.8]), each time that an inequality of the type

$$\left(\sum_{i_1 \cdots i_m} \|a_{i_1,\dots,i_m}\|^t \right)^{1/t} \le C\|T\| \tag{1}$$

holds for every m-linear mapping we automatically have an equivalent inequality (with the same exponent) for all m-homogeneous polynomials (see [**12**, Lemma 5] for more details). Littlewood showed in [**17**] that an inequality like (1) holds with $t = 4/3$ for bilinear forms on c_0. This result was generalised by Bohnenblust and Hille [**6**] to m-linear forms on c_0 and by Hardy and Littlewood [**15**] to bilinear forms on $\ell_p \times \ell_q$. In all these results the exponents in the respective inequalities were shown to be optimal. Praciano-Pereira gave in [**20**] inequalities for multilinear forms defined on $\ell_{p_1} \times \cdots \times \ell_{p_m}$, but he did not cover all possible cases and he did not deal with the optimality of the exponents. Recently there have been also some results on vector valued multilinear mappings defined on c_0 [**12, 11**]. Our result for polynomials will follow from the following more general result on m-linear mappings, that is our second main result.

Theorem 1.2. *Let Y be a cotype q space and $v\colon X \to Y$ an $(r,1)$-summing operator (with $1 \le r \le q$). For $1 \le p_1, \dots, p_m \le \infty$ with $\frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{r}$ we define*

$$\frac{1}{\lambda} = \frac{1}{r} - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_m} \right) \quad \text{and} \quad \frac{1}{\mu} = \frac{1}{m\lambda} + \frac{m-1}{mq}.$$

Then there exists $C > 0$ such that, for every m-linear $T\colon \ell_{p_1} \times \cdots \times \ell_{p_m} \to X$ with coefficients $(a_{i_1,\dots,i_m})$ we have:

(i) *If $\lambda \ge q$, then* $\displaystyle\left(\sum_{i_1,\dots,i_m=1}^{\infty} \|va_{i_1,\dots,i_m}\|^{\lambda} \right)^{1/\lambda} \le C\|T\|.$

(ii) *If $\lambda < q$, then* $\displaystyle\left(\sum_{i_1,\dots,i_m=1}^{\infty} \|va_{i_1,\dots,i_m}\|^{\mu} \right)^{1/\mu} \le C\|T\|.$

We can rewrite

$$\frac{1}{\mu} = \frac{q + (m+1)r}{mrq} + \frac{1}{m} \left(\frac{1}{p_1} + \cdots + \frac{1}{p_m} \right),$$

then we easily see that doing $p_1 = \cdots = p_m = \infty$ we recover (with the same exponent) [**11**, Corollary 5.2]. On the other hand, taking $X = Y = \mathbb{C}$ and v the identity we recover the classical result of Hardy and

Littlewood [**15**] in the bilinear case and we recover and complete with the remaining cases the results in [**20**] (see Proposition 4.1 below).

2. Definitions and preliminaries

We collect now some of the main definitions and results that we will be using along the paper. All the coming spaces will be complex Banach spaces. The open unit ball of X will be denoted by B_X and the dual of X by X^*. The space of continuous m-linear mappings on $X_1 \times \cdots \times X_m$ with values in Y will be denoted by $\mathcal{L}(^m X_1, \ldots, X_m; Y)$. With the norm

$$\|T\| = \sup\{\|T(x_1, \ldots, x_m)\| \colon x_j \in B_{X_j},\ j = 1, \ldots, m\}$$

it is a Banach space.

Every m-linear mapping T defined on $\ell_{p_1} \times \cdots \times \ell_{p_m}$ defines a set of coefficients given by $a_{i_1,\ldots,i_m} = T(e_{i_1}, \ldots, e_{i_m})$.

A mapping $P \colon X \to Y$ is a (continuous) m-homogeneous polynomial if there exists a (continuous) m-linear mapping $T \colon X \times \cdots \times X \to Y$ such that $P(x) = T(x, \ldots, x)$ for every x. The space of continuous m-homogeneous polynomials is denoted by $\mathcal{P}(^m X; Y)$ and with the norm $\|P\| = \sup\{\|P(x)\| : x \in B_X\}$ is a Banach space. Each polynomial has a unique associated symmetric m-linear mapping.

Given $1 \leq p \leq \infty$, the conjugate p' is defined by $1 = \frac{1}{p} + \frac{1}{p'}$.

A Banach space has cotype $2 \leq q < \infty$ (see e.g. [**13**, Chapter 11]) if there exists a constant $C > 0$ such that for every finite choice of elements $x_1, \ldots, x_N \in X$

$$\left(\sum_{k=1}^{N} \|x_k\|^q\right)^{1/q} \leq C \left(\int_0^1 \left\|\sum_{k=1}^{N} r_k(t) x_k\right\|^2 dt\right)^{1/2},$$

where r_k is the k-th Rademacher function. The smallest constant in this inequality is denoted by $C_q(X)$. Recall that ℓ_q has cotype $\max\{q, 2\}$.

We will use repeatedly the following easy fact: whenever X has cotype q and $s \geq q$ then $\ell_s^n(X)$ has cotype s with $C_s(\ell_s^n(X)) \leq C_s(X)$.

An operator between Banach spaces $v \colon X \to Y$ is (r, s)-summing (with $s \leq r \leq \infty$) [**13**, Chapter 10] if there exists $C > 0$ such that for every finite choice $x_1, \ldots, x_N \in X$

$$\left(\sum_{k=1}^{N} \|v x_k\|^r\right)^{1/r} \leq C \sup_{x^* \in B_{X^*}} \left(\sum_{k=1}^{N} |x^*(x_k)|^s\right)^{1/s}.$$

The smallest constant in this inequality is denoted by $\pi_{r,s}(v)$.

A straightforward computation shows that an operator $v\colon X \to Y$ is (r,s)-summing if and only if there exists $C > 0$ such that for every n and every operator $T\colon \ell_{s'}^n \to X$ we have

$$\left(\sum_{k=1}^{n} \|vT(e_k)\|_Y^r \right)^{1/r} \leq C\|T\|. \tag{2}$$

Also, it is well known that if a Banach space X has cotype q, then the identity $\mathrm{id}\colon X \to X$ is $(q,1)$-summing (see e.g. [**13**, Theorem 11.17]).

We will be using some facts about (r,s)-summing operators. The first one is the *Inclusion Theorem* [**13**, Theorem 10.4]: if $s_1 \leq s_2$, $r_1 \leq r_2$, and $\frac{1}{s_1} - \frac{1}{r_1} \leq \frac{1}{s_2} - \frac{1}{r_2}$ then every (r_1,s_1)-summing operator is (r_2,s_2)-summing and $\pi_{r_2,s_2}(v) \leq \pi_{r_1,s_1}(v)$.

Our second main fact are the celebrated *Bennett–Carl inequalities* [**4, 8**], that describe precisely how summing the inclusion mappings between ℓ_p spaces are: given $1 \leq u \leq q \leq \infty$ define the number

$$r = \begin{cases} \dfrac{2}{1 + 2(\frac{1}{u} - \frac{1}{q})} & \text{if } q < 2, \\ u & \text{if } q \geq 2. \end{cases}$$

Then the inclusion $\mathrm{id}\colon \ell_u \hookrightarrow \ell_q$ is $(r,1)$-summing and this r is optimal.

We will use the normed theory of tensor products as presented in [**9**]. The injective tensor norm will be denoted by ε. An operator $v\colon X \to Y$ is (r,s)-summing if and only if there is $C > 0$ such that $\|\mathrm{id} \otimes v\colon \ell_s^n \otimes_\varepsilon X \to \ell_r^n(Y)\| \leq C$ for every $n \in \mathbb{N}$; in this case $\pi_{r,s}(v) = \sup_n \|\mathrm{id} \otimes v\colon \ell_s^n \otimes_\varepsilon X \to \ell_r^n(Y)\|$.

Finally, we will be dealing with sums over indices $(i_1,\dots,i_m) \in \{1,\dots,n\}^m$. The symbol $\sum_{[i_k]}$ will mean that we are fixing the k-th index and summing over all the rest.

The cardinal of a set A will be denoted by $\sharp A$.

3. Proof of Theorem 1.2

The main tool for the proof of the main result will be the following inequality for mixed sums. For scalar valued mappings this is [**15**, (1.2.8)] in the bilinear case and [**20**, Theorem A] in the m-linear case. Our proof follows the guidelines of [**20**] and we present here an adapted version.

Proposition 3.1. *Let Y be a cotype q Banach space and $v\colon X \to Y$ an $(r,1)$-summing operator (with $1 \leq r \leq q$). Assume $1 \leq p_1,\dots,p_m \leq \infty$ are such that $\frac{1}{p_1} + \dots + \frac{1}{p_m} < \frac{1}{r} - \frac{1}{q}$ and let $\frac{1}{\lambda} = \frac{1}{r} - (\frac{1}{p_1} + \dots + \frac{1}{p_m})$.*

Then for every continuous m-linear mapping $T\colon \ell_{p_1}\times\cdots\times\ell_{p_m}\to X$ we have, for each $j=1,\dots,m$,

$$\left(\sum_{i_j}\left(\sum_{[i_j]}\|vT(e_{i_1},\dots,e_{i_m})\|^q\right)^{\lambda/q}\right)^{1/\lambda}\le\left(\sqrt{2}C_q(Y)\right)^{m-1}\pi_{r,1}(v)\|T\|.$$

Proof: If $p_1=\cdots=p_m=\infty$, then $\lambda=r$ and proceeding as in [**12**, Lemma 2] we easily get

$$\left(\sum_{i_j}\left(\sum_{[i_j]}\|vT(e_{i_1},\dots,e_{i_m})\|^q\right)^{r/q}\right)^{1/r}\le\left(\sqrt{2}C_q(Y)\right)^{m-1}\pi_{r,1}(v)\|T\|$$

for every m-linear $T\colon\ell_\infty\times\cdots\ell_\infty\to X$ and every $j=1,\dots,m$.

For the general case, we use induction in $\sharp\{i : p_i\neq\infty\}$. Let us suppose that the result is true for $\sharp\{i : p_i\neq\infty\}=k-1$ and let us prove it for $\sharp\{i : p_i\neq\infty\}=k$. We can suppose, without loss of generality, that $p_1,\dots,p_k$ are all different from ∞ and so fix $n\in\mathbb{N}$ and consider $T\in\mathcal{L}(^m\ell_{p_1}^n,\dots,\ell_{p_k}^n,\ell_\infty^n,\dots,\ell_\infty^n;X)$. We write the m-linear mapping as

$$T=\sum_{i_1,\dots,i_m=1}^{n}a_{i_1,\dots,i_m}e_{i_1,\dots,i_m},\quad\text{where } e_{i_1,\dots,i_m}=e'_{i_1}\cdots e'_{i_m}.$$

For each $x\in B_{\ell_{p_k}^n}$ let $T^{(x)}\in\mathcal{L}(^m\ell_{p_1}^n,\dots,\ell_{p_{k-1}}^n,\ell_\infty^n,\dots,\ell_\infty^n;X)$ be given by

$$T^{(x)}=\sum_{i_1,\dots,i_m=1}^{n}a_{i_1,\dots,i_m}x_{i_k}e_{i_1,\dots,i_m}.$$

Clearly, $\|T\|=\sup\{\|T^{(x)}\| : x\in B_{\ell_{p_k}^n}\}$. We can apply the inductive hypothesis to $T^{(x)}$: denoting $\frac{1}{\lambda^*}=\frac{1}{r}-(\frac{1}{p_1}+\cdots+\frac{1}{p_{k-1}})$, we know, for all $j=1,\dots,m$ and all $x\in B_{\ell_{p_k}^n}$,

$$\left(\sum_{i_j}\left(\sum_{[i_j]}\|va_{i_1,\dots,i_m}\|^q|x_{i_k}|^q\right)^{\frac{\lambda^*}{q}}\right)^{1/\lambda^*}\le K\|T^{(x)}\|\le K\|T\|. \tag{3}$$

First of all, if $j = k$ then we have, by the induction hypothesis,

$$\left(\sum_{i_k} \left(\sum_{[i_k]} \|va_{i_1,\dots,i_m}\|^q \right)^{\lambda/q} \right)^{1/\lambda}$$

$$= \left(\sum_{i_k} \left(\sum_{[i_k]} \|va_{i_1,\dots,i_m}\|^q \right)^{\frac{\lambda^*}{q}\cdot\frac{\lambda}{\lambda^*}} \right)^{1/\lambda}$$

$$= \sup_{x\in B_{\ell^n_{p_k}}} \left(\sum_{i_k} \left(\sum_{[i_k]} \|va_{i_1,\dots,i_m}\|^q |x_{i_k}|^q \right)^{\lambda^*} \right)^{1/\lambda^*} \le K\|T\|.$$

Let us suppose now $j \neq k$. We denote $S_j = (\sum_{[i_j]} \|va_{i_1,\dots,i_m}\|^q)^{1/q}$. Since $\lambda^* < \lambda < q$, some simple algebraic manipulations and the repeated use of Hölder's inequality yield

$$\sum_{i_j} \left(\sum_{[i_j]} \|va_{i_1,\dots,i_m}\|^q \right)^{\lambda/q}$$

$$= \sum_{i_j} S_j^{\lambda} = \sum_{i_j} S_j^{\lambda-q} S_j^q = \sum_{i_j} \sum_{[i_j]} \frac{\|va_{i_1,\dots,i_m}\|^q}{S_j^{q-\lambda}}$$

$$= \sum_{i_k} \sum_{[i_k]} \frac{\|va_{i_1,\dots,i_m}\|^q}{S_j^{q-\lambda}} = \sum_{i_k} \sum_{[i_k]} \frac{\|va_{i_1,\dots,i_m}\|^{\frac{q(q-\lambda)}{q-\lambda^*}}}{S_j^{q-\lambda}} \|va_{i_1,\dots,i_m}\|^{\frac{q(\lambda-\lambda^*)}{q-\lambda^*}}$$

$$\le \sum_{i_k} \left(\sum_{[i_k]} \frac{\|va_{i_1,\dots,i_m}\|^q}{S_j^{q-\lambda^*}} \right)^{\frac{q-\lambda}{q-\lambda^*}} \left(\sum_{[i_k]} \|va_{i_1,\dots,i_m}\|^q \right)^{\frac{\lambda-\lambda^*}{q-\lambda^*}}$$

$$\le \left[\sum_{i_k} \left(\sum_{[i_k]} \frac{\|va_{i_1,\dots,i_m}\|^q}{S_j^{q-\lambda^*}} \right)^{\lambda/\lambda^*} \right]^{\frac{(q-\lambda)\lambda^*}{(q-\lambda^*)\lambda}}$$

$$\times \left[\sum_{i_k} \left(\sum_{[i_k]} \|va_{i_1,\dots,i_m}\|^q \right)^{\lambda/q} \right]^{\frac{(\lambda-\lambda^*)q}{(q-\lambda^*)\lambda}}.$$

We have already seen when proving the case $j = k$ that the second factor of the last product is bounded by $(K\|T\|)^{\frac{(\lambda-\lambda^*)q}{q-\lambda^*}}$. Now we bound the first factor.

$$\left[\sum_{i_k}\left(\sum_{[i_k]}\frac{\|va_{i_1,\dots,i_m}\|^q}{S_j^{q-\lambda^*}}\right)^{\lambda/\lambda^*}\right]^{\lambda^*/\lambda} = \sup_{x\in B_{\ell_{p_k}^n}}\sum_{i_k}\sum_{[i_k]}\frac{\|va_{i_1,\dots,i_m}\|^q}{S_j^{q-\lambda^*}}|x_{i_k}|^{\lambda^*}$$

$$= \sup_{x\in B_{\ell_{p_k}^n}}\sum_{i_j}\sum_{[i_j]}\frac{\|va_{i_1,\dots,i_m}\|^{q-\lambda^*}}{S_j^{q-\lambda^*}}\|va_{i_1,\dots,i_m}\|^{\lambda^*}|x_{i_k}|^{\lambda^*}$$

$$\leq \sup_{x\in B_{\ell_{p_k}^n}}\sum_{i_j}\left(\sum_{[i_j]}\frac{\|va_{i_1,\dots,i_m}\|^q}{S_j^q}\right)^{\frac{q-\lambda^*}{q}}\left(\sum_{[i_j]}\|va_{i_1,\dots,i_m}\|^q|x_{i_k}|^q\right)^{\lambda^*/q}$$

$$= \sup_{x\in B_{\ell_{p_k}^n}}\sum_{i_j}\left(\sum_{[i_j]}\|va_{i_1,\dots,i_m}\|^q|x_{i_k}|^q\right)^{\lambda^*/q}$$

$$\leq (K\|T\|)^{\lambda^*}.$$

Since the n was arbitrary, this holds for every n and completes the proof. □

We can now address the proof of our result.

Proof of Theorem 1.2: Let us assume first that $\lambda \geq q$. We proceed by induction on m. For $m = 1$ we have $\frac{1}{\lambda} = \frac{1}{r} - \frac{1}{p_1}$. Since v is $(r,1)$-summing, then, by the Inclusion Theorem, it is also (λ, p_1')-summing. By (2) this gives, for every operator $T\colon \ell_{p_1} \to X$ and every n

$$\left(\sum_{j=1}^n \|vT(e_j)\|^\lambda\right)^{1/\lambda} \leq \pi_{r,1}(v)\|T\|.$$

For the inductive step we have $\frac{1}{\lambda} = \frac{1}{r} - (\frac{1}{p_1} + \cdots + \frac{1}{p_m})$ and we consider the exponent

$$\frac{1}{\lambda^*} = \frac{1}{r} - \left(\frac{1}{p_2} + \cdots + \frac{1}{p_m}\right).$$

We have now two possibilities, either $\lambda^* < q$ or $\lambda^* \geq q$. In the first case, given $T\colon \ell_{p_1}^n \times \ell_{p_2}^n \times \cdots \times \ell_{p_m}^m \to X$ with coefficients $(a_{i_1,\dots,i_m})$

we define $\widetilde{T}\colon \ell_\infty^n \times \ell_{p_2}^n \times \cdots \times \ell_{p_m}^m \to X$ in the same way as T. Since $\frac{1}{\lambda^*} = \frac{1}{r} - (\frac{1}{\infty} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}) > \frac{1}{q}$ we have, by Proposition 3.1

$$\left(\sum_{i_1} \left(\sum_{[i_1]} \|v\widetilde{T}(e_{i_1}, e_{i_2}, \ldots, e_{i_m})\|^q \right)^{\frac{\lambda^*}{q}} \right)^{\frac{1}{\lambda^*}} \leq K\|\widetilde{T}\|,$$

where $K = \left(\sqrt{2}C_q(Y)\right)^{m-1} \pi_{r,1}(v)$. This, by (2) means that the linear mapping $\mathcal{L}(^{m-1}\ell_{p_2}^n, \ldots, \ell_{p_m}^m; X) \to \ell_q^{n^{m-1}}(Y)$ given by $A \rightsquigarrow (vA(e_{i_2}, \ldots, e_{i_m}))_{i_2,\ldots,i_m}$ is $(\lambda^*, 1)$-summing. By the Inclusion Theorem this mapping is also (λ, p_1')-summing, which means, again by (2)

$$\left(\sum_{i_1} \left(\sum_{[i_1]} \|va_{i_1,\ldots,i_m}\|^q \right)^{\lambda/q} \right)^{1/\lambda} \leq K \sup_{y^{(j)} \in B_{\ell_{p_j}^n}} \left\| \sum_{i_1,\ldots,i_m} a_{i_1,\ldots,i_m} y_{i_1}^{(1)} y_{i_2}^{(2)} \cdots y_{i_m}^{(m)} \right\| = K\|T\|.$$

Finally, since $\lambda \geq q$ we have

$$\left(\sum_{i_2,\ldots,i_m} \|va_{i_1,\ldots,i_m}\|^q \right)^{1/q} \geq \left(\sum_{i_2,\ldots,i_m} \|va_{i_1,\ldots,i_m}\|^\lambda \right)^{1/\lambda}.$$

This completes the proof for this case.

Now, if $\lambda^* \geq q$ we have that Y has cotype λ^* and so also has $\ell_{\lambda^*}(Y)$. Then $\mathrm{id}\colon \ell_{\lambda^*}(Y) \to \ell_{\lambda^*}(Y)$ is $(\lambda^*, 1)$-summing and, by the ideal property (recall that $\lambda \geq \lambda^*$) [**13**, Proposition 10.2], $\mathrm{id}\colon \ell_{\lambda^*}(Y) \hookrightarrow \ell_\lambda(Y)$ is also $(\lambda^*, 1)$-summing. Then the Inclusion Theorem gives $\pi_{\lambda,p_1'}(\mathrm{id}\colon \ell_{\lambda^*}^{n^{m-1}}(Y) \hookrightarrow \ell_\lambda^{n^{m-1}}(Y)) \leq C$ for every n and m. This means that for every $(b_{i_2,\ldots,i_m}^{(k)})_{i_2,\ldots,i_m=1}^n \subseteq \ell_\lambda^{n^{m-1}}(Y)$, with $k = 1, \ldots, N$

$$\left(\sum_{k=1}^N \|(b_{i_2,\ldots,i_m}^{(k)})_{i_2,\ldots,i_m}\|_{\ell_\lambda(Y)}^\lambda \right)^{1/\lambda} \leq C \sup_{\gamma \in B_{\ell_{\lambda^*}^n(Y)^*}} \left(\sum_{k=1}^N |\gamma(b^{(k)})|^{p_1'} \right)^{1/p_1'}.$$

Then, if $T \in \mathcal{L}(^m\ell_{p_1}, \dots, \ell_{p_m}; X)$ with coefficients $(a_{i_1,\dots,i_m})$ we write $b^{(i_1)}_{i_2,\dots,i_m} = va_{i_1,\dots,i_m}$ and we have

$$\begin{aligned}
\left(\sum_{i_1,\dots,i_m} \|va_{i_1,\dots,i_m}\|^\lambda \right)^{1/\lambda} &= \left(\sum_{i_1} \|b^{(i_1)}\|^\lambda_{\ell_\lambda(Y)} \right)^{1/\lambda} \\
&\leq C \sup_{\gamma \in B_{\ell^n_{\lambda^*}(Y)^*}} \left(\sum_{i_1=1}^n |\gamma(b^{(i_1)})|^{p_1'} \right)^{1/p_1'} \\
&= C \sup_{\gamma \in B_{\ell^n_{\lambda^*}(Y)^*}} \sup_{x \in B_{\ell^n_{p_1}}} \left| \sum_{i_1=1}^n \gamma(b^{(i_1)}) x_{i_1} \right| \\
&= C \sup_{x \in B_{\ell^n_{p_1}}} \sup_{\gamma \in B_{\ell^n_{\lambda^*}(Y)^*}} \left| \gamma \left(\sum_{i_1=1}^n b^{(i_1)} x_{i_1} \right) \right| \\
&= C \sup_{x \in B_{\ell^n_{p_1}}} \left\| \sum_{i_1=1}^n va_{i_1,\dots,i_m} x_{i_1} \right\|_{\ell^n_{\lambda^*}(Y)} \\
&= C \sup_{x \in B_{\ell^n_{p_1}}} \left(\sum_{i_2,\dots,i_m} \left\| v \left(\sum_{i_1=1}^n a_{i_1,\dots,i_m} x_{i_1} \right) \right\|^{\lambda^*} \right)^{\frac{1}{\lambda^*}} .
\end{aligned}$$

We now apply the induction hypothesis with the $(m-1)$-linear mapping whose coefficients are $(\sum_{i_1=1}^n a_{i_1,\dots,i_m} x_{i_1})_{i_2,\dots,i_m}$ to have

$$\begin{aligned}
&\left(\sum_{i_2,\dots,i_m} \left\| v \left(\sum_{i_1=1}^n a_{i_1,\dots,i_m} x_{i_1} \right) \right\|^{\lambda^*} \right)^{1/\lambda^*} \\
&\qquad\qquad \leq K \sup_{y^{(j)} \in B_{\ell^n_{p_j}}} \left\| \sum_{i_2,\dots,i_m} \sum_{i_1=1}^n a_{i_1,\dots,i_m} x_{i_1} y^{(2)}_{i_2} \cdots y^{(m)}_{i_m} \right\|_X .
\end{aligned}$$

This completes the proof of (i).

We prove now (ii). If $m = 1$ we have $\mu = \lambda$ and then it follows as in the previous case. For a general m let us first note that the statement can be rephrased in terms of tensor products as

$$\sup_n \left\| \mathrm{id} \otimes v \colon \ell^n_{p_1'} \otimes_\varepsilon \cdots \otimes_\varepsilon \ell^n_{p_m'} \otimes_\varepsilon X \to \ell^{n^m}_\mu(Y) \right\| \leq K.$$

We are going to iterate a procedure of intertwining, transposition, and interpolation. First observe that $\lambda < q$ gives $\frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{r} - \frac{1}{q}$ and then, by Proposition 3.1 we have (denoting $K = \big(\sqrt{2}C_q(Y)\big)^{m-1}\pi_{r,1}(v)$)

$$\sup_n \Big\| \mathrm{id} \otimes v \colon \ell_{p_1'}^n \otimes_\varepsilon \ell_{p_2'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon X \to \ell_\lambda^n\big(\ell_q^{n^{m-1}}(Y)\big)\Big\| \le K, \tag{4}$$

and also

$$\sup_n \Big\| \mathrm{id} \otimes v \colon \ell_{p_2'}^n \otimes_\varepsilon \ell_{p_1'}^n \otimes_\varepsilon \ell_{p_3'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon X \to \ell_\lambda^n\big(\ell_q^{n^{m-1}}(Y)\big)\Big\| \le K.$$

We fix now n; by Minkowski's inequality (recall that $\lambda < q$), the transposition operator $\tau \colon \ell_\lambda^n\big(\ell_q^n(Y)\big) \to \ell_q^n\big(\ell_\lambda^n(Y)\big)$ has norm 1. The intertwining operator given by

$$\begin{array}{rcl} \rho_2 \colon \ell_{p_1'}^n \otimes_\varepsilon \ell_{p_2'}^n & \to & \ell_{p_2'}^n \otimes_\varepsilon \ell_{p_1'}^n \\ a \otimes b & \mapsto & b \otimes a \end{array}$$

also has norm 1.

So we have the following three operators:

- $\rho_2 \otimes \mathrm{id} \colon \ell_{p_1'}^n \otimes_\varepsilon \ell_{p_2'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon X \to \ell_{p_2'}^n \otimes_\varepsilon \ell_{p_1'}^n \otimes_\varepsilon \ell_{p_3'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon X$,
- $\mathrm{id} \otimes v \colon \ell_{p_2'}^n \otimes_\varepsilon \ell_{p_1'}^n \otimes_\varepsilon \ell_{p_3'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon X \to \ell_\lambda^n\big(\ell_q^{n^{m-1}}(Y)\big)$,
- $\tau \colon \ell_\lambda^n\big(\ell_q^{n^{m-1}}(Y)\big) \to \ell_q^n\Big(\ell_\lambda^n\big(\ell_q^{n^{m-2}}(Y)\big)\Big)$.

Composing them we have

$$\Big\| \mathrm{id} \otimes v \colon \ell_{p_1'}^n \otimes_\varepsilon \ell_{p_2'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon X \to \ell_q^n\Big(\ell_\lambda^n\big(\ell_q^{n^{m-2}}(Y)\big)\Big)\Big\| \le K. \tag{5}$$

We now use complex interpolation of (4) and (5) with $\theta = 1/2$ (see e.g. [**5**, Chapter 3]) to get

$$\Big\| \mathrm{id} \otimes v \colon \ell_{p_1'}^n \otimes_\varepsilon \ell_{p_2'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon X \to \ell_{\mu_2}^{n^2}\big(\ell_q^{n^{m-2}}(Y)\big)\Big\| \le K,$$

where $\frac{1}{\mu_2} = \frac{\frac{1}{2}}{\lambda} + \frac{\frac{1}{2}}{q}$.

Now, since $\mu_2 < q$, again we have that the first and third of the following mappings (defined in the obvious way) have norm 1, and the norm of the second one is bounded by K:

- $\rho_3 \otimes \mathrm{id} \colon \ell_{p_1'}^n \otimes_\varepsilon \ell_{p_2'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon X \to \ell_{p_2'}^n \otimes_\varepsilon \ell_{p_3'}^n \otimes_\varepsilon \ell_{p_1'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon X$,
- $\mathrm{id} \otimes v \colon \ell_{p_2'}^n \otimes_\varepsilon \ell_{p_3'}^n \otimes_\varepsilon \ell_{p_1'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon X \to \ell_{\mu_2}^{n^2}\big(\ell_q^{n^{m-2}}(Y)\big)$,
- $\tau \colon \ell_{\mu_2}^{n^2}\big(\ell_q^{n^{m-2}}(Y)\big) \to \ell_q^n\Big(\ell_{\mu_2}^{n^2}\big(\ell_q^{n^{m-3}}(Y)\big)\Big)$.

We compose these three mappings to obtain

$$\left\| \mathrm{id} \otimes v \colon \ell_{p_1'}^n \otimes_\varepsilon \ell_{p_2'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon X \to \ell_q^n \left(\ell_{\mu_2}^{n^2} \left(\ell_q^{n^{m-3}}(Y) \right) \right) \right\| \le K. \tag{6}$$

We again interpolate (4) and (6) with the complex method and $\theta = 1/3$,

$$\left\| \mathrm{id} \otimes v \colon \ell_{p_1'}^n \otimes_\varepsilon \ell_{p_2'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon X \to \ell_{\mu_3}^{n^3} \left(\ell_q^{n^{m-3}}(Y) \right) \right\| \le K,$$

where $\frac{1}{\mu_3} = \frac{\frac{1}{3}}{\lambda} + \frac{\frac{2}{3}}{q} = \frac{\frac{1}{3}}{q} + \frac{\frac{2}{3}}{\mu_2}$.

Following the same procedure we finally end up in

$$\left\| \mathrm{id} \otimes v \colon \ell_{p_1'}^n \otimes_\varepsilon \ell_{p_2'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon X \to \ell_{\mu_m}^{n^m}(Y) \right\| \le K,$$

where $\frac{1}{\mu_m} = \frac{\frac{1}{m}}{\lambda} + \frac{\frac{m-1}{m}}{q}$. □

4. Some consequences

We present now some results that follow immediately from Theorem 1.2. The first one is for scalar valued multilinear mappings and completes the result in [**20**] with the cases that were not considered there. We also show that the exponents are optimal.

Proposition 4.1. *Let $1 \le p_1, \dots, p_m \le \infty$ such that $\frac{1}{p_1} + \cdots + \frac{1}{p_m} < 1$. Consider the exponents*

$$\frac{1}{\lambda} = 1 - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_m} \right) \quad \textit{and} \quad \frac{1}{\mu} = \frac{1}{m\lambda} + \frac{m-1}{2m}.$$

Then there exists $C > 0$ such that, for every m-linear $T \colon \ell_{p_1} \times \cdots \times \ell_{p_m} \to \mathbb{C}$ with coefficients $(a_{i_1,\dots,i_m})$ we have

(i) *If $\frac{1}{2} \le \frac{1}{p_1} + \cdots + \frac{1}{p_m} < 1$, then* $\left(\sum_{i_1,\dots,i_m=1}^{\infty} |a_{i_1,\dots,i_m}|^\lambda \right)^{1/\lambda} \le C\|T\|$.

(ii) *If $0 \le \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{2}$, then* $\left(\sum_{i_1,\dots,i_m=1}^{\infty} |a_{i_1,\dots,i_m}|^\mu \right)^{1/\mu} \le C\|T\|$.

Moreover the exponents are optimal.

Proof: The inequalities follow from Theorem 1.2 using that $\mathbb{C}$ has cotype 2 and that the identity on $\mathbb{C}$ is $(1,1)$-summing. Let us assume now that t is such that for every $T \colon \ell_{p_1} \times \cdots \times \ell_{p_m} \to \mathbb{C}$ with $\frac{1}{2} \le \frac{1}{p_1} + \cdots + \frac{1}{p_m} < 1$ we have

$$\left(\sum_{i_1,\dots,i_m} |a_{i_1,\dots,i_m}|^t \right)^{1/t} \le C\|T\| \tag{7}$$

for some universal $C > 0$. Define $\Phi_n : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{C}$ by $\Phi_n(x^{(1)}, \ldots, x^{(m)}) = \sum_{i=1}^n x_i^{(1)} \cdots x_i^{(m)}$. Using Hölder's inequality it is easily seen that $\|\Phi_n\| \leq n^{1/\lambda}$. Then, if (7) holds then we have $n^{1/t} \leq C n^{1/\lambda}$ for every n, which gives $t \geq \lambda$.

For (ii) let us note first that the condition $0 \leq \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{2}$ implies $p_j > 2$ for every $j = 1, \ldots, m$. We show first that for $p_1, \ldots, p_m > 2$ there is a constant $K_m > 0$ such that if $(g_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^n$ are independent Gaussian random variables we have

$$\text{(8)} \quad \int \Big\| \sum_{i_1,\ldots,i_m=1}^n g_{i_1,\ldots,i_m}(\omega) e_{i_1} \otimes \cdots \otimes e_{i_m} \Big\|_{\ell_{p_1'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n} d\omega \leq K_m n^{\frac{1}{\lambda} + \frac{m-1}{2}}.$$

We proceed by induction. It is well known (see e.g. [**10**, (4)]) that for $m = 1$ there is $K_1 > 0$ such that $\int \| \sum_{i=1}^n g_i(\omega) e_i \|_{\ell_{p_1'}^n} d\omega \leq K_1 n^{1/p_1'}$. We assume that (8) holds for an $(m-1)$-fold tensor product and take families of independent Gaussian random variables $(g_{i_1,\ldots,i_{m-1}})$ and (g_k). By Chevét's inequality (see [**21**, (43.2)]) there is a constant $C > 0$ such that

$$\int \Big\| \sum_{i_1,\ldots,i_m=1}^n g_{i_1,\ldots,i_m}(\omega) e_{i_1} \otimes \cdots \otimes e_{i_m} \Big\|_{\ell_{p_1'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n} d\omega$$
$$\leq C \Bigg(\big\| \mathrm{id} \colon \ell_2^n \hookrightarrow \ell_{p_m'}^n \big\| \times \int \Big\| \sum_{i_1,\ldots,i_{m-1}=1}^n g_{i_1,\ldots,i_{m-1}}(\omega) e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \Big\|_{\ell_{p_1'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_{m-1}'}^n} d\omega$$
$$+ \big\| \mathrm{id} \colon \ell_2^{n^{m-1}} \hookrightarrow \ell_{p_1'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_{m-1}'}^n \big\| \int \Big\| \sum_{k=1}^n g_k(\omega) e_k \Big\|_{\ell_{p_m'}^n} d\omega \Bigg).$$

By the metric mapping property of ε we have

$$\left\|\mathrm{id}\colon \ell_2^{n^{m-1}} \hookrightarrow \ell_{p_1'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_{m-1}'}^n\right\| \le \prod_{i=1}^{m-1} \left\|\mathrm{id}\colon \ell_2^n \hookrightarrow \ell_{p_i'}^n\right\|$$
$$= \prod_{i=1}^{m-1} n^{\frac{1}{p_i'}-\frac{1}{2}} = n^{\sum_{i=1}^{m-1} \frac{1}{2}-\frac{1}{p_i}} = n^{\frac{m-1}{2}-\sum_{i=1}^{m-1}\frac{1}{p_i}}.$$

With this, the induction hypothesis and the case $m=1$, we have

$$\int \left\| \sum_{i_1,\dots,i_m=1}^n g_{i_1,\dots,i_m}(\omega) e_{i_1}\otimes\cdots\otimes e_{i_m}\right\|_{\ell_{p_1'}^n\otimes_\varepsilon\cdots\otimes_\varepsilon \ell_{p_m'}^n} d\omega$$
$$\le C\left(n^{\frac{1}{p_m'}-\frac{1}{2}} K_{m-1} n^{\frac{1}{\lambda^*}+\frac{m-2}{2}} + n^{\frac{m-1}{2}-\sum_{i=1}^{m-1}\frac{1}{p_i}} K_1 n^{\frac{1}{p_m'}}\right),$$

where $\frac{1}{\lambda^*} = 1-(\frac{1}{p_1}+\cdots+\frac{1}{p_{m-1}})$. Noting that $\frac{1}{p_m'}-\frac{1}{2}+\frac{1}{\lambda^*}+\frac{m-2}{2} = \frac{m-1}{2}-\sum_{i=1}^{m-1}\frac{1}{p_i}+\frac{1}{p_m'} = \frac{1}{\lambda}+\frac{m-1}{2}$ we finally have (8).

It is a well known fact that Bernoulli averages are dominated by Gaussian averages [**13**, Proposition 12.11], then there is a constant $K>0$ such that for all n

$$\int \left\| \sum_{i_1,\dots,i_m=1}^n \varepsilon_{i_1,\dots,i_m}(\omega) e_{i_1}\otimes\cdots\otimes e_{i_m}\right\|_{\ell_{p_1'}^n\otimes_\varepsilon\cdots\otimes_\varepsilon \ell_{p_m'}^n} d\omega \le K n^{\frac{1}{\lambda}+\frac{m-1}{2}}.$$

Then for each n there is a choice of signs $\varepsilon_{i_1,\dots,i_m}=\pm1$ such that $z=\sum_{i_1,\dots,i_m}\varepsilon_{i_1,\dots,i_m} e_{i_1}\otimes\cdots\otimes e_{i_m}$ satisfies $\|z\|_{\ell_{p_1'}^n\otimes_\varepsilon\cdots\otimes_\varepsilon \ell_{p_m'}^n} \le K n^{\frac{1}{\lambda}+\frac{m-1}{2}}$. Since $(\sum_{i_1,\dots,i_m=1}^n |\varepsilon_{i_1,\dots,i_m}|^t)^{1/t} = n^{m/t}$, if (7) holds for $p_1,\dots,p_m$ satisfying (ii) we have $n^{m/t}\le K n^{\frac{1}{\lambda}+\frac{m-1}{2}}$, which implies $t\ge\mu$. □

Remark 4.2. The condition $\frac{1}{p_1}+\cdots+\frac{1}{p_m}<1$ is necessary in Proposition 4.1. Indeed, if $\frac{1}{p_1}+\cdots+\frac{1}{p_m}\ge1$ then the mapping $\Phi\colon \ell_{p_1}\times\cdots\times\ell_{p_m}\to\mathbb{C}$ given by $\Phi_n(x^{(1)},\dots,x^{(m)})=\sum_{i=1}^\infty x_i^{(1)}\cdots x_i^{(m)}$ is well defined and has infinitely many coefficients equal to 1. Hence, there is no exponent t satisfying an inequality like in Proposition 4.1.

If X is a Banach space with cotype q then the identity is $(q,1)$-summing and we obtain from Theorem 1.2:

Proposition 4.3. *Let $2 \leq p_1, \ldots, p_m \leq \infty$ and $q \geq 2$ such that $\frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{q}$. Define*

$$\frac{1}{\lambda} = \frac{1}{q} - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_m} \right).$$

Then for each Banach space X with cotype q there exists $C > 0$ such that for every continuous, m-linear $T \colon \ell_{p_1} \times \cdots \times \ell_{p_m} \to X$ with coefficients $(a_{i_1,\ldots,i_m})$ we have

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} \|a_{i_1,\ldots,i_m}\|_X^{\lambda} \right)^{1/\lambda} \leq C\|T\|.$$

We can now give the following result, from which Theorem 1.1 readily follows. Let us note that by [**12**, Lemma 5] the fact that an exponent is optimal in an inequality for m-linear mappings implies that it is also optimal for the corresponding inequality for m-homogeneous polynomials. Hence, the optimality of the exponents in Theorem 1.1 also follows.

Proposition 4.4. *Let $1 \leq p_1, \ldots, p_m \leq \infty$ and $1 \leq u \leq q \leq \infty$. Then there is $C > 0$ such that, for every continuous m-linear $T \colon \ell_{p_1} \times \cdots \times \ell_{p_m} \to \ell_u$ with coefficients $(a_{i_1,\ldots,i_m})$ we have*

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} \|a_{i_1,\ldots,i_m}\|_{\ell_q}^{\rho} \right)^{1/\rho} \leq C\|T\|,$$

where ρ is given by

(i) *If $1 \leq u \leq q \leq 2$, and*
  (a) *if $0 \leq \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{u} - \frac{1}{q}$, then* $\rho = \frac{2m}{m+2(1/u-1/q-(1/p_1+\cdots+1/p_m))}$;
  (b) *if $\frac{1}{u} - \frac{1}{q} \leq \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{2} + \frac{1}{u} - \frac{1}{q}$, then* $\rho = \frac{2}{1+2(1/u-1/q-(1/p_1+\cdots+1/p_m))}$.

(ii) *If $1 \leq u \leq 2 \leq q$, and*
  (a) *if $0 \leq \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{u} - \frac{1}{2}$, then* $\rho = \frac{2m}{m+2(1/u-1/2-(1/p_1+\cdots+1/p_m))}$;
  (b) *if $\frac{1}{u} - \frac{1}{2} \leq \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{u}$, then* $\rho = \frac{1}{1/u-(1/p_1+\cdots+1/p_m)}$.

(iii) *If* $2 \leq u \leq q \leq \infty$ *and* $0 \leq \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{u}$, *then* $\rho = \frac{1}{1/u-(1/p_1+\cdots+1/p_m)}$.

Moreover, the exponents in the cases (ia), (iib), *and* (iii) *are optimal. Also, the exponent in* (ib) *is optimal for* $\frac{1}{u} - \frac{1}{q} \leq \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{2}$.

Let us remark that, by doing $p_1 = \cdots = p_m = \infty$, we again find the exponents in [**12**, Theorem 1].

Proof: The case (i) follows immediately from Theorem 1.2, taking $v = \mathrm{id}\colon \ell_u \hookrightarrow \ell_q$ that is $(r,1)$-summing with $\frac{1}{r} = \frac{1}{2} + \frac{1}{u} - \frac{1}{q}$ (by the Bennett–Carl inequalities) and that ℓ_q has cotype 2.

The case (ii) follows from the previous one with $\mathrm{id}\colon \ell_u \hookrightarrow \ell_2$ and the fact that $\| \ \|_q \leq \| \ \|_2$.

Finally, the case (iii) follows from Proposition 4.3 (since ℓ_u has cotype u) and the fact that $\| \ \|_q \leq \| \ \|_u$.

To see that the exponent is optimal, let us suppose that $t \geq 1$ is such that for every $T \in \mathcal{L}({}^m\ell_{p_1}, \ldots, \ell_{p_m}; \ell_u)$ we have

$$\left(\sum_{i_1,\ldots,i_m=1}^{n} \|a_{i_1,\ldots,i_m}\|_{\ell_q}^t \right)^{1/t} \leq C\|T\|, \tag{9}$$

for some universal $C > 0$. Equivalently,

$$\sup_n \left\| \mathrm{id}\colon \ell_{p_1'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon \ell_u^n \to \ell_t^{n^m}\left(\ell_q^n\right) \right\| \leq C.$$

In (ia) we can proceed as in (8) (taking into account that we have $\frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{2}$ and $u' \geq 2$) to find a choice of signs $\varepsilon_{i_1,\ldots,i_{m+1}} = \pm 1$ such that $z = \sum_{i_1,\ldots,i_m} \varepsilon_{i_1,\ldots,i_{m+1}} e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes e_{i_{m+1}}$ satisfies

$$\|z\|_{\ell_{p_1'}^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_{p_m'}^n \otimes_\varepsilon \ell_u^n} \leq n^{1-(\frac{1}{p_1}+\cdots+\frac{1}{p_m})-\frac{1}{u'}+\frac{m}{2}}.$$

On the other hand, proceeding as in [**12**, Section 3.1] we have $\|z\|_{\ell_t(\ell_q)} = n^{m/t+1/q}$. Then, if (9) holds, this implies $\frac{m}{t} + \frac{1}{q} \leq \frac{1}{u} - (\frac{1}{p_1} + \cdots + \frac{1}{p_m}) + \frac{m}{2}$, which gives

$$\frac{1}{t} \leq \frac{1}{2} + \frac{1}{m}\left(\frac{1}{u} - \frac{1}{q} - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_m}\right)\right) \text{ and so, } t \geq \rho.$$

Now, if $\frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{u}$ we consider $T\colon \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \ell_u^n$ given by $T(x^{(1)}, \ldots, x^{(m)}) = \sum_{j=1}^n x_j^{(1)} \cdots x_j^{(m)} e_j$. Taking $x^{(i)} \in B_{\ell_{p_i}}$ for $i = 1, \ldots, m$ we have

$$\begin{aligned}
\|T(x^{(1)}, \ldots, x^{(m)})\|_{\ell_u} &= \left(\sum_{j=1}^n |x_j^{(1)} \cdots x_j^{(m)}|^u \right)^{1/u} \\
&= \sup_{y \in B_{\ell_{u'}}} \left| \sum_{j=1}^n x_j^{(1)} \cdots x_j^{(m)} y_j \right| \\
&\leq \left(\sum_j |x_j^{(1)}|^{p_1} \right)^{1/p_1} \cdots \left(\sum_j |x_j^{(m)}|^{p_1} \right)^{1/p_m} \\
&\quad \times \sup_{y \in B_{\ell_{u'}}} \left(\sum_j |y_j|^{u'} \right)^{1/u'} \left(\sum_j 1 \right)^{1 - \frac{1}{u'} - (\frac{1}{p_1} + \cdots + \frac{1}{p_m})} \\
&\leq n^{\frac{1}{u} - (\frac{1}{p_1} + \cdots + \frac{1}{p_m})}.
\end{aligned}$$

On the other hand, $T(e_{i_1}, \ldots, e_{i_m}) = e_i$ if $i_1 = \ldots i_m = i$ and the null vector otherwise. Then $(\sum \|T(e_{i_1}, \ldots, e_{i_m})\|_{\ell_q}^t)^{1/t} = n^{1/t}$ and if (9) holds we have

$$\frac{1}{t} \leq \frac{1}{u} - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_m} \right).$$

Thus, $t \geq \rho$ in the cases (iib) and (iii).

For $\frac{1}{u} - \frac{1}{q} \leq \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{2}$ (and $1 \leq u \leq q \leq 2$) we consider the Fourier $n \times n$ matrix $a_{kl} = e^{\frac{2\pi i k l}{n}}$ and define $T\colon \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \ell_u^n$ by $T(x^{(1)}, \ldots, x^{(m)}) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_j^{(1)} \cdots x_j^{(m)} e_i$. For $x^{(i)} \in B_{\ell_{p_i}}$ with

$i = 1, \dots, m$ we have

$$\begin{aligned}
\|T(x^{(1)}, \dots, x^{(m)})\|_{\ell_u} &= \left(\sum_{i=1}^n \left| \sum_{j=1}^n a_{ij} x_j^{(1)} \cdots x_j^{(m)} \right|^u \right)^{1/u} \\
&= \sup_{y \in B_{\ell_{u'}}} \left| \sum_{i,j=1}^n a_{ij} x_j^{(1)} \cdots x_j^{(m)} y_i \right| \\
&\leq \left(\sum_j |x_j^{(1)}|^{p_1} \right)^{1/p_1} \cdots \left(\sum_j |x_j^{(m)}|^{p_1} \right)^{1/p_m} \\
&\quad \times \sup_{y \in B_{\ell_{u'}}} \left(\sum_j \left| \sum_i a_{ij} y_i \right|^s \right)^{1/s} \\
&\leq \sup_{y \in B_{\ell_{u'}}} \left(\sum_j \left| \sum_i a_{ij} y_i \right|^2 \right)^{1/2} n^{1/s - 1/2},
\end{aligned}$$

where $\frac{1}{s} = 1 - (\frac{1}{p_1} + \cdots + \frac{1}{p_m})$ and noting that $s < 2$. Since $\sum_{j=1}^n a_{kj} \overline{a}_{lj} = n\delta_{kl}$ we have, for each $y \in B_{\ell_{u'}}$,

$$\begin{aligned}
\left(\sum_{j=1}^n \left| \sum_i a_{ij} y_i \right|^2 \right)^{1/2} &= \left(\sum_{j=1}^n \sum_{i_1, i_2 = 1}^n a_{i_1 j} \overline{a}_{i_2 j} y_{i_1} \overline{y}_{i_2} \right)^{1/2} \\
&= \left(\sum_{i_1, i_2 = 1}^n \sum_{j=1}^n a_{i_1 j} \overline{a}_{i_2 j} y_{i_1} \overline{y}_{i_2} \right)^{1/2} \\
&= n^{1/2} \left(\sum_{i=1}^n |y_i|^2 \right)^{1/2} \\
&\leq n^{1/2} \left(\sum_{i=1}^n |y_i|^{u'} \right)^{1/u'} n^{1/2 - 1/u'} \leq n^{1/u}.
\end{aligned}$$

This altogether gives $\|T\| \leq n^{\frac{1}{2} + \frac{1}{u} - (\frac{1}{p_1} + \cdots + \frac{1}{p_m})}$. On the other hand, $T(e_{i_1}, \dots, e_{i_m}) = (a_{1i}, \dots, a_{ni})$ if $i_1 = \dots i_m = i$ and the null vector, otherwise, then $(\sum \|T(e_{i_1},\dots,e_{i_m})\|_{\ell_q}^t)^{1/t} = n^{1/t+1/q}$ and, if (9) holds we have

$$\frac{1}{t} \leq \frac{1}{2} + \frac{1}{u} - \frac{1}{q} - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_m}\right).$$

Hence, $t \geq \rho$ in the case (ib) under the assumption $\frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{2}$. □

By a deep result of Kwapień [**16**] we know that every operator $v\colon \ell_1 \to \ell_q$ is $(r,1)$-summing with $\frac{1}{r} = 1 - |\frac{1}{q} - \frac{1}{2}|$, and this r is optimal. For $q = 2$ this is Grothendieck's theorem. A straightforward application of Theorem 1.2 with this gives the following.

Proposition 4.5. *Let $1 \leq p_1,\dots,p_m \leq \infty$ and $1 \leq q \leq \infty$. Then there is $C > 0$ such that, for every continuous m-linear $T\colon \ell_{p_1} \times \cdots \times \ell_{p_m} \to \ell_1$ with coefficients $(a_{i_1,\dots,i_m})$ and every operator $v\colon \ell_1 \to \ell_q$ we have*

$$\left(\sum_{i_1,\dots,i_m=1}^{\infty} \|v a_{i_1,\dots,i_m}\|^{\rho}\right)^{1/\rho} \leq C\|T\|,$$

where ρ is given by

(i) *If $1 \leq q \leq 2$ and*
   (a) *if $0 \leq \frac{1}{p_1}+\cdots+\frac{1}{p_m} < 1-\frac{1}{q}$, then $\rho = \frac{2m}{m+2-2(1/q-(1/p_1+\cdots+1/p_m))}$;*
   (b) *if $1-\frac{1}{q} \leq \frac{1}{p_1}+\cdots+\frac{1}{p_m} < \frac{3}{2}-\frac{1}{q}$, then $\rho = \frac{2}{3-2(1/q+(1/p_1+\cdots+1/p_m))}$.*

(ii) *If $2 \leq q$ and*
   (a) *if $0 \leq \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{2}$, then $\rho = \frac{m}{1/2+m/q-(1/p_1+\cdots+1/p_m)}$;*
   (b) *if $\frac{1}{2} \leq \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{2} + \frac{1}{q}$, then $\rho = \frac{1}{1/2+1/q-(1/p_1+\cdots+1/p_m)}$.*

5. Final comments

An m-linear mapping between Banach spaces $T\colon X_1 \times \cdots \times X_m \to Y$ is multiple $(t; r_1,\dots,r_m)$-summing (see e.g. [**18, 7**]) if there is $K > 0$ such that for every $(x_{i_j}^{(j)})_{i_j=1}^{N_j} \subseteq X_j$, for $j = 1,\dots,m$ we have

$$\left(\sum_{i_1,\dots,i_m}^{N_1,\dots,N_m} \|T(x_{i_1}^{(1)},\dots,x_{i_m}^{(m)})\|^t\right)^{1/t} \leq K \prod_{j=1}^{m} \sup_{x_j^* \in B_{X_j^*}} \left(\sum_{i_j=1}^{N_j} |x_j^*(x_{i_j}^{(j)})|^{r_j}\right)^{1/r_j}.$$

We denote by $\mathcal{L}_{\mathrm{ms}(t;r_1,\dots,r_m)}({}^m X_1,\dots,X_m;Y)$ the space of multiple $(t;r_1,\dots,r_m)$-summing m-linear mapppings. Proceeding as in [**19**, Corollary 3.20] one gets that the following two statements are equivalent:

- There is a constant $C>0$ such that for every $T\in\mathcal{L}(^m\ell_{p_1},\dots,\ell_{p_m};Y)$ the following holds
$$\left(\sum_{i_1,\dots,i_m}\|T(e_{i_1},\dots,e_{i_m})\|^t\right)^{1/t}\leq C\|T\|.$$
- For all Banach spaces $X_1,\dots,X_m$ we have
$$\mathcal{L}(^mX_1,\dots,X_m;Y)=\mathcal{L}_{\mathrm{ms}(t;p_1',\dots,p_m')}(^mX_1,\dots,X_m;Y).$$

Then all our results have a straightforward interpretation as coincidence results for multiple summing multilinear mappings.

We have recently learned that some particular cases of some of our results (more precisely Proposition 3.1 for $q=2$ and the case (ia) in Proposition 4.4) have been independently obtained in [**1**].

On the other hand, after we made this note public the same authors produced an alternative proof of Theorem 1.2 and have show that the exponent is also optimal in Proposition 4.4(iia). For the remaining case in Proposition 4.4(ia) lower and upper estimates for the optimal exponent are given. All this can be found in [**2**].

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