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HYBRID BOUNDS FOR TWISTS OF $GL(3)$ L-FUNCTIONS

Qingfeng Sun

Abstract: Let π be a Hecke–Maass cusp form for $SL(3,\mathbb{Z})$ and $\chi=\chi_1\chi_2$ a Dirichlet character with χ_i primitive modulo M_i. Suppose that M_1, M_2 are primes such that $\max\{(M|t|)^{1/3+2\delta/3}, M^{2/5}|t|^{-9/20}, M^{1/2+2\delta}|t|^{-3/4+2\delta}\}(M|t|)^{\varepsilon}<M_1<\min\{(M|t|)^{2/5}, (M|t|)^{1/2-8\delta}\}(M|t|)^{-\varepsilon}$ for any $\varepsilon>0$, where $M=M_1M_2$, $|t|\geq 1$, and $0<\delta<1/52$. Then we have

$$L\left(\frac{1}{2}+it,\pi\otimes\chi\right)\ll_{\pi,\varepsilon}(M|t|)^{3/4-\delta+\varepsilon}.$$

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1. Introduction

Let π be a Hecke–Maass cusp form for $SL(3,\mathbb{Z})$ with normalized Fourier coefficients $\lambda(n_1,n_2)$ such that $\lambda(1,1)=1$. Let χ be a primitive

Dirichlet character modulo M. The L-function attached to the twisted form $\pi \otimes \chi$ is given by the Dirichlet series

$$L(s, \pi \otimes \chi) = \sum_{n=1}^{\infty} \lambda(1, n)\chi(n)n^{-s}$$

for $\mathrm{Re}(s) > 1$, which can be continued to an entire function with a functional equation of arithmetic conductor M^3. Thus by the Phragmen–Lindelöf principle one derives the convexity bound $L(1/2+it, \pi\otimes\chi) \ll_{\pi,\varepsilon} (M(1+|t|))^{3/4+\varepsilon}$, where $\varepsilon > 0$ is arbitrary. The important challenge for us is to prove a sub-convexity bound which improves the convexity bound by providing a smaller exponent. There has been great progress for the sub-convexity problem of $L(s, \pi \otimes \chi)$ in the works [**1**], [**5**], and [**12**]–[**16**] (also see [**7**], [**9**], and [**17**] for the t-aspect sub-convexity for $L(s, \pi)$). In [**1**], Blomer established the bound

$$L\left(\frac{1}{2} + it, \pi \otimes \chi\right) \ll_{\pi,t,\varepsilon} M^{3/4-1/8+\varepsilon}$$

for π self-dual and χ a quadratic character modulo prime M. This was extended by Huang in [**5**], where by combining the methods in [**1**] and [**7**], he showed that

$$L\left(\frac{1}{2} + it, \pi \otimes \chi\right) \ll_{\pi,\varepsilon} (M(1+|t|))^{3/4-1/46+\varepsilon}$$

for the same form $\pi \otimes \chi$ as in [**1**]. For general $GL(3)$ Hecke–Maass cusp forms, the sub-convexity results have recently been established in several cases by Munshi in a series of papers [**13**]–[**16**]. In the t-aspect, Munshi proved in [**14**] that

$$L\left(\frac{1}{2} + it, \pi\right) \ll_{\pi,\varepsilon} (1+|t|)^{3/4-1/16+\varepsilon}. \tag{1.1}$$

For χ a primitive Dirichlet character modulo prime M, he proved in [**15**], [**16**] that

$$L\left(\frac{1}{2}, \pi \otimes \chi\right) \ll_{\pi,\varepsilon} M^{3/4-1/308+\varepsilon}.$$

For $\chi = \chi_1\chi_2$ a Dirichlet character with χ_i primitive modulo prime M_i such that $\sqrt{M_2}M^{4\vartheta} < M_1 < M_1M^{-3\vartheta}$, he showed in [**13**] that

$$L\left(\frac{1}{2}, \pi \otimes \chi\right) \ll_{\pi,\varepsilon} M^{3/4-\vartheta+\varepsilon},$$

where $M = M_1M_2$ and $0 < \vartheta < 1/28$.

In this paper we want to extend some results by Munshi in [**13**] and [**14**]. Our main result is the following:

Theorem 1. *Let π be a Hecke–Maass cusp form for $SL(3,\mathbb{Z})$ and $\chi = \chi_1\chi_2$ a Dirichlet character with χ_i primitive modulo M_i. Suppose that M_1, M_2 are primes such that*

$$\max\{(M|t|)^{1/3+2\delta/3}, M^{2/5}|t|^{-9/20}, M^{1/2+2\delta}|t|^{-3/4+2\delta}\}(M|t|)^{\varepsilon} < M_1 < \min\{(M|t|)^{2/5}, (M|t|)^{1/2-8\delta}\}(M|t|)^{-\varepsilon}$$

for any $\varepsilon > 0$, where $M = M_1M_2$, $|t| \geq 1$, and $0 < \delta < 1/52$. Then we have

$$L\left(\frac{1}{2}+it, \pi\otimes\chi\right) \ll_{\pi,\varepsilon} (M|t|)^{3/4-\delta+\varepsilon}.$$

We also have a result which can be compared with (1.1).

Theorem 2. *Let π be a Hecke–Maass cusp form for $SL(3,\mathbb{Z})$ and $\chi = \chi_1\chi_2$ a Dirichlet character with χ_i primitive modulo M_i. Suppose that M_1, M_2 are primes such that*

$$\max\{M^{3/8-2\delta/3}|t|^{3/8}, M^{2/5}|t|^{-9/20}, M^{5/8-2\delta}|t|^{-5/8}\}(M|t|)^{\varepsilon} < M_1 < \min\{(M|t|)^{2/5}, M^{8\delta}\}(M|t|)^{-\varepsilon}$$

for any $\varepsilon > 0$, where $M = M_1M_2$, $|t| \geq 1$, and $0 < \delta \leq 1/16$. Then we have

$$L\left(\frac{1}{2}+it, \pi\otimes\chi\right) \ll_{\pi,\varepsilon} M^{\delta}(M|t|)^{3/4-1/16+\varepsilon}.$$

Remark 1. Theorems 1 and 2 give us a sub-convexity bound for $L\left(\frac{1}{2}+it, \pi\otimes\chi\right)$ for M and t in some range. For example, if $|t| > M^{1/5}$ and $(M|t|)^{1/3+2\delta/3+\varepsilon} < M_1 < (M|t|)^{2/5-\varepsilon}$ with $0 < \delta \leq 1/80$, then we have

$$L\left(\frac{1}{2}+it, \pi\otimes\chi\right) \ll_{\pi,\varepsilon} (M|t|)^{3/4-\delta+\varepsilon}.$$

If $|t| > M^{1/4}$ and $(M|t|)^{3/8+\varepsilon}M^{-2\delta/3} < M_1 < M^{8\delta-\varepsilon}$ with $0 < \delta \leq 1/16$, then we have

$$L\left(\frac{1}{2}+it, \pi\otimes\chi\right) \ll_{\pi,\varepsilon} M^{\delta}(M|t|)^{3/4-1/16+\varepsilon}.$$

To prove Theorems 1 and 2, we will use the same method as in [**13**] and [**14**]. Suppose that $t \geq 1$. Then by the approximate functional equation we have

$$L\left(\frac{1}{2}+it, \pi\otimes\chi\right) \ll_{\pi,\varepsilon} (Mt)^{\varepsilon} \sup_{N\leq (Mt)^{3/2+\varepsilon}} \frac{|\mathcal{S}(N)|}{\sqrt{N}}, \tag{1.2}$$

where

$$\mathcal{S}(N) = \sum_{n=1}^{\infty} \lambda(1,n)\chi(n)n^{-it}V\left(\frac{n}{N}\right)$$

for some smooth function V supported in $[1,2]$, normalized such that $\int_{\mathbb{R}} V(v)\,\mathrm{d}v = 1$ and satisfying $V^{(\ell)}(x) \ll_\ell 1$. Note that, by the Cauchy's inequality and the Rankin–Selberg estimate $\sum_{n\le x} |\lambda(1,n)|^2 \ll_\pi x$ (see [**11**]), we have the trivial bound $\mathcal{S}(N) \ll_{\pi,\varepsilon} N$. Thus Theorem 1 (resp. Theorem 2) is true for $N \ll (Mt)^{3/2-2\delta}$ (resp. $N \ll (Mt)^{11/8} M^{2\delta}$). In the following, we will estimate $\mathcal{S}(N)$ in the range

$$(Mt)^{3/2-2\delta} < N \le (Mt)^{3/2+\varepsilon} \quad (\text{resp. } (Mt)^{11/8} M^{2\delta} < N \le (Mt)^{3/2+\varepsilon}). \tag{1.3}$$

The first step is to separate the Fourier coefficients $\lambda(1,n)$ and $\chi(n)n^{-it}$. Let $\delta(n)$ be equal to 1 if $n=0$ and 0 otherwise. Like in [**13**] and [**14**] we apply Kloosterman's version of the circle method, which states that for any $n \in \mathbb{Z}$ and $Q \in \mathbb{R}^+$, we have

$$\delta(n) = 2\mathrm{Re} \int_0^1 \sum_{1\le q\le Q} \sum_{\substack{Q<a\le q+Q \\ (a,q)=1}} \frac{1}{aq} e\left(\frac{n\overline{a}}{q} - \frac{n\zeta}{aq}\right) \mathrm{d}\zeta, \tag{1.4}$$

where, throughout the paper, $e(z) = e^{2\pi i z}$ and $\overline{a}$ denotes the multiplicative inverse of a modulo q.

To construct a conductor lowering system to take care of both the t-aspect and the M-aspect, we introduce a parameter K satisfying $(Mt)^\varepsilon < K < t$ and write

$$\mathcal{S}(N) = \frac{1}{K} \int_{\mathbb{R}} V\left(\frac{v}{K}\right) \sum_{n=1}^{\infty} \lambda(1,n) V\left(\frac{n}{N}\right) \times \sum_{\substack{m\in\mathbb{Z} \\ M_1 | n-m}} \chi(m) m^{-it} U\left(\frac{m}{N}\right) \delta\left(\frac{n-m}{M_1}\right) \left(\frac{n}{m}\right)^{iv} \mathrm{d}v,$$

where U is a smooth function supported in $[1/2, 5/2]$, $U(x) = 1$ for $x \in [1,2]$, and $U^{(\ell)}(x) \ll_\ell 1$. Applying (1.4) and choosing

$$Q = \sqrt{\frac{N}{KM_1}}$$

we get

$$\mathcal{S}(N) = \mathcal{S}^+(N) + \mathcal{S}^-(N),$$

where

$$\mathcal{S}^\pm(N) = \frac{1}{K} \int_{\mathbb{R}} \int_0^1 V\left(\frac{v}{K}\right) \sum_{n=1}^{\infty} \lambda(1,n) n^{iv} V\left(\frac{n}{N}\right) \sum_{\substack{m\in\mathbb{Z} \\ M_1 | n-m}} \chi(m) m^{-i(t+v)} U\left(\frac{m}{N}\right) \times \sum_{1\le q\le Q} \sum_{\substack{Q<a\le q+Q \\ (a,q)=1}} \frac{1}{aq} e\left(\pm \frac{\overline{a}(n-m)}{qM_1} \mp \frac{(n-m)\zeta}{aqM_1}\right) \mathrm{d}v\, \mathrm{d}\zeta.$$

In the rest of the paper we will estimate $\mathcal{S}^+(N)$ (and the same analysis holds for $\mathcal{S}^-(N)$). Denote by $\mathcal{S}^\flat(N)$ and $\mathcal{S}^\sharp(N)$ the contribution to $\mathcal{S}^+(N)$ from $M_1|q$ and $(M_1, q) = 1$, respectively. Then Theorems 1 and 2 follow from (1.2), (1.3), and the following propositions:

Proposition 1. *Assume* $K < \min\{t, NM_1/M^2\}(Mt)^{-\varepsilon}$*. Then we have*

$$\mathcal{S}^\flat(N) \ll N\sqrt{Mt}/M_1^{3/2}.$$

Proposition 2. *Assume* $(Mt)^{6/5}/(NM_1)^{3/5} \le K < \min\{t, (Mt)^2/NM_1, NM_1/M^2\}(Mt)^{-\varepsilon}$*. Then we have*

$$\mathcal{S}^\sharp(N) \ll \begin{cases} N^{5/8}(Mt)^{1/2} & \text{if } (Mt)^{24/17}M_1^{8/17} < N \le (Mt)^{3/2+\varepsilon}, \\ N^{1/5}(Mt)^{11/10}M_1^{1/5} & \text{if } N \le (Mt)^{24/17}M_1^{8/17}. \end{cases}$$

For our purpose we choose the optimal K as

$$K = \max\left\{\frac{N^{1/4}}{M_1}, \frac{(Mt)^{6/5}}{(NM_1)^{3/5}}\right\}. \tag{1.5}$$

Propositions 1 and 2 will be proved by summation formulas of Voronoi's type and stationary phase method, which are listed in Section 2.

Remark 2. With K as in (1.5), one sees that the assumptions for K in Propositions 1 and 2 are fulfilled if M_1 is in the range of Theorem 1 or Theorem 2.

Remark 3. In the appendix of [**13**], Munshi showed that Kloosterman's circle method with suitable conductor lowering mechanism also works for χ with a prime power modulus. For hybrid bounds in the t and the M aspects, we will study this in a separate paper.

Notation. Throughout the paper, the letters q, m, and n, with or without subscript, denote integers. The letter ε is an arbitrarily small positive constant, not necessarily the same at different occurrences. The symbol $\ll_{a,b,c}$ denotes that the implied constant depends at most on a, b, and c. The symbols $q \sim C$ and $q \asymp C$ mean that $C < q \le 2C$ and $c_1C \le q \le c_2C$ for some absolute constants c_1, c_2, respectively. Finally, fractional numbers such as $\frac{ab}{cd}$ will be written as ab/cd, and $a/b + c$ or $c + a/b$ mean $\frac{a}{b} + c$.

2. Voronoi formula and stationary phase method

2.1. *GL*(3) cusp forms and Voronoi formula. Let π be a Hecke–Maass cusp form of type $\nu = (\nu_1, \nu_2)$ for $SL(3, \mathbb{Z})$, which has a Fourier–Whittaker expansion (see [**3**]) with Fourier coefficients $\lambda(n_1, n_2)$, normalized so that $\lambda(1,1)=1$. By Rankin–Selberg theory, the Fourier coefficients $\lambda(n_1,n_2)$ satisfy

$$\sum\sum_{n_1^2n_2\le x}|\lambda(n_1,n_2)|^2\ll_{\pi,\varepsilon}x^{1+\varepsilon}. \tag{2.1}$$

Denote the Langlands parameters by

$$\mu_1=-\nu_1-2\nu_2+1,\quad \mu_2=-\nu_1+\nu_2,\quad \mu_3=2\nu_1+\nu_2-1.$$

The generalized Ramanujan conjecture asserts that $\mathrm{Re}(\mu_j)=0$, $1\le j\le 3$, while the current record bound due to Luo, Rudnick, and Sarnak [**8**] is $|\mathrm{Re}(\mu_j)|\le 1/2-1/10$, $1\le j\le 3$. For $\ell=0,1$ we define

$$\gamma_\ell(s)=\frac{1}{2\pi^{3(s+1/2)}}\prod_{j=1}^{3}\frac{\Gamma((1+s+\mu_j+\ell)/2)}{\Gamma((-s-\mu_j+\ell)/2)}$$

and set $\gamma_\pm(s)=\gamma_0(s)\mp i\gamma_1(s)$. Then for $\sigma\ge -1/2$,

$$\gamma_\pm(\sigma+i\tau)\ll_{\pi,\sigma}(1+|\tau|)^{3(\sigma+1/2)} \tag{2.2}$$

and, for $|\tau|\gg (Mt)^\varepsilon$, we can apply Stirling's formula to get (see [**14**])

$$\gamma_\pm\left(-\frac{1}{2}+i\tau\right)=\left(\frac{|\tau|}{e\pi}\right)^{3i\tau}\Psi_\pm(\tau),\quad \text{where}\quad \Psi_\pm'(\tau)\ll\frac{1}{|\tau|}. \tag{2.3}$$

Let $\phi(x)$ be a smooth function compactly supported on $(0,\infty)$ and denote by $\widetilde{\phi}(s)$ the Mellin transform of $\phi(x)$. Let

$$\Phi_\phi^\pm(x)=\frac{1}{2\pi i}\int_{(\sigma)}x^{-s}\gamma_\pm(s)\widetilde{\phi}(-s)\,\mathrm{d}s,$$

where $\sigma>\max_{1\le j\le 3}\{-1-\mathrm{Re}(\mu_j)\}$. Then we have the following Voronoi-type formula (see [**4**], [**10**]):

Lemma 1. *Suppose that $\phi(x)\in C_c^\infty(0,\infty)$. Let $a,q\in\mathbb{Z}$ with $q\ge 1$, $(a,q)=1$, and $a\overline{a}\equiv 1 \pmod q$. Then*

$$\sum_{n=1}^\infty\lambda(1,n)e\left(\frac{an}{q}\right)\phi(n)=q\sum_\pm\sum_{n_1|q}\sum_{n_2=1}^\infty\frac{\lambda(n_2,n_1)}{n_1n_2}S\left(\overline{a},\pm n_2;\frac{q}{n_1}\right)\Phi_\phi^\pm\left(\frac{n_1^2n_2}{q^3}\right),$$

where $S(m,n;c)$ is the classical Kloosterman sum.

2.2. Exponential integral and stationary phase method. Here we collect relevant results from [**2**], [**6**], [**14**], and [**18**] that will be used to estimate some exponential integrals in this paper. First we need the stationary phase estimates from [**6**] which will be used to derive asymptotic expansion of the exponential integral

$$\mathcal{I}=\int_a^b g(v)e(f(v))\,\mathrm{d}v,$$

where f, g are smooth real valued functions and $\mathrm{Supp}(g) \subset [a,b]$. The following result can be found in Huxley [**6**].

Lemma 2. *Assume that* $\Theta_f, \Omega_f \gg b-a$ *and*

$$f^{(i)}(v) \ll \Theta_f \Omega_f^{-i}, \quad g^{(j)}(v) \ll \Omega_g^{-j} \tag{2.4}$$

for $i=2,3$ *and* $j=0,1,2$.

(1) *Suppose* f' *and* f'' *do not vanish in* $[a,b]$. *Let* $\Lambda = \min_{[a,b]} |f'(v)|$. *Then we have*

$$\mathcal{I} \ll \frac{\Theta_f}{\Omega_f^2 \Lambda^3}\left(1+\frac{\Omega_f}{\Omega_g}+\frac{\Omega_f^2}{\Omega_g^2}\frac{\Lambda}{\Theta_f/\Omega_f}\right).$$

(2) *Suppose* f' *changes sign from negative to positive at the unique point* $v_0 \in (a,b)$. *Let* $\kappa = \min\{b-v_0, v_0-a\}$. *Further, suppose* (2.4) *holds for* $i=4$ *and*

$$f^{(2)}(v) \gg \Theta_f/\Omega_f^2.$$

Then

$$\mathcal{I} = \frac{g(v_0)e(f(v_0)+1/8)}{\sqrt{f''(v_0)}} + O\left(\frac{\Omega_f^4}{\Theta_f^2\kappa^3}+\frac{\Omega_f}{\Theta_f^{3/2}}+\frac{\Omega_f^3}{\Theta_f^{3/2}\Omega_g^2}\right).$$

For the special exponential integral

$$U^\dagger(r,s) = \int_0^\infty U(x)e(-rx)x^{s-1}\,\mathrm{d}x,$$

where U is a smooth real valued function with $\mathrm{Supp}(U) \subset [a,b] \subset (0,\infty)$, we quote the following result from [**14**] which is derived from Lemma 2.

Lemma 3. *Suppose* $U^{(j)}(x) \ll_{a,b,j} 1$. *Let* $r \in \mathbb{R}$ *and* $s = \sigma + i\beta \in \mathbb{C}$. *We have*

$$\begin{aligned} U^\dagger(r,s) &= \frac{\sqrt{2\pi}e(1/8)}{\sqrt{-\beta}} U\left(\frac{\beta}{2\pi r}\right)\left(\frac{\beta}{2\pi r}\right)^{\sigma}\left(\frac{\beta}{2\pi e r}\right)^{i\beta} \\ &\quad + O\Big(\min\{|\beta|^{-3/2}, |r|^{-3/2}\}\Big), \end{aligned} \tag{2.5}$$

where the implied constant depends only on a, b, *and* σ. *We also have*

$$U^\dagger(r,s) \ll_{a,b,\sigma,j} \min\left\{\left(\frac{1+|\beta|}{|r|}\right)^j, \left(\frac{1+|r|}{|\beta|}\right)^j\right\}. \tag{2.6}$$

In applications, the O-term in (2.5) is not essential. For our purpose, we will also use the following more precise asymptotic expansion to simplify computations (see [**2**, Proposition 8.2]). For a proof, see also [**18**].

Lemma 4. *Let $r \in \mathbb{R}$ and $s = \sigma + i\beta \in \mathbb{C}$ such that $x_0 = \beta/(2\pi r) \in [a/2, 2b]$. Then we have*

$$U^\dagger(r,s) = \frac{\sqrt{2\pi}e(1/8)}{\sqrt{-\beta}} U^*\left(\frac{\beta}{2\pi r}\right)\left(\frac{\beta}{2\pi r}\right)^{\sigma}\left(\frac{\beta}{2\pi e r}\right)^{i\beta} + O\Big(\min\{|\beta|^{-5/2}, |r|^{-5/2}\}\Big), \tag{2.7}$$

where $U^*(x_0) = x_0^{1-\sigma}\sum_{n=0}^{5} p_n(x_0)$ and

$$p_n(x_0) = \frac{1}{n!}\left(\frac{i}{2h''(x_0)}\right)^n G^{(2n)}(x_0).$$

Here $h(x) = -2\pi r x + \beta \log x$, $G(x) = U(x)x^{\sigma-1}e^{iH(x)}$, and

$$H(x) = h(x) - h(x_0) - \frac{1}{2!}h''(x_0)(x-x_0)^2.$$

Moreover, $G^{(2n)}(x_0)$ is a linear combination of terms of the form $(U(x)x^{\sigma-1})^{(\ell_0)}|_{x=x_0} H^{(\ell_1)}(x_0)\cdots H^{(\ell_i)}(x_0)$, where $\ell_0+\ell_1+\cdots+\ell_i = 2n$, so that $U^{*(\ell)}(x_0) \ll_{\sigma,a,b,\ell} 1$.

3. Estimating $\mathcal{S}^\flat(N)$

Recall that

$$\mathcal{S}^\flat(N) = \frac{1}{K}\int_{\mathbb{R}}\int_0^1 V\left(\frac{v}{K}\right)\sum_{n=1}^{\infty}\lambda(1,n)n^{iv}V\left(\frac{n}{N}\right) \times \sum_{1\le q\le Q/M_1}\ \sum_{\substack{Q<a\le qM_1+Q\\(a,qM_1)=1}} \frac{1}{aqM_1} e\left(\frac{\bar{a}n}{qM_1^2} - \frac{n\zeta}{aqM_1^2}\right) \times \sum_{\substack{m\in\mathbb{Z}\\ M_1|n-m}} \chi(m)m^{-i(t+v)}U\left(\frac{m}{N}\right)e\left(-\frac{\bar{a}m}{qM_1^2} + \frac{m\zeta}{aqM_1^2}\right)\mathrm{d}v\,\mathrm{d}\zeta.$$

Applying Poisson summation formula with modulus $qM_1^2M_2$ on the sum over m we get

$$\sum_{\substack{m\in\mathbb{Z}\\ M_1|n-m}} \chi(m)m^{-i(t+v)}U\left(\frac{m}{N}\right)e\left(-\frac{\bar{a}m}{qM_1^2} + \frac{m\zeta}{aqM_1^2}\right) = \frac{N^{1-i(t+v)}}{qM_1^2M_2}\sum_{m\in\mathbb{Z}}\mathscr{E}(a,m,q)U^\dagger\left(\frac{N(ma-\zeta M_2)}{aqM_1^2M_2}, 1-i(t+v)\right),$$

where $U^\dagger(r,s)$ is defined in Section 2 and

$$\mathscr{E}(a,m,q) = \sum_{\substack{c \bmod qM_1^2M_2\\ c\equiv n \bmod M_1}} \chi(c)e\left(\frac{(m-M_2\bar{a})c}{qM_1^2M_2}\right).$$

Lemma 5. *Let $q = q_0 M_1^j M_2^k$, $(q_0, M_1M_2) = 1$ with $j, k \geq 0$. We have*

$$\mathscr{E}(a, m, q) = \varepsilon_2 q M_1 \sqrt{M_2} \chi_1(q_0 M_2^{k+1} n) \chi_2(q_0 M_1 \overline{m^*}) e(m^* M_2^k n / M_1)$$

if $m \equiv M_2 \overline{a} \bmod qM_1$, and is zero otherwise. Here $\varepsilon_2 \sqrt{M_2}$ is the value of the Gauss sum corresponding to the character χ_2, and

$$m^* = (m - M_2\overline{a})/M_1^{j+1} M_2^k.$$

In particular, we have $a \equiv \overline{m} M_2 \bmod qM_1$ if $k = 0$. If $k \geq 1$, we have $M_2 | m$ and $a \equiv \overline{(m/M_2)} \bmod qM_1/M_2$.

Proof: We have

$$\begin{aligned}\mathscr{E}(a, m, q) = &\sum_{c_1 \bmod q_0} e\left(\frac{(m - M_2\overline{a})c_1}{q_0}\right) \\ &\times \sum_{\substack{c_2 \bmod M_1^{j+2} \\ c_2 \equiv n \bmod M_1}} \chi_1(q_0 M_2^{k+1} c_2) e\left(\frac{(m - M_2\overline{a})c_2}{M_1^{j+2}}\right) \\ &\times \sum_{c_3 \bmod M_2^{k+1}} \chi_2(q_0 M_1^{j+2} c_3) e\left(\frac{(m - M_2\overline{a})c_3}{M_2^{k+1}}\right),\end{aligned}$$

where the first sum vanishes unless $m \equiv M_2\overline{a} \bmod q_0$, in which case it is q_0. The second sum vanishes unless $m \equiv M_2\overline{a} \bmod M_1^{j+1}$, in which case it equals

$$\chi_1(q_0 M_2^{k+1} n) e\left(\frac{m^* M_2^k n}{M_1}\right) M_1^{j+1},$$

where $m^* = (m - M_2\overline{a})/M_1^{j+1} M_2^k$. Finally, the last sum equals

$$\varepsilon_2 \chi_2(q_0 M_1) \overline{\chi_2}(m^*) M_2^k \sqrt{M_2}$$

if $m \equiv M_2\overline{a} \bmod M_2^k$, and is zero otherwise, where $\varepsilon_2\sqrt{M_2}$ is the value of the Gauss sum corresponding to the character χ_2. □

Note that, if $m = 0$ we have $k \geq 1$ and $(m, qM_1) = M_2$. Then

$$\frac{N|0 - \zeta M_2|}{aqM_1^2 M_2} \leq \frac{N}{QM_2M_1^2} < (Mt)^{-\varepsilon} t.$$

For $|m| \geq 1$, we have (recall $a > Q$)

$$\frac{N|ma - \zeta M_2|}{aqM_1^2 M_2} \asymp \frac{N|m|}{qM_1^2 M_2}.$$

Applying (2.6) one sees that the contribution from $m = 0$ and $|m| \geq qM_1(Mt)^{1+\varepsilon}/N$ is negligibly small. For smaller nonzero m, by the second derivative bound for the exponential integral, we have

$$U^\dagger\left(\frac{N(ma - \zeta M_2)}{aqM_1^2 M_2}, 1 - i(t + v)\right) \ll t^{-1/2}.$$

Therefore, using (2.1),

$$\mathcal{S}^{\flat}(N) \ll \frac{N}{M_1\sqrt{M_2 t}} \sum_{n\le 2N} |\lambda(1,n)| \sum_{\substack{1\le q\le Q/M_1\\(q,M_2)=1}} \frac{1}{QqM_1} \frac{qM_1(Mt)^{1+\varepsilon}}{N}$$

$$+ \frac{N}{M_1\sqrt{M_2 t}} \sum_{n\le 2N} |\lambda(1,n)| \sum_{\substack{1\le q\le Q/M_1\\M_2|q}} \frac{M_2}{QqM_1} \frac{qM_1(Mt)^{1+\varepsilon}}{N}$$

$$\ll N\sqrt{Mt}/M_1^{3/2}.$$

This completes the proof of Proposition 1. □

4. Estimating $\mathcal{S}^{\sharp}(N)$-I

First we detect the congruence $m \equiv n \bmod M_1$ using exponential sums to get (recall M_1 is a prime)

$$\mathcal{S}^{\sharp}(N) = \mathcal{S}_0(N) + \mathcal{S}_1(N),$$

where

$$\mathcal{S}_0(N) = \frac{1}{KM_1} \int_{\mathbb{R}} \int_0^1 V\left(\frac{v}{K}\right) \sum_{\substack{1\le q\le Q\\(q,M_1)=1}} \sum_{\substack{Q<a\le q+Q\\(a,q)=1}} \frac{1}{aq}$$

$$\times \sum_{n=1}^{\infty} \lambda(1,n) e\left(\frac{\overline{aM_1}n}{q}\right) n^{iv} V\left(\frac{n}{N}\right) e\left(-\frac{n\zeta}{aqM_1}\right)$$

$$\times \sum_{m\in\mathbb{Z}} \chi(m) e\left(\frac{-\overline{aM_1}m}{q}\right) m^{-i(t+v)} U\left(\frac{m}{N}\right) e\left(\frac{m\zeta}{aqM_1}\right) \mathrm{d}v\, \mathrm{d}\zeta$$

and

$$\mathcal{S}_1(N) = \frac{1}{KM_1} \int_{\mathbb{R}} \int_0^1 V\left(\frac{v}{K}\right) \sum_{\substack{1\le q\le Q\\(q,M_1)=1}} \sum_{\substack{Q<a\le q+Q\\(a,q)=1}} \sideset{}{^*}\sum_{b \bmod M_1} \frac{1}{aq}$$

$$\times \sum_{n=1}^{\infty} \lambda(1,n) e\left(\frac{(\overline{aM_1}M_1+bq)n}{qM_1}\right) n^{iv} V\left(\frac{n}{N}\right) e\left(-\frac{n\zeta}{aqM_1}\right) \tag{4.1}$$

$$\times \sum_{m\in\mathbb{Z}} \chi(m) e\left(\frac{-(\overline{aM_1}M_1+bq)m}{qM_1}\right) m^{-i(t+v)} U\left(\frac{m}{N}\right) e\left(\frac{m\zeta}{aqM_1}\right) \mathrm{d}v\, \mathrm{d}\zeta,$$

where the $*$ denotes the condition $(b, M_1) = 1$. In the rest of the paper, we will estimate $\mathcal{S}_1(N)$. The analysis for $\mathcal{S}_0(N)$ is similar, and following the proof for $\mathcal{S}_1(N)$, one can see that it is smaller.

Applying Poisson summation with modulus $qM_1M_2 = qM$ on the sum over m in (4.1) we get

$$\sum_{m\in\mathbb{Z}} \chi(m) e\left(\frac{-(\overline{aM_1}M_1+bq)m}{qM_1}\right) m^{-i(t+v)} U\left(\frac{m}{N}\right) e\left(\frac{m\zeta}{aqM_1}\right)$$
$$= \frac{N^{1-i(t+v)}}{qM} \sum_{m\in\mathbb{Z}} \mathscr{D}(a,b,m,q) U^{\dagger}\left(\frac{N(ma-\zeta M_2)}{aqM}, 1-i(t+v)\right),$$

where

$$\mathscr{D}(a,b,m,q) = \sum_{c \bmod qM} \chi(c) e\left(\frac{cm}{qM} - \frac{c(\overline{aM_1}M_1+bq)}{qM_1}\right).$$

Lemma 6. *Let $q = q_0M_2^k$, $(q_0, M_1M_2) = 1$ with $k \geq 0$. We have*

$$\mathscr{D}(a,b,m,q) = \varepsilon_1\varepsilon_2 q\sqrt{M}\chi_2(q_0M_1)\overline{\chi_1}(\overline{qM_2}m - b)\overline{\chi_2}(m_0)$$

if $m \equiv M_2\overline{a} \bmod q$, and is zero otherwise. Here, $\varepsilon_i\sqrt{M_i}$ is the value of the Gauss sum corresponding to the character χ_i and $m_0 = (m - M_2\overline{a})/M_2^k$. In particular, we have $a \equiv \overline{m}M_2$ if $k = 0$. If $k \geq 1$, we have $M_2|m$ and $a \equiv \overline{(m/M_2)} \bmod q/M_2$.

Proof: Note that

$$\mathscr{D}(a,b,m,q) = \sum_{c_1 \bmod q_0} e\left(\frac{(m-M_2\overline{a})c_1}{q_0}\right)$$
$$\times \sum_{c_2 \bmod M_2^{k+1}} \chi_2(q_0M_1c_2) e\left(\frac{(m-M_2\overline{a})c_2}{M_2^{k+1}}\right)$$
$$\times \sum_{c_3 \bmod M_1} \chi_1(q_0M_2^{k+1}c_3) e\left(\frac{(\overline{qM_2}m-b)q_0M_2^{k+1}c_3}{M_1}\right),$$

where the first sum vanishes unless $m \equiv M_2\overline{a} \bmod q_0$, in which case it is q_0. The second sum equals $\varepsilon_2\chi_2(q_0M_1)\overline{\chi_2}(m_0)M_2^k\sqrt{M_2}$ with $m_0 = (m - M_2\overline{a})/M_2^k$ if $m \equiv M_2\overline{a} \bmod M_2^k$, and is zero otherwise. Here $\varepsilon_i\sqrt{M_i}$ is the value of the Gauss sum corresponding to the character χ_i. Thus the lemma follows. □

As before, by Lemma 6 and (2.6), one sees that the contribution from $m = 0$ and $|m| \geq q(Mt)^{1+\varepsilon}/N$ is negligibly small. For $1 \leq |m| < q(Mt)^{1+\varepsilon}/N$, we have $N/(Mt)^{1+\varepsilon} < q \leq Q$. Taking a dyadic subdivision for the sum over q and denoting $C/2 < q \leq C$ by $q \sim C$, we have the following:

Lemma 7. *Suppose $K < \min\{t, NM_1/M^2\}(Mt)^{-\varepsilon}$. We have*

$$\mathcal{S}_1(N) = \varepsilon_1\varepsilon_2\chi_2(M_1)N^{-it} \sum_{\substack{N/(Mt)^{1+\varepsilon}<C\leq Q \\ C \text{ dyadic}}} \mathcal{S}_1(N,C) + O((Mt)^{-1000}),$$

where

$$\mathcal{S}_1(N,C) = \frac{N}{KM_1\sqrt{M}} \int_{\mathbb{R}} \int_0^1 V\left(\frac{v}{K}\right) N^{-iv} \sum_{\substack{q=q_0M_2^k\sim C \\ (q_0,M)=1}} \frac{\chi_2(q_0)}{q} \sum_{\substack{Q<a\leq q+Q \\ (a,q)=1}} \frac{1}{a}$$

$$\times \sideset{}{^*}\sum_{b \bmod M_1} \sum_{\substack{1\leq |m|\leq q(Mt)^{1+\varepsilon}/N \\ m\equiv M_2\overline{a} \bmod q}} \overline{\chi_1}(\overline{qM_2}m-b)\overline{\chi_2}(m_0)U^\dagger\left(\frac{N(ma-\zeta M_2)}{aqM}, 1-i(t+v)\right)$$

$$\times \sum_{n=1}^\infty \lambda(1,n) e\left(\frac{(\overline{aM_1}M_1+bq)n}{qM_1}\right) n^{iv} V\left(\frac{n}{N}\right) e\left(-\frac{n\zeta}{aqM_1}\right) \mathrm{d}v\,\mathrm{d}\zeta.$$

Applying the $GL(3)$ Voronoi formula in Lemma 1 with $\phi(y) = y^{iv}V(y/N)e(-\zeta y/aqM_1)$ we have

$$\sum_{n=1}^\infty \lambda(1,n) e\left(\frac{(\overline{aM_1}M_1+bq)n}{qM_1}\right) n^{iv} V\left(\frac{n}{N}\right) e\left(-\frac{n\zeta}{aqM_1}\right)$$

$$= qM_1N^{iv} \sum_{\pm} \sum_{n_1|qM_1} \sum_{n_2=1}^\infty \frac{\lambda(n_2,n_1)}{n_1n_2} S\left(\overline{\overline{aM_1}M_1+bq}, \pm n_2; \frac{qM_1}{n_1}\right)$$

$$\times \mathcal{J}_\pm\left(\frac{n_1^2n_2}{q^3M_1^3}, \frac{\zeta}{aqM_1}\right),$$

where

$$\mathcal{J}_\pm(x,y) = \frac{1}{2\pi i}\int_{(\sigma)} (Nx)^{-s}\gamma_\pm(s)V^\dagger(Ny, -s+iv)\,\mathrm{d}s.$$

By (2.6),

$$V^\dagger\left(\frac{\zeta N}{aqM_1}, -s+iv\right) \ll_j \min\left\{1, \left(\frac{1}{q|v-\tau|}\sqrt{\frac{NK}{M_1}}\right)^j\right\}$$

for any $j \geq 0$. Then shifting the contour to $\sigma = \ell$ (a large positive integer) and taking $j = 3\ell+3$ (in view of (2.2)) one has

$$\mathcal{J}_\pm\left(\frac{n_1^2n_2}{q^3M_1^3}, \frac{\zeta}{aqM_1}\right) \ll \left(\frac{1}{q}\sqrt{\frac{NK}{M_1}}\right)^{5/2} \left(\frac{n_1^2n_2}{N^{1/2}K^{3/2}M_1^{3/2}}\right)^{-\ell}.$$

Thus the contribution from $n_1^2 n_2 \geq N^{1/2+\varepsilon} K^{3/2} M_1^{3/2}$ is negligible. For $n_1^2 n_2 < N^{1/2+\varepsilon} K^{3/2} M_1^{3/2}$ we shift the contour to $\sigma = -1/2$, and obtain

$$\mathcal{J}_{\pm}\left(\frac{n_1^2 n_2}{q^3 M_1^3}, \frac{\zeta}{aqM_1}\right) = \sum_{J \in \mathscr{J}} \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{N n_1^2 n_2}{q^3 M_1^3}\right)^{1/2-i\tau} \gamma_{\pm}\left(-\frac{1}{2} + i\tau\right)$$
$$\times V^{\dagger}\left(\frac{N\zeta}{aqM_1}, \frac{1}{2} + i(v-\tau)\right) W_J(\tau)\, \mathrm{d}\tau + O((Mt)^{-1000}),$$

where as in [**14**], $\mathscr{J}$ is a collection of $O(\log(Mt))$ many real numbers in the interval $[-(Mt)^{\varepsilon} C^{-1}\sqrt{NK/M_1}, (Mt)^{\varepsilon} C^{-1}\sqrt{NK/M_1}]$, and W_J is a smooth partition of unity such that, for $J = 0$, the function $W_0(x)$ is supported in $[-1, 1]$ and satisfies $W_0^{(\ell)}(x) \ll_{\ell} 1$, for each $J > 0$ (resp. $J < 0$), the function $W_J(x)$ is supported in $[J, 4J/3]$ (resp. $[4J/3, J]$) and satisfies $y^{\ell} W_J^{(\ell)}(x) \ll_{\ell} 1$ for all $\ell \geq 0$, and finally

$$\sum_{J \in \mathscr{J}} W_J(x) = 1, \quad \text{for} \quad x \in \left[-\frac{(Mt)^{\varepsilon}}{C}\sqrt{\frac{NK}{M_1}}, \frac{(Mt)^{\varepsilon}}{C}\sqrt{\frac{NK}{M_1}}\right].$$

We conclude with the following:

Lemma 8. *Let K be as in Lemma 7. We have*

$$\mathcal{S}_1(N, C) = \sum_{\substack{1 \leq L < N^{1/2+\varepsilon} K^{3/2} M_1^{3/2} \\ L \text{ dyadic}}} \sum_{J \in \mathscr{J}} \sum_{\pm} \mathcal{S}_1(N, C, L, J, \pm) + O((Mt)^{-100}),$$

where

$$\mathcal{S}_1(N, C, L, J, \pm) = \frac{N^{3/2}}{\sqrt{MM_1^3}} \sum\sum_{n_1^2 n_2 \sim L} \frac{\lambda(n_2, n_1)}{\sqrt{n_2}} \sum_{\substack{q = q_0 M_2^k \sim C \\ (q_0, M) = 1 \\ n_1 | qM_1}} \frac{\chi_2(q_0)}{q^{3/2}} \sum_{\substack{Q < a \leq q+Q \\ (a,q)=1}} \frac{1}{a}$$
$$\times \sum_{\substack{1 \leq |m| \leq q(Mt)^{1+\varepsilon}/N \\ m \equiv M_2 \overline{a} \bmod q}} \overline{\chi_2}(m_0) \mathscr{B}(n_1, \pm n_2, m, a, q) \mathcal{J}_{J,\pm}^{*}(q, m, n_1^2 n_2),$$

where

$$\mathscr{B}(n_1, n_2, m, a, q) = \sideset{}{^*}\sum_{b \bmod M_1} \overline{\chi_1}(\overline{qM_2} m - b) S\left(\overline{\overline{aM_1} M_1 + bq}, n_2; \frac{qM_1}{n_1}\right) \tag{4.2}$$

and

$$\mathcal{J}_{J,\pm}^{*}(q, m, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{Ny}{q^3 M_1^3}\right)^{-i\tau} \gamma_{\pm}\left(-\frac{1}{2} + i\tau\right) \mathcal{J}^{**}(q, m, \tau) W_J(\tau)\, \mathrm{d}\tau \tag{4.3}$$

with

$$\mathcal{J}^{**}(q,m,\tau) = \int_{\mathbb{R}}\int_0^1 V(v)V^\dagger\left(\frac{N\zeta}{aqM_1}, \frac{1}{2}+i(Kv-\tau)\right) \times U^\dagger\left(\frac{N(ma-\zeta M_2)}{aqM}, 1-i(t+Kv)\right) \mathrm{d}v\,\mathrm{d}\zeta. \tag{4.4}$$

Let $n = n_1'l$, $(n_1', M_1) = 1$, and $l|M_1$. Since M_1 is a prime, we have $l = M_1$ or 1. For $l = M_1$, by Weil's bound for Klooertman sums $\mathscr{B}(n_1'M_1, n_2, m, a, q) \ll (q/n_1)^{1/2}$. Trivially, we have $\mathcal{J}^*_{J,\pm}(q, m, n_1'^2 M_1^2 n_2) \ll C^{-1}\sqrt{NK/M_1 t}$. Thus the contribution from $l = M_1$ to $\mathcal{S}_1(N, C, L, J, \pm)$ is at most $N^{3/4}K^{7/4}(Mt)^{1/2}M_1^{-5/4}$, which is admissible by the range of M_1. For $l = 1$, we will need extra cancellation from the character sum $\mathscr{B}(n_1, n_2, m, a, q)$ and the integral $\mathcal{J}^*_{J,\pm}(q, m, n_1'^2 M_1^2 n_2)$. Then, the rest of the paper is devoted to estimating

$$\mathcal{S}_1^*(N,C,L,J,\pm) = \frac{N^{3/2}}{\sqrt{MM_1^3}} \sum\sum_{n_1^2 n_2\sim L} \frac{\lambda(n_2,n_1)}{\sqrt{n_2}} \sum_{\substack{q=q_0M_2^k\sim C\\ (q_0,M)=1\\ n_1|q}} \frac{\chi_2(q_0)}{q^{3/2}} \sum_{\substack{Q<a\leq q+Q\\ (a,q)=1}} \frac{1}{a} \times \sum_{\substack{1\leq|m|\leq q(Mt)^{1+\varepsilon}/N\\ m\equiv M_2\overline{a} \bmod q}} \overline{\chi_2}(m_0)\mathscr{B}(n_1,\pm n_2,m,a,q)\mathcal{J}^*_{J,\pm}(q,m,n_1^2 n_2). \tag{4.5}$$

5. A decomposition of the integral $\mathcal{J}^{**}(q,m,\tau)$

The aim of this section is to give a decomposition of $\mathcal{J}^{**}(q,m,\tau)$ for $|\tau|\leq(Mt)^\varepsilon C^{-1}\sqrt{NK/M_1}$. Since we are working on both the variables M and t, we need more precise estimates than those used by Munshi.

5.1. Stationary phase expansion for $U^\dagger$ and $V^\dagger$. Applying (2.7) we get

$$U^\dagger\left(\frac{N(ma-\zeta M_2)}{aqM}, 1-i(t+Kv)\right) = \frac{e(1/8)}{\sqrt{2\pi}}\,\frac{aqM\sqrt{t+Kv}}{N(\zeta M_2-ma)} \times U^*\left(\frac{(t+Kv)aqM}{2\pi N(\zeta M_2-ma)}\right)\left(\frac{(t+Kv)aqM}{2\pi eN(\zeta M_2-ma)}\right)^{-i(t+Kv)} + O(t^{-5/2}).$$

By (2.5) we have

$$V^\dagger\left(\frac{N\zeta}{aqM_1}, \frac{1}{2}+i(Kv-\tau)\right) = \frac{e(1/8)}{\sqrt{\tau-Kv}} \times V\left(\frac{(Kv-\tau)aqM_1}{2\pi N\zeta}\right)\left(\frac{(Kv-\tau)aqM_1}{N\zeta}\right)^{1/2}\left(\frac{(Kv-\tau)aqM_1}{2\pi eN\zeta}\right)^{i(Kv-\tau)} + O\left(\min\left\{|Kv-\tau|^{-3/2}, \left(\frac{N\zeta}{qQM_1}\right)^{-3/2}\right\}\right).$$

Plugging the above asymptotic expansions into (4.4) we obtain

$$(5.1)\quad \begin{aligned}\mathcal{J}^{**}(q,m,\tau) = c_1 M_2 \left(\frac{aqM_1}{N}\right)^{3/2} \\ \times \int_{\mathbb{R}}\int_0^1 V(v)\frac{\sqrt{t+Kv}}{\zeta^{1/2}(\zeta M_2 - ma)} U^*\left(\frac{(t+Kv)aqM}{2\pi N(\zeta M_2 - ma)}\right) \\ \times \left(\frac{(t+Kv)aqM}{2\pi e N(\zeta M_2 - ma)}\right)^{-i(t+Kv)} V\left(\frac{(Kv-\tau)aqM_1}{2\pi N\zeta}\right) \\ \times \left(\frac{(Kv-\tau)aqM_1}{2\pi e N\zeta}\right)^{i(Kv-\tau)} \mathrm{d}v\,\mathrm{d}\zeta + O(t^{-5/2} + E^{**})\end{aligned}$$

for some absolute constant c_1, where

$$E^{**} = \frac{1}{\sqrt{t}}\int_0^1\int_1^2 \min\left\{|Kv-\tau|^{-3/2}, \left(\frac{N\zeta}{qQM_1}\right)^{-3/2}\right\} \mathrm{d}v\,\mathrm{d}\zeta.$$

To estimate the error term E^{**}, we split the integral over v into two pieces: $|Kv-\tau| < N\zeta/aqM_1$ and $|Kv-\tau| \geq N\zeta/aqM_1$ as in [**14**] to get

$$E^{**} \ll \frac{(Mt)^{\varepsilon}}{t^{1/2}K^{3/2}} \min\left\{1, \frac{10K}{|\tau|}\right\}.$$

We also note that, by our choice K in (1.5) and $|\tau| \leq (Mt)^{\varepsilon} C^{-1}\sqrt{NK/M_1}$, we have

$$t^{-5/2} \ll \frac{(Mt)^{\varepsilon}}{t^{1/2}K^{3/2}} \min\left\{1, \frac{10K}{|\tau|}\right\}.$$

5.2. Stationary phase expansion for the v-integral. Now we will study the integral over v in (5.1). Note that the weight function restricts the v-integral to a range of length $(Mt)^{\varepsilon}N\zeta/aqKM_1$. Thus, for $\zeta < K^{-1}$ we can bound the integral over v trivially to get the bound $(Mt)^{\varepsilon}t^{-1/2}K^{-5/2}(N/aqM_1)^{1/2}$. Denote by $\mathcal{I}^{**}(q,m,\tau)$ the integral in (5.1). Then

$$(5.2)\quad \begin{aligned}\mathcal{I}^{**}(q,m,\tau) = c_1\left(\frac{aqM_1}{Nt}\right)^{1/2}\int_{K^{-1}}^1\int_{\mathbb{R}} g(v)e(f(v))\,\mathrm{d}v\frac{\mathrm{d}\zeta}{\sqrt{\zeta}} \\ + O\left(\frac{(Mt)^{\varepsilon}}{t^{1/2}K^{5/2}}\left(\frac{N}{qQM_1}\right)^{1/2}\right),\end{aligned}$$

where

$$g(v) = \frac{aqM\sqrt{t(t+Kv)}}{N(\zeta M_2 - ma)} U^*\left(\frac{(t+Kv)aqM}{2\pi N(\zeta M_2 - ma)}\right) V\left(\frac{(Kv-\tau)aqM_1}{2\pi N\zeta}\right) V(v)$$

and

$$f(v) = -\frac{t+Kv}{2\pi}\log\frac{(t+Kv)aqM}{2\pi e N(\zeta M_2 - ma)} + \frac{Kv-\tau}{2\pi}\log\frac{(Kv-\tau)aqM_1}{2\pi e N\zeta}.$$

By explicit computations,

$$f'(v) = \frac{K}{2\pi} \log \frac{(Kv-\tau)(\zeta M_2 - ma)}{(t+Kv)\zeta M_2},$$

and for $j \geq 2$,

$$f^{(j)}(v) = \frac{(-1)^j (j-2)!}{2\pi} \left(\frac{K^j}{(Kv-\tau)^{j-1}} - \frac{K^j}{(Kv+t)^{j-1}} \right).$$

The stationary phase is given by

$$v_0 = \frac{(t+\tau)M_2\zeta - \tau ma}{-Kma}.$$

In the support of the integral, we have

$$g^{(j)}(v) \ll \left(1 + \frac{aqKM_1}{N\zeta}\right)^j, \quad j \geq 0,$$

and by the range of K,

$$f^{(j)}(v) \asymp \frac{N\zeta}{aqM_1} \left(\frac{aqKM_1}{N\zeta}\right)^j, \quad j \geq 2.$$

Moreover, if $v_0 \notin [0.5, 3]$, then in the support of the integral we also have

$$\begin{aligned} f'(v) &= \frac{K}{2\pi} \log\left(1 + \frac{K(v_0 - v)}{t + Kv}\right) - \frac{K}{2\pi} \log\left(1 + \frac{K(v_0 - v)}{Kv - \tau}\right) \\ &\asymp K \log\left(1 + \frac{K(v_0 - v)}{Kv - \tau}\right) \gg K \min\left\{1, \frac{aqKM_1}{N\zeta}\right\}. \end{aligned}$$

According to the lower bound of $f'(v)$, we distinguish two cases.

Case a. $N\zeta/aqKM_1 \geq 1$. If $v_0 \notin [0.5, 3]$, then the length of the integral is $b - a = 1$. Applying Lemma 2(1) with

$$\Theta_f = \frac{N\zeta}{aqM_1}, \quad \Omega_f = \frac{N\zeta}{aqKM_1}, \quad \Omega_g = 1, \quad \text{and} \quad \Lambda = \frac{aqK^2M_1}{N\zeta},$$

we obtain

$$\int_{\mathbb{R}} g(v) e(f(v))\, \mathrm{d}v \ll \frac{1}{K^2} \left(\frac{N}{qQKM_1}\right)^3.$$

If $v_0 \in [0.5, 3]$, then treating the integral as a finite integral over the range $[0.1, 4]$ and applying Lemma 2(2), it follows that

$$\int_{\mathbb{R}} g(v) e(f(v))\, \mathrm{d}v = \frac{g(v_0) e(f(v_0) + 1/8)}{\sqrt{f''(v_0)}} + O\left(\left(\frac{N}{qQK^2M_1}\right)^{3/2}\right).$$

Thus, for K as in (1.5), we have

$$
\begin{aligned}
&\left(\frac{aqM_1}{Nt}\right)^{1/2}\int_{K^{-1}}^{1} 1_{\frac{N\zeta}{aqKM_1}\geq 1}\int_{\mathbb{R}} g(v)e(f(v))\,\mathrm{d}v\frac{\mathrm{d}\zeta}{\sqrt{\zeta}}\\
&\qquad = \left(\frac{aqM_1}{Nt}\right)^{1/2}\int_{K^{-1}}^{1} 1_{\frac{N\zeta}{aqKM_1}\geq 1}\frac{g(v_0)e(f(v_0)+1/8)}{\sqrt{f''(v_0)}}\frac{\mathrm{d}\zeta}{\sqrt{\zeta}}\\
&\qquad\qquad + O\left(\frac{N}{qQK^3M_1\sqrt{t}}\right),
\end{aligned}
\tag{5.3}
$$

where 1_S denotes the characteristic function of the set S.

Case b. $N\zeta/aqKM_1 < 1$. In this case $[a,b] = [\tau/K - 2\pi N\zeta/aqKM_1, \tau/K + 4\pi N\zeta/aqKM_1]$ and we apply Lemma 2 with

$$
\Theta_f = \frac{N\zeta}{aqM_1},\quad \Omega_f = \frac{N\zeta}{aqKM_1},\quad \Omega_g = \frac{N\zeta}{aqKM_1},\quad \text{and}\quad \Lambda = K.
$$

If $v_0 \notin [a,b]$, then

$$
\int_{\mathbb{R}} g(v)e(f(v))\,\mathrm{d}v \ll \frac{1}{K^2\Omega_f}.
$$

If $v_0 \in [a,b]$, treating the integral as a finite integral over $[\tau/K - 3\pi N\zeta/aqKM_1, \tau/K + 5\pi N\zeta/aqKM_1]$, then

$$
\int_{\mathbb{R}} g(v)e(f(v))\,\mathrm{d}v = \frac{g(v_0)e(f(v_0)+1/8)}{\sqrt{f''(v_0)}} + O\left(\frac{1}{K^2\Omega_f} + \frac{1}{K^{3/2}\Omega_f^{1/2}}\right).
$$

Recall that $\zeta > K^{-1}$. We have $\Omega_f > K^{-1}$ and the O-term above is at most $K^{-1}\sqrt{aqM_1/N\zeta}$. Thus

$$
\begin{aligned}
&\left(\frac{aqM_1}{Nt}\right)^{\frac{1}{2}}\int_{K^{-1}}^{1} 1_{\frac{N\zeta}{aqKM_1}< 1}\int_{\mathbb{R}} g(v)e(f(v))\,\mathrm{d}v\frac{\mathrm{d}\zeta}{\sqrt{\zeta}}\\
&= \left(\frac{aqM_1}{Nt}\right)^{1/2}\int_{K^{-1}}^{1} 1_{\frac{N\zeta}{aqKM_1}< 1}\frac{g(v_0)e(f(v_0)+1/8)}{\sqrt{f''(v_0)}}\frac{\mathrm{d}\zeta}{\sqrt{\zeta}} + O\left(\frac{qQM_1}{KN\sqrt{t}}\right).
\end{aligned}
\tag{5.4}
$$

Note that the O-terms in (5.2) and (5.4) are dominated by the O-term in (5.3). By (5.2)–(5.4) we obtain

$$
\begin{aligned}
\mathcal{I}^{**}(q,m,\tau) = c_1\left(\frac{aqM_1}{Nt}\right)^{1/2}\int_{K^{-1}}^{1} \frac{g(v_0)e(f(v_0)+1/8)}{\sqrt{f''(v_0)}}\frac{\mathrm{d}\zeta}{\sqrt{\zeta}}\\
+ O\left(\frac{N}{qQK^3M_1\sqrt{t}}\right).
\end{aligned}
\tag{5.5}
$$

Finally, we compute the main term. We have

$$f(v_0) = -\frac{t+\tau}{2\pi}\log\left(\frac{-(t+\tau)qM}{2\pi eNm}\right), \quad f''(v_0) = \frac{(Kma)^2}{2\pi(t+\tau)(\zeta M_2 - ma)\zeta M_2}$$

and

$$g(v_0) = \frac{aqM}{N}\left(\frac{-t(t+\tau)}{ma(\zeta M_2 - ma)}\right)^{1/2} V\left(\frac{(t+\tau)qM}{-2\pi Nm}\right) \times U^*\left(\frac{(t+\tau)qM}{-2\pi Nm}\right) V\left(\frac{\tau}{K} - \frac{(t+\tau)M_2\zeta}{Kma}\right).$$

Plugging these into (5.5) we have

$$\mathcal{I}^{**}(q,m,\tau) = c_2\frac{t+\tau}{K}\left(\frac{qM}{-mN}\right)^{3/2} V\left(\frac{(t+\tau)qM}{-2\pi Nm}\right) U^*\left(\frac{(t+\tau)qM}{-2\pi Nm}\right) \times \left(-\frac{(t+\tau)qM}{2\pi eNm}\right)^{-i(t+\tau)} \int_{K^{-1}}^{1} V\left(\frac{\tau}{K} - \frac{(t+\tau)M_2\zeta}{Kma}\right) \mathrm{d}\zeta + O\left(\frac{N}{qQK^3M_1\sqrt{t}}\right)$$

for some absolute constant c_2. Extending the integral to the interval $[0,1]$ at a cost of an error term dominated by the O-term in (5.1), we conclude the following:

Lemma 9. *We have*

$$\mathcal{J}^{**}(q,m,\tau) = \mathcal{J}_1(q,m,\tau) + \mathcal{J}_2(q,m,\tau),$$

where

$$\mathcal{J}_1(q,m,\tau) = \frac{c_3}{K\sqrt{t+\tau}}\left(-\frac{(t+\tau)qM}{2\pi eNm}\right)^{3/2-i(t+\tau)} V\left(\frac{(t+\tau)qM}{-2\pi Nm}\right) \times U^*\left(\frac{(t+\tau)qM}{-2\pi Nm}\right)\int_0^1 V\left(\frac{\tau}{K} - \frac{(t+\tau)M_2\zeta}{Kma}\right)\mathrm{d}\zeta, \tag{5.6}$$

and

$$\mathcal{J}_2(q,m,\tau) = \mathcal{J}^{**}(q,m,\tau) - \mathcal{J}_1(q,m,\tau) = O(\mathcal{B}(C,\tau)(Mt)^{\varepsilon}), \tag{5.7}$$

where

$$\mathcal{B}(C,\tau) = \frac{1}{t^{1/2}K^{3/2}}\min\left\{1, \frac{10K}{|\tau|}\right\} + \frac{N^{1/2}}{t^{1/2}K^{5/2}M_1^{1/2}C}. \tag{5.8}$$

6. Estimating $\mathcal{S}^\sharp(N)$-II

Denote by $\mathcal{J}_{\ell,J,\pm}(q,m,n_1^2n_2)$ and $\mathcal{S}_{1,\ell}(N,C,L,J,\pm)$ the contribution of $\mathcal{J}_\ell(q,m,\tau)$ to $\mathcal{J}^*_{J,\pm}(q,m,n_1^2n_2)$ in (4.3) and $\mathcal{S}_1^*(N,C,L,J,\pm)$ in (4.5), respectively.

6.1. Estimating $\mathcal{S}_{1,1}(N,C,L,J,\pm)$. By the Cauchy inequality and the Rankin–Selberg estimate in (2.1), $\mathcal{S}_{1,1}(N,C,L,J,\pm)$ is bounded by

$$\begin{aligned}&\frac{N^{3/2}}{\sqrt{MM_1^3}}\sum_{0\le k\le \log C}\sum\sum_{n_1^2n_2\sim L}\frac{|\lambda(n_2,n_1)|}{\sqrt{n_2}}\Bigg|\sum_{\substack{q=q_0M_2^k\sim C\\(q_0,M)=1\\n_1|q}}\frac{\chi_2(q_0)}{q^{3/2}}\sum_{\substack{Q<a\le q+Q\\(a,q)=1}}\frac{1}{a}\\
&\times\sum_{\substack{1\le|m|\le q(Mt)^{1+\varepsilon}/N\\m\equiv M_2\overline{a}\bmod q}}\overline{\chi_2}(m_0)\mathscr{B}(n_1,\pm n_2,m,a,q)\mathcal{J}_{1,J,\pm}(q,m,n_1^2n_2)\Bigg| \qquad (6.1)\\
&\le\sqrt{\frac{N^3L}{M_1^3M}}\sum_{0\le k\le\log C}\sqrt{\mathcal{T}(k)},\end{aligned}$$

where, temporarily,

$$\begin{aligned}\mathcal{T}(k)=&\sum_{n_1}\sum_{n_2}\frac{1}{n_2}W\left(\frac{n_1^2n_2}{L}\right)\Bigg|\sum_{\substack{q=q_0M_2^k\sim C\\(q_0,M)=1\\n_1|q}}\frac{\chi_2(q_0)}{q^{3/2}}\sum_{\substack{Q<a\le q+Q\\(a,q)=1}}\frac{1}{a}\\
&\times\sum_{\substack{1\le|m|\le q(Mt)^{1+\varepsilon}/N\\m\equiv M_2\overline{a}\bmod q}}\overline{\chi_2}(m_0)\mathscr{B}(n_1,\pm n_2,m,a,q)\mathcal{J}_{1,J,\pm}(q,m,n_1^2n_2)\Bigg|^2\end{aligned}$$

with m_0 defined in Lemma 6 and W a smooth function supported on $[1/2,3]$, which equals 1 on $[1,2]$ and satisfies $W^{(\ell)}(x)\ll_\ell 1$. Opening the absolute square and interchanging the order of summations we get

$$\begin{aligned}\mathcal{T}(k)=&\sum_{n_1\le\sqrt{3L}}\sum_{\substack{q=q_0M_2^k\sim C\\(q_0,M)=1\\n_1|q}}\frac{\chi_2(q_0)}{q^{3/2}}\sum_{\substack{Q<a\le q+Q\\(a,q)=1}}\frac{1}{a}\sum_{\substack{1\le|m|\le q(Mt)^{1+\varepsilon}/N\\m\equiv M_2\overline{a}\bmod q}}\overline{\chi_2}(m_0)\\
&\times\sum_{\substack{q'=q_0'M_2^k\sim C\\(q_0',M)=1\\n_1|q'}}\frac{\overline{\chi_2}(q_0')}{q'^{3/2}}\sum_{\substack{Q<a'\le q'+Q\\(a',q')=1}}\frac{1}{a'}\sum_{\substack{1\le|m'|\le q'(Mt)^{1+\varepsilon}/N\\m'\equiv M_2\overline{a'}\bmod q'}}\chi_2(m_0')T^*, \qquad (6.2)\end{aligned}$$

where

$$\begin{aligned}T^*=&\sum_{n_2}\frac{1}{n_2}W\left(\frac{n_1^2n_2}{L}\right)\mathcal{J}_{1,J,\pm}(q,m,n_1^2n_2)\overline{\mathcal{J}_{1,J,\pm}(q',m',n_1^2n_2)}\\
&\times\mathscr{B}(n_1,\pm n_2,m,a,q)\overline{\mathscr{B}(n_1,\pm n_2,m',a',q')}.\end{aligned}$$

Denote $\widehat{q} = q/n_1$. Then $\mathscr{B}(n_1, n_2, m, a, q)$ in (4.2) is

$$\mathscr{B}(n_1, n_2, m, a, q) = \chi_1(q) S(a\overline{M_1}, n_2\overline{M_1}; \widehat{q}) \sideset{}{^*}\sum_{b \bmod M_1} \overline{\chi_1}(m\overline{M_2} - b) S(\overline{b\widehat{q}}, n_2\overline{\widehat{q}}; M_1).$$

Applying Poisson summation formula with modulus $\widehat{q}\widehat{q'}M_1$ we obtain

$$T^* = \frac{n_1^2}{qq'M_1} \sum_{n_2 \in \mathbb{Z}} \mathscr{C}^*(n_2) \mathcal{I}^*(n_2), \tag{6.3}$$

where

$$\mathscr{C}^*(n_2) = \sum_{c \bmod \widehat{q}\widehat{q'}M_1} \mathscr{B}(n_1, c, m, a, q) \overline{\mathscr{B}(n_1, c, m', a', q')} e\left(\frac{n_2 c}{\widehat{q}\widehat{q'}M_1}\right) \tag{6.4}$$

and

$$\mathcal{I}^*(n_2) = \int_{\mathbb{R}} W(y) \mathcal{J}_{1,J,\pm}(q, m, Ly) \overline{\mathcal{J}_{1,J,\pm}(q', m', Ly)} e\left(-\frac{n_2 L y}{qq'M_1}\right) \frac{\mathrm{d}y}{y}. \tag{6.5}$$

Lemma 10. *We have $\mathcal{I}^*(n_2)$ is arbitrarily small unless*

$$|n_2| \leq (Mt)^{\varepsilon} C\sqrt{NKM_1}/L \quad \text{and} \quad \mathcal{I}^*(n_2) \ll (Mt)^{\varepsilon} B^*(n_2),$$

where $B^*(n_2)$ is given by

$$B^*(n_2) = \begin{cases} \frac{N^{1/2}}{tK^{3/2}M_1^{1/2}C} & \text{if } n_2 = 0, \\ \frac{N^{1/2}}{tK^{3/2}(|n_2|L)^{1/2}} & \text{if } n_2 \neq 0. \end{cases}$$

The following estimate for the character sum $\mathscr{C}^*(n_2)$ was proved in [**14**] by using Deligne's bound.

Lemma 11. *For $n_2 \neq 0$ we have*

$$\mathscr{C}^*(n_2) \ll \widehat{q}\widehat{q'}(\widehat{q}, \widehat{q'}, n_2) M_1^{5/2} (M_1, n_2, m\widehat{q}^2 - m'\widehat{q'}^2)^{1/2},$$

and for $n_2 = 0$ the sum vanishes unless $\widehat{q} = \widehat{q'}$ (i.e., $q = q'$) in which case

$$\mathscr{C}^*(0) \ll \widehat{q}^2 R_{\widehat{q}}(a - a') M_1^{5/2} (M_1, m - m')^{1/2},$$

where $R_c(u) = \sideset{}{^*}\sum_{\gamma \bmod c} e(u\gamma/c)$ is the Ramanujan sum.

By (6.2), (6.3), and Lemma 10, we have, up to an arbitrarily small error term,

$$\mathcal{T}(k) \ll \frac{(Mt)^\varepsilon}{M_1C^5} \sum_{n_1\le\sqrt{3L}} n_1^2 \sum_{\substack{q=q_0M_2^k\sim C\\(q_0,M)=1\\n_1|q}} \sum_{\substack{Q<a\le q+Q\\(a,q)=1}} \frac{1}{a}$$

$$\times \sum_{\substack{1\le|m|\le q(Mt)^{1+\varepsilon}/N\\m\equiv M_2\overline{a} \bmod q}} \sum_{\substack{q'=q_0'M_2^k\sim C\\(q_0',M)=1\\n_1|q'}} \sum_{\substack{Q<a'\le q'+Q\\(a',q')=1}} \frac{1}{a'}$$

$$\times \sum_{\substack{1\le|m'|\le q'(Mt)^{1+\varepsilon}/N\\m'\equiv M_2\overline{a'} \bmod q'}} \sum_{|n_2|\le(Mt)^\varepsilon C\sqrt{NKM_1}/L} |\mathscr{C}^*(n_2)|B^*(n_2).$$

Note that, for $(q,M_2)=1$ the condition $m\equiv M_2\overline{a} \bmod q$ implies that $a\equiv\overline{m}M_2 \bmod q$. By Lemmas 10 and 11, the contribution from $k=0$ is

(6.6)

$$\frac{(Mt)^\varepsilon}{M_1C^5} \sum_{n_1\le\sqrt{3L}} n_1^2 \sum_{\substack{q\sim C\\(q,M)=1\\n_1|q}} \sum_{1\le|m|\le q(Mt)^{1+\varepsilon}/N} \sum_{\substack{Q<a\le q+Q\\a\equiv M_2\overline{m} \bmod 1}} \frac{1}{a}$$

$$\times \sum_{\substack{q'\sim C\\(q',M)=1\\n_1|q'}} \sum_{1\le|m'|\le q'(Mt)^{1+\varepsilon}/N} \sum_{\substack{Q<a'\le q'+Q\\a'\equiv M_2\overline{m'} \bmod q'}} \frac{1}{a'} \sum_{|n_2|\le(Mt)^\varepsilon C\sqrt{NKM_1}/L} |\mathscr{C}^*(n_2)|B^*(n_2)$$

$$\ll \frac{(Mt)^\varepsilon}{Q^2M_1C^5}\frac{N^{1/2}}{tK^{3/2}M_1^{1/2}C} \sum_{n_1\le\sqrt{3L}} n_1^2 \sum_{\substack{q\sim C\\(q,M)=1\\n_1|q}} \sum_{1\le|m|\le q(Mt)^{1+\varepsilon}/N}$$

$$\times \sum_{1\le|m'|\le q'(Mt)^{1+\varepsilon}/N} \widehat{q}^2(m-m',\widehat{q})M_1^{5/2}(M_1,m-m')^{1/2}$$

$$+ \frac{(Mt)^\varepsilon}{Q^2M_1C^5} \sum_{n_1\le\sqrt{3L}} n_1^2 \sum_{\substack{q\sim C\\(q,M)=1\\n_1|q}} \sum_{\substack{q'\sim C\\(q',M)=1\\n_1|q'}} \sum_{1\le|m|\le q(Mt)^{1+\varepsilon}/N} \sum_{1\le|m'|\le q'(Mt)^{1+\varepsilon}/N}$$

$$\times \sum_{1\le|n_2|\le(Mt)^\varepsilon C\sqrt{NKM_1}/L} \widehat{q}\widehat{q'}(\widehat{q},\widehat{q'},n_2)M_1^{5/2}(M_1,n_2)^{1/2}\frac{N^{1/2}}{tK^{3/2}(|n_2|L)^{1/2}}$$

$$\ll \frac{M_1^{5/2}M^2t}{N^{5/2}K^{1/2}} + \frac{M_1^2M^2t}{N^{3/2}KL}.$$

Note that, for $k \geq 1$ the condition $m \equiv M_2\overline{a} \bmod q$ implies that $M_2|m$ and $a \equiv \overline{(m/M_2)} \bmod q/M_2$. Thus

$$\sum_{\substack{Q<a\leq q+Q\\ m\equiv M_2\overline{a} \bmod q}} \frac{1}{a} = \sum_{i=0}^{M_2-1} \sum_{\substack{Q+iq/M_2<a\leq Q+(i+1)q/M_2\\ a\equiv\overline{(m/M_2)} \bmod q/M_2}} \frac{1}{a} = \sum_{i=0}^{M_2-1} \frac{1}{a_i(m,q)} \asymp \frac{M_2}{Q},$$

where $a_i(m,q)$ is the unique solution of $a \equiv \overline{(m/M_2)} \bmod q/M_2$ in $Q + iq/M_2 < a \leq Q + (i+1)q/M_2$. Bounding similarly as in the case $k = 0$, one sees that the contribution from $k \neq 0$ is dominated by (6.6). Therefore

$$\mathcal{T}(k) \ll \frac{M_1^{5/2}M^2t}{N^{5/2}K^{1/2}} + \frac{M_1^2M^2t}{N^{3/2}KL},$$

and by (6.1) (also recall that $L \leq N^{1/2+\varepsilon}K^{3/2}M_1^{3/2}$),

$$\begin{aligned}\mathcal{S}_{1,1}(N,C,L,J,\pm) &\ll \sqrt{\frac{N^3L}{M_1^3M}}\left(\frac{M_1^{5/4}M\sqrt{t}}{N^{5/4}K^{1/4}} + \frac{M_1M\sqrt{t}}{N^{3/4}\sqrt{KL}}\right)\\ &\ll (Mt)^{\varepsilon}N^{3/4}(Mt)^{1/2}\left(\frac{M_1^{1/2}K^{1/2}}{N^{1/4}} + \frac{1}{M_1^{1/2}K^{1/2}}\right).\end{aligned} \tag{6.7}$$

6.2. Bounding $\mathcal{S}_{1,2}(N,C,L,J,\pm)$. Applying the Cauchy inequality and (2.1), we have

$$\begin{aligned}\mathcal{S}_{1,2}(N,C,L,J,\pm) &\ll \sqrt{\frac{N^3L}{M_1^3M}}\\ &\quad\times \sum_{0\leq k\leq \log C} \int_{|\tau|\leq (Mt)^{\varepsilon}C^{-1}\sqrt{NK/M_1}} \sqrt{\mathcal{R}(k,\tau)}\,\mathrm{d}\tau,\end{aligned} \tag{6.8}$$

where, temporarily,

$$\begin{aligned}\mathcal{R}(k,\tau) = \sum_{n_1}\sum_{n_2}\frac{1}{n_2}W\left(\frac{n_1^2n_2}{L}\right)\Bigg|\sum_{\substack{q=q_0M_2^k\sim C\\ (q_0,M)=1\\ n_1|q}} \frac{\chi_2(q_0)}{q^{3/2}} \sum_{\substack{Q<a\leq q+Q\\ (a,q)=1}} \frac{1}{a}\\ \times \sum_{\substack{1\leq|m|\leq q(Mt)^{1+\varepsilon}/N\\ m\equiv M_2\overline{a} \bmod q}} \overline{\chi_2}(m_0)\mathscr{B}(n_1,\pm n_2,m,a,q)\mathcal{J}_2(q,m,\tau)\Bigg|^2.\end{aligned}$$

As before, we open the absolute square and interchange the order of summations to get

$$\mathcal{R}(k,\tau) = \sum_{n_1 \le \sqrt{3L}} \sum_{\substack{q=q_0 M_2^k \sim C \\ (q_0,M)=1 \\ n_1 | q}} \frac{\chi_2(q_0)}{q^{3/2}} \sum_{\substack{Q<a\le q+Q \\ (a,q)=1}} \frac{1}{a}$$
$$\times \sum_{\substack{1\le |m| \le q(Mt)^{1+\varepsilon}/N \\ m\equiv M_2 \overline{a} \bmod q}} \overline{\chi_2}(m_0) \mathcal{J}_2(q,m,\tau)$$
$$\times \sum_{\substack{q'=q_0' M_2^k \sim C \\ (q_0',M)=1 \\ n_1 | q'}} \frac{\overline{\chi_2}(q_0')}{q'^{3/2}} \sum_{\substack{Q<a'\le q'+Q \\ (a',q')=1}} \frac{1}{a'}$$
$$\times \sum_{\substack{1\le |m'| \le q'(Mt)^{1+\varepsilon}/N \\ m'\equiv M_2 \overline{a'} \bmod q'}} \chi_2(m_0') \overline{\mathcal{J}_2(q',m',\tau)} R^*,$$

where

$$R^* = \sum_{n_2} \frac{1}{n_2} W\left(\frac{n_1^2 n_2}{L}\right) \mathscr{B}(n_1, \pm n_2, m, a, q) \overline{\mathscr{B}(n_1, \pm n_2, m', a', q')}.$$

Applying Poisson summation with modulus $\widehat{q}\widehat{q'}M_1$, we obtain

$$R^* = \frac{n_1^2}{qq'M_1} \sum_{n_2\in\mathbb{Z}} \mathscr{C}^*(n_2) W^\dagger\left(\frac{n_2 L}{qq'M_1}, 0\right),$$

where $\mathscr{C}^*(n_2)$ is defined in (6.4). By (2.6), the integral is arbitrarily small if $|n_2| \gg (Mt)^\varepsilon C^2 M_1/L$. By (5.7),

$$\mathcal{R}(k,\tau) \ll (Mt)^\varepsilon \frac{\mathcal{B}(C,\tau)^2}{M_1 C^5} \sum_{n_1\le 2C} n_1^2 \sum_{\substack{q=q_0 M_2^k \sim C \\ (q_0,M)=1 \\ n_1 | q}} \sum_{\substack{Q<a\le q+Q \\ (a,q)=1}} \frac{1}{a}$$
$$\times \sum_{\substack{1\le |m| \le q(Mt)^{1+\varepsilon}/N \\ m\equiv M_2 \overline{a} \bmod q}} \sum_{\substack{q'=q_0' M_2^k \sim C \\ (q_0',M)=1 \\ n_1 | q'}} \sum_{\substack{Q<a'\le q'+Q \\ (a',q')=1}} \frac{1}{a'}$$
$$\times \sum_{\substack{1\le |m'| \le q'(Mt)^{1+\varepsilon}/N \\ m'\equiv M_2 \overline{a'} \bmod q'}} \sum_{|n_2|\le (Mt)^\varepsilon C^2 M_1/L} |\mathscr{C}^*(n_2)|,$$

where $\mathcal{B}(C,\tau)$ is defined in (5.8). By Lemmas 10 and 11, we have

$$R(0,\tau)\ll(Mt)^{\varepsilon}\frac{\mathcal{B}(C,\tau)^2}{M_1Q^2C^5}\sum_{n_1\leq 2C}n_1^2\sum_{\substack{q\sim C\\(q,M)=1\\n_1|q}}\sum_{1\leq|m|\leq C(Mt)^{1+\varepsilon}/N}$$

$$\times\sum_{1\leq|m'|\leq C(Mt)^{1+\varepsilon}/N}\widehat{q}^2(\widehat{q},m-m')M_1^{5/2}(M_1,m-m')^{1/2}$$

$$+(Mt)^{\varepsilon}\frac{\mathcal{B}(C,\tau)^2}{M_1Q^2C^5}\sum_{n_1\leq 2C}n_1^2\sum_{\substack{q\sim C\\(q,M)=1[1pt]\ n_1|q}}\sum_{\substack{q'\sim C\\(q',M)=1\\n_1|q'}}\sum_{1\leq|m|\leq C(Mt)^{1+\varepsilon}/N}\tag{6.9}$$

$$\times\sum_{1\leq|m'|\leq C(Mt)^{1+\varepsilon}/N}\ \sum_{1\leq|n_2|\leq(Mt)^{\varepsilon}C^2M_1/L}\widehat{q}\widehat{q'}(\widehat{q},\widehat{q}',n_2)M_1^{5/2}(M_1,n_2)^{1/2}$$

$$\ll(Mt)^{\varepsilon}\mathcal{B}(C,\tau)^2\left(\frac{KM_1^3Mt}{N^2}+\frac{KC^3M_1^{7/2}(Mt)^2}{N^3L}\right),$$

and similarly the contribution from $k\neq 0$ is dominated by (6.9). Thus by (6.8),

$$\mathcal{S}_{1,2}(N,C,L,J,\pm)\ll\sqrt{\frac{N^3L}{M_1^3M}}\left(\frac{K^{1/2}M_1^{3/2}(Mt)^{1/2}}{N}+\frac{K^{1/2}C^{3/2}M_1^{7/4}Mt}{N^{3/2}L^{1/2}}\right)$$

$$\times\int_{|\tau|\leq(Mt)^{\varepsilon}C^{-1}\sqrt{NK/M_1}}\mathcal{B}(C,\tau)\,\mathrm{d}\tau,$$

where by (5.8)

$$\int_{|\tau|\leq(Mt)^{\varepsilon}C^{-1}\sqrt{NK/M_1}}\mathcal{B}(C,\tau)\,\mathrm{d}\tau\ll\frac{(Mt)^{\varepsilon}}{t^{1/2}K^{1/2}}\left(1+\frac{N}{C^2K^{3/2}M_1}\right).$$

Thus (note that $L\ll N^{1/2+\varepsilon}K^{3/2}M_1^{3/2}$ and $N/(Mt)^{1+\varepsilon}\leq C\leq\sqrt{N/KM_1}$)

$$\mathcal{S}_{1,2}(N,C,L,J,\pm)$$
$$\ll(Mt)^{\varepsilon}N^{3/4}\left(K^{3/4}M_1^{3/4}+\frac{(Mt)^2}{NK^{3/4}M_1^{1/4}}+\frac{(Mt)^{1/2}}{K^{3/4}M_1^{1/2}}+\frac{Mt}{N^{1/4}K^{3/2}M_1^{3/4}}\right),$$

where the second term dominates the last two terms by the range of M_1 and our choice of K in (1.5). Therefore

$$\mathcal{S}_{1,2}(N,C,L,J,\pm)\ll(Mt)^{\varepsilon}N^{3/4}\left(K^{3/4}M_1^{3/4}+\frac{(Mt)^2}{NK^{3/4}M_1^{1/4}}\right).\tag{6.10}$$

Under the assumptions $(Mt)^{6/5}/(NM_1)^{3/5} \le K \le (Mt)^2/NM_1$, we see that the bound in (6.10) can be controlled by (6.7). By (6.7), Lemmas 7 and 8 we conclude that

$$\mathcal{S}_1(N) \ll (Mt)^\varepsilon N^{3/4}(Mt)^{1/2}\left(\frac{M_1^{1/2}K^{1/2}}{N^{1/4}} + \frac{1}{M_1^{1/2}K^{1/2}}\right).$$

Then Proposition 2 follows in view of our choice of K in (1.5).

6.3. Proof of Lemma 10. We follow closely [**14**]. By (4.3) and (6.5), $\mathcal{I}^*(n_2)$ is

$$\begin{aligned}(6.11)\quad &\frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{NL}{q^3M_1^3}\right)^{-i\tau}\left(\frac{NL}{q'^3M_1^3}\right)^{i\tau'}\gamma_\pm\left(-\frac{1}{2}+i\tau\right)\overline{\gamma_\pm\left(-\frac{1}{2}+i\tau'\right)}\\ &\times\mathcal{J}_1(q,m,\tau)\overline{\mathcal{J}_1(q',m',\tau')}W_J(\tau)W_J(\tau')W^\dagger\left(\frac{n_2L}{qq'M_1},-i(\tau-\tau')\right)\mathrm{d}\tau\,\mathrm{d}\tau'.\end{aligned}$$

By (2.6), the integral $W^\dagger(n_2L/qq'M_1, -i(\tau-\tau'))$ is negligible if $|n_2| \ge (Mt)^\varepsilon C\sqrt{NKM_1}/L$. For smaller $|n_2|$, we plug (5.6) into (6.11) to get

$$\begin{aligned}\mathcal{I}^*(n_2) = &\frac{|c_3|^2}{4\pi^2K^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{NL}{q^3M_1^3}\right)^{-i\tau}\left(\frac{NL}{q'^3M_1^3}\right)^{i\tau'}\gamma_\pm\left(-\frac{1}{2}+i\tau\right)\\ &\times\overline{\gamma_\pm\left(-\frac{1}{2}+i\tau'\right)}\left(-\frac{(t+\tau)qM}{2\pi eNm}\right)^{-i(t+\tau)}\left(-\frac{(t+\tau')q'M}{2\pi eNm'}\right)^{i(t+\tau')}\\ &\times H_J(q,m,a,\tau)H_J(q',m',a',\tau')W^\dagger\left(\frac{n_2L}{qq'M_1},-i(\tau-\tau')\right)\mathrm{d}\tau\,\mathrm{d}\tau',\end{aligned}$$

where

$$\begin{aligned}H_J(q,m,a,\tau) = &\frac{1}{\sqrt{t+\tau}}\left(-\frac{(t+\tau)qM}{2\pi eNm}\right)^{3/2}V\left(\frac{(t+\tau)qM}{-2\pi Nm}\right)\\ &\times U^*\left(\frac{(t+\tau)qM}{-2\pi Nm}\right)W_J(\tau)\int_0^1 V\left(\frac{\tau}{K}-\frac{(t+\tau)M_2\zeta}{Kma}\right)\mathrm{d}\zeta\end{aligned}$$

satisfies the bound

$$H_J(q,m,a,\tau) \ll t^{-1/2}, \quad \frac{\partial}{\partial\tau}H_J(q,m,a,\tau) \ll \frac{(Mt)^\varepsilon}{t^{1/2}(1+|\tau|)}.$$

For $n_2 = 0$, by (2.6) we have $W^\dagger(0, -i(\tau-\tau'))$ is arbitrarily small if $|\tau-\tau'| \ge (Mt)^\varepsilon$. For $|\tau-\tau'| \le (Mt)^\varepsilon$, we have $W^\dagger(0, -i(\tau-\tau')) \ll 1$ and

$$\mathcal{I}^*(n_2) \ll (Mt)^\varepsilon\frac{N^{1/2}}{tK^{3/2}M_1^{1/2}C}.$$

For $n_2 \neq 0$ we apply (2.5) to get

$$W^\dagger\left(\frac{n_2 L}{qq'M_1}, -i(\tau-\tau')\right) = \frac{c_4}{\sqrt{\tau'-\tau}} W\left(\frac{(\tau'-\tau)qq'M_1}{2\pi n_2 L}\right)\left(\frac{(\tau'-\tau)qq'M_1}{2\pi e n_2 L}\right)^{i(\tau'-\tau)} + O\left(\min\left\{\frac{1}{|\tau'-\tau|^{3/2}}, \left(\frac{C^2 M_1}{|n_2|L}\right)^{3/2}\right\}\right)$$

for some absolute constant c_4. The contribution from the above O-term towards $\mathcal{I}^*(n_2)$ is bounded by

$$\frac{1}{K^2 t}\int_{|\tau|\leq 1+2|J|}\int_{|\tau'|\leq 1+2|J|} \min\left\{\frac{1}{|\tau'-\tau|^{3/2}}, \left(\frac{C^2 M_1}{|n_2|L}\right)^{3/2}\right\} \mathrm{d}\tau\,\mathrm{d}\tau' \ll (Mt)^\varepsilon \frac{N^{1/2}}{tK^{3/2}(|n_2|L)^{1/2}}.$$

For the main term, we write by Fourier inversion

$$\left(\frac{2\pi n_2 L}{(\tau'-\tau)qq'M_1}\right)^{1/2} W\left(\frac{(\tau'-\tau)qq'M_1}{2\pi n_2 L}\right) = \int_{\mathbb{R}} W^\dagger\left(r, \frac{1}{2}\right) e\left(\frac{(\tau'-\tau)qq'M_1}{2\pi n_2 L} r\right) \mathrm{d}r.$$

Then $\mathcal{I}^*(n_2)$ can be written as

$$\frac{c_5}{K^2}\left(\frac{qq'M_1}{|n_2|L}\right)^{1/2}\int_{\mathbb{R}} W^\dagger\left(r,\frac{1}{2}\right)\int_{\mathbb{R}}\int_{\mathbb{R}} \gamma_\pm\left(-\frac{1}{2}+i\tau\right)\overline{\gamma_\pm\left(-\frac{1}{2}+i\tau'\right)} H_J(q,m,a,\tau)$$
$$\times H_J(q',m',a',\tau')\left(\frac{NL}{q^3M_1^3}\right)^{-i\tau}\left(\frac{NL}{q'^3M_1^3}\right)^{i\tau'}$$
$$\times\left(-\frac{(t+\tau)qM}{2\pi e N m}\right)^{-i(t+\tau)}\left(-\frac{(t+\tau')q'M}{2\pi e N m'}\right)^{i(t+\tau')}$$
$$\times\left(\frac{(\tau'-\tau)qq'M_1}{2\pi e n_2 L}\right)^{i(\tau'-\tau)} e\left(\frac{(\tau'-\tau)qq'M_1}{2\pi n_2 L} r\right)\mathrm{d}\tau\,\mathrm{d}\tau'\,\mathrm{d}r + O((Mt)^\varepsilon B^*(n_2))$$

for some absolute constant c_5 where, for $n_2 \neq 0$,

$$B^*(n_2) = \frac{N^{1/2}}{tK^{3/2}(|n_2|L)^{1/2}}.$$

Note that, for $J=0$ we have trivially $\mathcal{I}^*(n_2) \ll N^{1/2}/tK^{5/2}(|n_2|L)^{1/2}$, which is dominated by $B^*(n_2)$. In the following, for notational simplicity we only consider the case of $J>0$. The same analysis holds for $J<0$.

By (2.3), we write

$$\begin{aligned}\mathcal{I}^*(n_2) = \frac{c_5}{K^2}\left(\frac{qq'M_1}{|n_2|L}\right)^{1/2} &\int_{\mathbb{R}} W^\dagger\left(r, \frac{1}{2}\right) \\ &\times \int_{\mathbb{R}}\int_{\mathbb{R}} g(\tau,\tau')e(f(\tau,\tau'))\,\mathrm{d}\tau\,\mathrm{d}\tau'\,\mathrm{d}r + O((Mt)^\varepsilon B^*(n_2)),\end{aligned} \tag{6.12}$$

where

$$g(\tau,\tau') = \Psi_\pm(\tau)\overline{\Psi_\pm(\tau')}H_J(q,m,a,\tau)H_J(q',m',a',\tau')$$

and

$$\begin{aligned}2\pi f(\tau,\tau') = 3\tau\log\left(\frac{\tau}{e\pi}\right) - 3\tau'\log\left(\frac{\tau'}{e\pi}\right) - \tau\log\left(\frac{NL}{q^3M_1^3}\right) + \tau'\log\left(\frac{NL}{q'^3M_1^3}\right) \\ - (t+\tau)\log\left(-\frac{(t+\tau)qM}{2\pi eNm}\right) + (t+\tau')\log\left(-\frac{(t+\tau')q'M}{2\pi eNm'}\right) \\ + (\tau'-\tau)\log\left(\frac{(\tau'-\tau)qq'M_1}{2\pi en_2L}\right) + \frac{(\tau'-\tau)qq'M_1\ell^2}{n_2L}r.\end{aligned}$$

For the double integral over τ, τ' in (6.12), Munshi [**14**] showed that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} g(\tau,\tau')e(f(\tau,\tau'))\,\mathrm{d}\tau\,\mathrm{d}\tau' \ll Jt^{-1+\varepsilon}.$$

Then using $W^\dagger\left(r, \frac{1}{2}\right) \ll_j |r|^{-j}$ we obtain

$$\mathcal{I}^*(n_2) \ll (Mt)^\varepsilon B^*(n_2).$$

This completes the proof of Lemma 10. □

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School of Mathematics and Statistics, Shandong University, Weihai, Weihai, Shandong 264209, China
E-mail address: qfsun@sdu.edu.cn

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