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POSITIVE NEIGHBORHOODS OF CURVES

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Abstract: In this work we study when neighborhoods of curves in holomorphic surfaces with positive self-intersection number can be embedded in the projective plane.

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Contents

1. Introduction

In this paper we study neighborhoods of compact, smooth, holomorphic curves of complex surfaces which have positive self intersection number. Our main purpose is to give a condition that guarantees the existence of an embedding of a neighborhood of the curve into the projective plane. The first example of a result on this problem comes from [**3**]. In that paper the authors showed that if the curve has genus 0 and self intersection number equal to 1, then the existence of three different fibrations over it implies that some neighborhood is diffeomorphic to a neighborhood of the line in the projective plane. In the present paper we consider curves of self-intersection n^2 with $n \geq 2$.

Since a fibration over a curve of genus 0 is defined by a local submersion over $\mathbb{P}^1$ (that is, defined in a neighborhood of the curve), we may wonder if in the case of higher genus the existence of a number of local submersions is enough to guarantee an embedding into the projective plane $\mathbb{P}^2$. This is in fact a necessary condition.

In order to discuss this, let us suppose that a curve C contained in some surface S can be embedded in $\mathbb{P}^2$ as a curve C_0 of degree $n \geq 2$ (we have of course to start with $C \cdot C = n^2$ in S). It is easy to find infinitely many submersions in a neighborhood of C_0. For example, we take two curves $\{A = 0\}$ and $\{B = 0\}$ of the same degree $l \in \mathbb{N}$ which cross each other in l^2 distinct points not in C_0. It can be seen that the map A/B, which is well defined outside $\{A = 0\} \cap \{B = 0\}$, has no multiple fibers so that it has only a finite number of critical points. If C_0 avoids all these points, then A/B is a submersion in some neighborhood of C_0 and the restriction of A/B to C_0 is a ramified map from C_0 to $\mathbb{P}^1$ of degree $l.n$. We will be particularly interested in the case $l = 1$, that is, $A = 0$ and $B = 0$ are lines whose common point is not in C_0; the submersion A/B will be called a *pencil submersion* and the restriction of A/B to C_0 is a ramified map of degree n (any local submersion that leaves such a trace in C_0 is in fact a pencil submersion). We see that to be equivalent to a neighborhood of C_0, a neighborhood of C has to carry also many submersions to $\mathbb{P}^1$. However, this is not enough to guarantee the embedding of a neighborhood of the curve into $\mathbb{P}^2$ as we can see in the following example.

Example 1.1. Consider the rational curve in $\mathbb{P}^2$ defined in affine coordinates by the equation $y^2 = x^2(x+1)$. It is a smooth rational curve except for the node at the point $(0, 0)$. We blow up first at a point in the curve different from $(0, 0)$, and then we blow up at $(0, 0)$. The strict transform is a smooth rational curve C of self intersection number equal to 4 with many local submersions (which come from submersions constructed in the plane as above), but its neighborhood can not be embedded in the plane: given a submersion constructed using $l = 1$ as above (before blow ups), we notice that it induces a ramified map from C to $\mathbb{P}^1$ of degree 3; but for a conic C_0 in the plane (which has of course self intersection number equal to 4), the ramified map induced by any local submersion is of even degree.

The surprising feature in [**3**], in case $n = 1$, is that only three submersions are needed. A natural question would be: can we obtain an embedding once it is assumed the existence of three local submersions in a neighborhood of C whose restrictions to C are meromorphic maps of degree n, or a multiple of n? We give a partial negative answer in Section 2.

We introduce then an extra condition (also a necessary one). A curve C that has an embedding $\phi\colon C \to C_0 \subset \mathbb{P}^2$ carries naturally a special set of meromorphic maps $\mathbf{G}_\phi = \{G|_{C_0} \circ \phi$, G a pencil submersion$\}$. A set $\{F_i\}$ of submersions defined in a neighborhood of C whose restrictions to C have no common critical points is *projective at* C if $F_i|_C \in \mathbf{G}_\phi$. The submersions are called *independant* if the singularities of the correspondent pencils on $\mathbb{P}^2$ are not aligned. We may state then our main result:

Theorem A. *The existence of a projective triple of independant submersions at C implies the existence of an embedding of a neighborhood of C into the projective plane.*

The submersions in the statement of the theorem are supposed to produce different fibrations; we remark that if F is a submersion over $\mathbb{P}^1$ and T is a Moebius transformation, then F and $T \circ F$ induce the same fibration.

Remark 1.2. The fibers of a submersion define a regular foliation in a neighborhood of C, which is generically transverse to C with tangency points at the critical points of the restriction to the curve. The submersion is a meromorphic first integral for the foliation. The converse does not hold, that is, this type of foliation may not have a first integral (see [**5**]).

We mention that the study of neighborhoods of curves has already been pursued when the self-intersection is not positive as we can see in [**4**], [**8**], and [**9**].

This paper is organized as follows: Section 2 presents some examples and it is followed by Section 3 where we discuss how to build meromophic maps starting from two different pencil submersions. This allows (Section 4) to show the existence of foliations defined in a neighborhood of the curve which have this curve as an invariant set and, finally, in Section 5 we prove our theorem.

2. Examples

This section has two parts. In the first part we give examples of surfaces containing smooth curves of self-intersection number n^2 which are not embeddable in the plane, although they are fibered by submersions whose restrictions to the curves are meromorphic functions of a degree multiple of n. Once this is done, we give examples which satisfy the extra condition of our Theorem A but have only one or two fibrations and do not embed them in the plane.

2.1. Separating branches and examples with 3 fibrations. We will use the following construction. Let us consider a curve H with an ordinary singularity P with m branches $L_1, \ldots, L_m$. For each branch L_j, we take a neighborhood V_j which is biholomorphic to a bidisc D_j by means of a biholomorphism $\phi_j \colon D_j \to V_j$. We assume that $\delta L_j \cap V_i = \emptyset$ for all $i \neq j$. We fix a neighborhood V of $H \setminus \cup_1^m L_j$. Finally, we take the disjoint union of V with all the D_j and glue D_j to V using the restriction of the map ϕ_j to $\phi_j^{-1}(V \cap V_j)$. In this way the union of the sets V_j, which contains P, is replaced by m copies of the bidisc, and there is a new curve H' replacing H inside a new surface without the ordinary singularity. As for the self-intersection number $H' \cdot H'$, we have that $H' \cdot H' = H \cdot H - m(m-1) = (H \cdot H - m^2) + m$. Also, any holomorphic foliation $\mathcal{F}$ defined in $V \cup_1^m V_j$ induces naturally a holomorphic foliation in the new surface which is $\mathcal{F}$ in V and $\phi_j^*(\mathcal{F})$ in each D_j. We refer to this construction as *separating branches of H at P*. This is the simplest way of desingularizing such a curve.

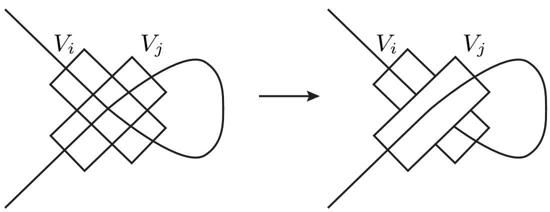

FIGURE 1. Separating branches.

Let us consider then a smooth plane curve C' of degree n' and genus $g(C') = \frac{(n'-1)(n'-2)}{2}$. It can be also immersed in the plane as a curve C of degree n for any $n > 2g(C')$ with a number s of nodal points such that $n^2 - 3n - 2s = n'^2 - 3n'$. We choose $n - n' = e^2$ for some $e \in \mathbb{N}$ such that

(i) n' divides $e^3 - e^2$,
(ii) n' does not divide $2e^2$;

we choose also three pencils $d\left(\frac{u_j}{v_j}\right) = 0$ of curves of degree e whose sets of e^2 base points lie in the regular part of C and are two by two disjoint. After blowing up at these $3e^2$ points and separating branches at the nodal points of C we get a curve $\tilde{C}$ contained in some surface with self-intersection number equal to $n^2 - 3e^2 - 2s = n'^2$. The maps $\frac{u_j}{v_j}$ become submersions whose restrictions to $\tilde{C}$ are meromorphic maps of degree $e.n - e^2$, which is a multiple of n' because of (i).

A neighborhood of $\tilde{C}$ is not equivalent to a neighborhood of the curve C' in the plane. In fact, let us take a linear pencil $\mathcal{L}$ in the plane with base point outside C and transverse to the branches at each nodal point (this is before blow ups and separation of branches). We have $2n = \mathrm{Tang}(\mathcal{L}, C) + \chi(C)$. Since $\mathrm{tang}(\mathcal{L}, C, P) = 2$ for each nodal point P, we get $2n = 2s + \chi(C) + \mathrm{tang}(\mathcal{L}, C)$, where the last term counts the tangencies with the regular part of C. These tangencies persist when we blow up and separate branches. Therefore, if a neighborhood of $\tilde{C}$ is equivalent to a neighborhood of C', we get in this neighborhood a foliation $\mathcal{L}'$ with $\mathrm{Tang}(\mathcal{L}', C') = \mathrm{tang}(\mathcal{L}, C) = 2n - 2s - \chi(C)$. It follows that $(\deg(\mathcal{L}') + 2)n' = 2n - 2s - \chi(C) + \chi(C') = 2n - 2s + 2g(C) - 2g(C')$ and since $g(C) - s = g(C')$, we conclude that $(\deg(\mathcal{L}') + 2)n' = 2n = 2n' + 2e^2$, a contradiction because of (ii).

We remark that when $n = 4$, $n' = 3$ or $n = 3$, $n' = 2$ the construction can be done with $e = 1$ because $n > 2g(C')$ (and obviously (i) is satisfied in both cases). When $n = 4$, $n' = 3$ we have also that (ii) holds true. In the general case $n' > 3$ we may choose $e = n' + 1$ for example in order to get both (i) and (ii) satisfied.

The special case $n = 3$, $n' = 2$ (and $e = 1$) can be treated with a small difference in what concerns the proof that the neighborhood of $\tilde{C}$ is not equivalent to a neighborhood of C: we select $\mathcal{L}$ as the pencil whose base point is the node point P of C. Since $\mathrm{Tang}(\mathcal{L}, C) = 6 = \mathrm{tang}(\mathcal{L}, C, P)$, we see that there is no other point of tangency between $\mathcal{L}$ and C. We get then $(\deg(\mathcal{L}') + 2).2 = 4 + 2$ (after separating the branches at P we obtain 2 radial singularities belonging to $\tilde{C}$ and a fortiori to C') and therefore $\deg(\mathcal{L}') = 1$. But it is impossible for a foliation of degree 1 in the plane to have 2 radial singularities.

It would be nice to have examples where the degree induced by the submersions on the curve is exactly n'.

2.2. Special examples. Given a meromorphic function of a curve, we can realize it as the restriction of a submersion defined in some surface containing the curve (we will say that the submersion is a *lifting* of the function). This is a construction already presented in [**7**] which we present here again. We start with a line bundle of Chern class $n^2 \in \mathbb{N}$ over a curve C. The lines of the bundle define a foliation $\mathcal{L}$ in the total space of the bundle. Let $f \colon C \to \mathbb{P}^1$ be a ramified map with simple critical points and let p be one of these points. There exists an involution i defined in a neighborhood of p in C by $f(q) = f(i(q))$ for q close to p.

We fix a neighborhood U of p and a holomorphic diffeomorphism $G \colon U \to \mathbb{D} \times \mathbb{D}$ such that: 1) $G(p) = (0, 0)$; 2) $G(C \cap U) = \{(z_1, 0) \in \mathbb{D} \times$

$\mathbb{D}\}$; 3) G takes $\mathcal{L}$ to the foliation $dz_1 = 0$, and 4) $g := G|_{C\cap U}$ conjugates i to the involution $z \mapsto -z$, that is, $g(i(q)) = -g(q)$. We take also a biholomorphism ψ from $\mathbb{D} \times \mathbb{D}$ to a neighborhood of $(0,0)$ with the properties: 1) $\psi(0,0) = (0,0)$; 2) $\psi(z_1, 0) = (z_1, 0)$; 3) $\psi(\{1/2 < |z_1| < 1\} \times \mathbb{D})$ is saturated by leaves of the foliation $dZ_2 - Z_1\, dZ_1 = 0$, and 4) ψ is a holomorphic diffeomorphism when restricted to $\{1/2 < |z_1| < 1\} \times \mathbb{D}$ that sends the foliation $dz_1 = 0$ to the foliation $dZ_2 - Z_1\, dZ_1 = 0$. Put $G_1 = \psi \circ G$.

We remove from the total space of the line bundle the fibers over the points of $g^{-1}(\{|z_1| \leq 1/2\})$ and glue $\psi(\mathbb{D} \times \mathbb{D})$ to the remaining set using G_1. In this way we get a new holomorphic surface which contains C (the same curve we started with) and a holomorphic foliation transverse to C except at p, where the tangency is simple. Furthermore, the “local holonomy” of the new foliation at p is exactly i. We repeat the same procedure for all critical points of f. At the end we have a holomorphic surface that contains C and a holomorphic foliation which is transverse to C except at the critical points of f. We may even assume that the self-intersection number of C is $n^2 \in \mathbb{N}$. The map f can be extended along the leaves (because of its compatibility with the involutions involved). This finishes the construction of the desired lifting. A similar construction can be made if the critical points are not simple.

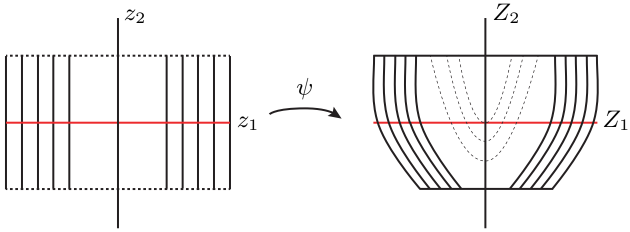

FIGURE 2. Glueing map.

Let us give two examples of pairs (curve, surface) which are not embeddable in the projective plane.

Example 2.1. We have already noticed that, in order to be embeddable in the projective plane, all the submersions defined in the neighborhood of the curve must have restriction maps whose degrees are multiple of n (here n^2 is the self-intersection number of the curve in the surface). This does not happen in the example given in the introduction. We give now another example of a different nature. Take $C \subset \mathbb{P}^2$. We start by claiming that there exists a ramified cover $f : C \to \mathbb{P}^1$ of degree $(n-1)n$ such that the set of poles is not contained in any curve of degree $n-1$. In order to see this, let us start with a ramified cover $f_0 : C \to \mathbb{P}^1$ defined as the restriction of $\frac{1}{Q_0}$ to C, where Q_0 is a polynomial of degree $n-1$ which intersects C transversely at $l=(n-1)n$ different points $P_1,\dots,P_l$. Let us consider nearby points $P'_1,\dots,P'_l$ and apply Riemann–Roch's Theorem to $D=P'_1+\cdots+P'_l$. Hence $l(D)\geq(n-1)n-g+1$. If we want to have $l(D)>1$, we ask for $(n-1)n-g+1>1$, or $(n-1)n>\frac{(n-1)(n-2)}{2}$, which is always true when $n>1$. In fact, from the proof of Riemann–Roch's Theorem, since $(P'_1,\dots,P'_l)$ is close to $(P_1,\dots,P_l)$, we may choose a meromorphic function close to f_0, so that its polar locus is D. On the other hand, the points $(P'_1,\dots,P'_l)$ which belong to a curve of degree $n-1$ are contained in a subvariety of dimension $\frac{n(n+1)}{2}-1$ and all we have to do is to check if $(n-1)n>\frac{n(n+1)}{2}-1$, which is obvious if $n\geq 3$ (we remark that there are not two different curves of degree $n-1$ passing through the $(n-1)n$ points $P'_1,\dots,P'_l$). We select then $P'_1,\dots,P'_l$ outside this subvariety in order to get the ramification map f and take a lifting F defined in a surface S. We prove then the statement: there is no embedding $\Phi\colon S\to\mathbb{P}^2$. In fact, the submersion $F\circ\Phi^{-1}$ defined in a neighborhood of $C_0\subset\mathbb{P}^2$ extends to $\mathbb{P}^2$ as a meromorphic function (holomorphic in a neighborhood of C_0). We observe that, for $n\geq 4$, given two embeddings $\phi_i\colon C\to\mathbb{P}^2$, $i=1,2$, there exists an automorphism $T\in\mathrm{Aut}(\mathbb{P}^2)$ such that $T(\phi_1(C))=\phi_2(C)$ (see the Appendix). Then the map $\Phi|_C\colon C\to C_0$ comes from a linear map on $\mathbb{P}^2$ and poles of f are the intersection of C with a curve of degree $n-1$, which is impossible.

Example 2.2. We present now an example of a non embeddable pair (curve, surface) with a set of two fibrations which is projective at the curve. We start with a projective, smooth curve C and select two pencil submersions. Let f_1 and f_2 be the associated ramification maps of C. The tangencies between the pencils are obviously pieces of the common line. We will replace one of these pieces by a non-invariant curve of tangencies between two new foliations. The idea is the same used above to realize ramification maps; the homeomorphism ψ is going to be changed. The point p this time is a point of tangency, and the coordinate chart G sends the foliations associated to the submersions to two foliations $(dz_1=0,\mathcal{H})$. We consider in $\mathbb{C}^2$ a couple of foliations $(dZ_1=0,\mathcal{H}':d(Z_1-Z_2(Z_2-Z_1))=0)$, which have $Z_1=2Z_2$ as non-invariant line of tangencies. The homeomorphism ψ is chosen in order to satisfy: 1) $\psi(0,0)=(0,0)$ and $\psi(z_1,0)=(z_1,0)$; 2) $\psi|_{\{1/2<|z_1|<1\}\times\mathbb{D}}$ is a holomorphic diffeomorphism over its image that sends $(dz_1=0,\mathcal{H})$ to $(dZ_1=0,\mathcal{H}')$. We put again $G_1=\psi\circ G$, which is the new glueing map. We can see that the self-intersection C^2 does not change and so the germ of surface is not isomorphic to $(C,\mathbb{P}^2)$.

We could also use in the construction the pair of foliations $(dZ_1 = 0, \mathcal{H}' : d(Z_1 - Z_2(Z_2 - Z_1^{e+1})) = 0)$ for $e \in \mathbb{N}$, but the curve C will have self-intersection number equal to $n^2 - e$.

3. Constructing meromorphic maps

Let us once more describe the setting we are going to analyse. We have a curve C contained in some surface S with $C \cdot C = n^2$ and $n \in \mathbb{N}$. There exist three submersions F, G, and H defined in S and taking values in $\mathbb{P}^1$ which define foliations $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ generically transversal to C whose leaves are the level curves. In order to simplify the exposition, we assume that all tangencies with C are simple and distinct (when we look to the tangencies for any pair of foliations). We denote $f = F|_C$, $g = G|_C$, and $h = H|_C$, all of them ramification maps from C to $\mathbb{P}^1$ whose ramification points correspond to the tangency points of the foliations (because F, G, and H are submersions). Furthermore, we assume that C embeds into $\mathbb{P}^2$ by a map $\phi\colon C \to C_0$, where C_0 is a smooth algebraic curve of degree n. In order to complete the picture, we select pencil submersions F_0, G_0, and H_0 (with associated foliations $\mathcal{F}_0$, $\mathcal{G}_0$, and $\mathcal{H}_0$), with singular points not aligned, which restrict to C_0 as n to 1 maps f_0, g_0, and h_0 to $\mathbb{P}^1$ and ask $\{f, g, h\}$ to be conjugated by ϕ to $\{f_0, g_0, h_0\}$: $f_0 \circ \phi = f$, $g_0 \circ \phi = g$, and $h_0 \circ \phi = h$. We remark that $\phi(\operatorname{tang}(\mathcal{F}, C)) = \operatorname{tang}(\mathcal{F}_0, C_0)$, once more because these tangency points are exactly the ramification points of f and f_0 (we have also that $\phi(\operatorname{tang}(\mathcal{G}, C)) = \operatorname{tang}(\mathcal{G}_0, C_0)$ and $\phi(\operatorname{tang}(\mathcal{H}, C)) = \operatorname{tang}(\mathcal{H}_0, C_0)$). For simplicity, we will assume that $F|_C$, $G|_C$, and $H|_C$ have only simple critical points.

Lemma 3.1. *Any pair of foliations defined by projective submersions at C are generically transverse to each other along C.*

Proof: Let $\mathcal{F}$ and $\mathcal{G}$ be two projective submersions at C. The tangency divisor $\operatorname{tang}(\mathcal{F}, \mathcal{G})$ between the foliations is given by

$$\operatorname{tang}(\mathcal{F}, \mathcal{G}) \cdot C = N_{\mathcal{F}} \cdot C + N_{\mathcal{G}} \cdot C + K_S \cdot C,$$

where $N_{\mathcal{F}}$ (resp. $N_{\mathcal{G}}$) is the normal bundle associated to $\mathcal{F}$ (resp. $\mathcal{G}$) and K_S is the canonical bundle of S. Since, from [**2**, Section 2.2], one has

$$\begin{aligned} N_{\mathcal{F}} \cdot C &= \chi(C) + \operatorname{tang}(\mathcal{F}, C) = 3n - n^2 + n^2 - n, \\ N_{\mathcal{G}} \cdot C &= \chi(C) + \operatorname{tang}(\mathcal{G}, C) = 3n - n^2 + n^2 - n, \\ -K_S \cdot C &= \chi(C) + C \cdot C = 3n - n^2 + n^2, \end{aligned}$$

we conclude that $\text{tang}(\mathcal{F},\mathcal{G})\cdot C = n$ so that $\mathcal{F}$ and $\mathcal{G}$ are not tangent to each other along C, as otherwise $\text{tang}(\mathcal{F},\mathcal{G}) = eC + D$ with $e > 0$, D effective, and $\text{tang}(\mathcal{F},\mathcal{G})\cdot C \geq en^2$. □

We observe that the lemma is not true for $n = 1$ (see [**3**]).

In this section we will see how to associate to a pair of submersions, say F, G, a meromorphic map $\Phi_{F,G}$. It is defined initially as a biholormorphism from a neighborhood of the set $C\setminus(A\cup\phi^{-1}(A_0))$ to a neighborhood of $C_0\setminus(A_0\cup\phi(A))$, where $A = \text{tang}(\mathcal{F},C)\cup\text{tang}(\mathcal{G},C)\cup(\text{tang}(\mathcal{F},\mathcal{G})\cap C)$ and $A_0 = \text{tang}(\mathcal{F}_0,C_0)\cup\text{tang}(\mathcal{G}_0,C_0)\cup(\text{tang}(\mathcal{F}_0,\mathcal{G}_0)\cap C_0)$. Given a point $p \in C\setminus(A\cup\phi^{-1}(A_0))$, the foliations $\mathcal{F}$ and $\mathcal{G}$ are transverse to each other and to C in a neighborhood of this point, and the foliations $\mathcal{F}_0$ and $\mathcal{G}_0$ are transverse to each other and to C_0 in a neighborhood of $\phi(p)$. Therefore, for $q \in S$ close to p we may associate the points $q_{\mathcal{F}}$ and $q_{\mathcal{G}}$ where the leaves of $\mathcal{F}$ and $\mathcal{G}$ intersect C. The leaves of $\mathcal{F}_0$ and $\mathcal{G}_0$ through $\phi(q_{\mathcal{F}})$ and $\phi(q_{\mathcal{G}})$ will intersect (by definition) at the point $\Phi_{F,G}(q)$. It can be seen that this maps extends biholomorphically to the points of $\text{tang}(\mathcal{F},C)$ and $\text{tang}(\mathcal{G},C)$, essentially because the foliations $\mathcal{F}$ and $\mathcal{G}$ are transverse to each other at those points. From now on we change A and A_0 to $A = \text{tang}(\mathcal{F},\mathcal{G})\cap C$ and $A_0 = \text{tang}(\mathcal{F}_0,\mathcal{G}_0)\cap C_0$ and analyse the behavior of $\Phi_{F,G}$ at points of $A\cup\phi^{-1}(A_0)$. We distinguish two cases:

(A) $\phi(p)\in\text{tang}(\mathcal{F}_0,\mathcal{G}_0)\cap C_0$.
(B) $\phi(p)\notin\text{tang}(\mathcal{F}_0,\mathcal{G}_0)\cap C_0$.

Proposition 3.2. *$\Phi_{F,G}$ extends meromorphically to a neighborhood of C.*

Proof: Case (A): We may assume, choosing conveniently the coordinates (x,y) around p and affine coordinates (X,Y), that

- $p = (0,0)$, C is $y = 0$, and $\mathcal{F}$ is defined by $dx = 0$;
- $\phi(p) = (0,0)$, $\mathcal{F}_0$ is defined by $dX = 0$, and $\mathcal{G}_0$ is the radial pencil with $(0,1)$ as base point ($X = 0$ is a common fiber of $\mathcal{F}_0$ and $\mathcal{G}_0$);
- C_0 is defined by $Y = h(X)$ with $h(0) = 0$, $h'(0) = 0$, and $\phi(x) = (x,h(x))$.

The leaf of $\mathcal{F}$ (resp. $\mathcal{G}$) through a point (x,y) crosses the x-axis at x (resp. $\xi(x,y)$) for a holomorphic function ξ such that $\xi(x,0) = x$. It follows that

$$\Phi_{F,G}(x,y) = \left(x, 1 - \frac{x(1-h(\xi(x,y)))}{\xi(x,y)}\right) = \left(x, \frac{u(x,y)}{\xi(x,y)}\right).$$

The expression defines a meromorphic map in a neighborhood of $(0,0)$. There are two possible cases:

Case (A1): The germs x and ξ are relatively prime. The line of poles of $\Phi_{F,G}(x,y)$ is $\xi(x,y)=0$ and has multiplicity 1. We write $\xi(x,y)-x = y\,A_1(x,y)$ for some holomorphic function $A_1(x,y)$; the $\mathcal{G}$-fiber may be transversal to the $\mathcal{F}$-fiber (when $A_1(0,0)\neq 0$) or tangent to it (in which case $A_1(0,0)\neq 0$).

Case (A2): The germs x and ξ have a common factor. Write $\xi(x,y) = x(1+y\,A_2(x,y))$. Thus $\Phi_{F,G}(x,y)$ is a holomorphic map ($\mathcal{F}$ and $\mathcal{G}$ have $x=0$ as a common fiber), but it may be non-injective (unless $A_2(0,0)\neq 0$).

Case (B): We assume:

- $p=(0,0)$, C is $y=0$, and $\mathcal{F}$ is defined by $dx=0$;
- $\phi(p) = (0,0)$, $\mathcal{F}_0$ is defined by $dX = 0$, and $\mathcal{G}_0$ is defined by $dY-dX=0$ (in affine coordinates);
- C_0 is defined by $Y=h(X)$ with $h(0)=0$, $h'(0)=0$, and $\phi(x) = (x,h(x))$.

We have then

$$\Phi_{F,G}(x,y) = (x, \xi(x,y) - x + h(\xi(x,y))).$$

It follows that $\Phi_{F,G}$ is a holomorphic map in a neighborhood of p. Writing $\xi(x,y)-x = y\,B(x,y)$, we see that $\Phi_{F,G}$ is a local biholomorphism when $B(0,0)\neq 0$, that is, the fibers of $\mathcal{F}$ and $\mathcal{G}$ are transversal at p. □

An important consequence for us is that the pull-back by $\Phi_{F,G}$ of a holomorphic foliation $\mathcal{L}$ on $\mathbb{P}^2$ is also a holomorphic foliation in S. In the next section we describe the singularities of $\Phi^*_{F,G}(\mathcal{L})$.

4. New foliations on S

Let us take a foliation $\mathcal{L}$ on $\mathbb{P}^2$ defined by $\omega = L\,dP - n.P\,dL = 0$, where $P(X,Y) = \sum_{i+j\leq n} a_{ij}X^iY^j$ is a polynomial of degree n such that $C_0 = \{P=0\}$ (we may assume $a_{0n}\neq 0$) and L is a linear polynomial such that $L=0$ is transverse to C_0. The singularities of $\mathcal{L}$ contained in C_0 are supposed to be disjoint of $A_0\cup\phi(A)$.

We proceed to compute the multiplicity $Z(\mathcal{L}^*,C,p)$ along C of p as a singularity of $\mathcal{L}^* = \Phi^*_{F,G}(\mathcal{L})$ at the points where $\Phi_{F,G}$ maybe fails to be a biholomorphism. In order to make the computation easier, we take $L(X,Y) = X+b$, where b is a constant different from 0.

Proposition 4.1. *With the notation of the proof of Proposition 3.2, we have*
Case (A1): $Z(\mathcal{L}^*,C,p) = n + \mathrm{mult}_0(A_1(x,0))$.
Case (A2): $Z(\mathcal{L}^*,C,p) = \mathrm{mult}_0(A_2(x,0))$.
Case (B): $Z(\mathcal{L}^*,C,p) = \mathrm{mult}_0(B(x,0))$.

Proof: Case (A1): x and ξ are relatively prime. It follows that

$$P(\Phi_{F,G}(x,y)) = \frac{yv(x,y)}{\xi(x,y)^n}.$$

In fact, $P(\Phi_{F,G}(x,0)) = 0$ and $P(X,Y) = a_{0n}Y^n + \sum_{j\leq n-1} a_{ij}X^iY^j$, and therefore

$$P(\Phi_{F,G}(x,y)) = a_{0n}\frac{u^n}{\xi^n} + \frac{\sum_{n-j\geq 1} a_{ij}x^iu^j\xi^{n-j}}{\xi^n}.$$

In particular, $v(x,0) = x^{n-1}A_1(x,0)+\cdots$. We have also $L(\Phi_{F,G}(x,y)) = x+b$, $b\neq 0$, so that

$$\Phi^*_{F,G}\,\omega = \frac{1}{\xi^{n+1}}[(x+b)\xi(y\,dv + v\,dy) - n.yv((x+b)\,d\xi + \xi\,dx)].$$

Therefore, $\mathcal{L}^*$ is defined by $(x+b)\xi(y\,dv+v\,dy)-n.yv((x+b)\,d\xi+\xi\,dx) = 0$ near the point p and

$$Z(\mathcal{L}^*, C, p) = 1 + \mathrm{mult}_0(v(x,0)) = n + \mathrm{mult}_0(A_1(x,0)).$$

We observe that $Z(\mathcal{L}^*, C, p) > 0$ when the case (A1) holds.

Case (A2): ξ divides x ($\mathcal{F}$ and $\mathcal{G}$ share the leaf passing through p). Let us write as before $\xi(x,y) = x(1+yA_2(x,y))$. It follows that

$$\Phi_{F,G}(x,y) = \left(x, \frac{yA_2(x,y)+h(\xi(x,y))}{1+yA_2(x,y)}\right).$$

Writing $P(\Phi_{F,G}(x,y)) = yv(x,y)$, we see that $v(x,0) = A_2(x,0)+\cdots$ and

$$\Phi^*_{F,G}\,\omega = (x+b)(v\,dy + y\,dv) - n.yv\,dx.$$

We conclude that

$$Z(\mathcal{L}^*, C, p) = \mathrm{mult}_0(v(x,0)) = \mathrm{mult}_0(A_2(x,0)).$$

Let us notice that $Z(\mathcal{L}^*, C, p) = 0$ implies $A_2(0,0)\neq 0$, that is, $\Phi_{F,G}(x,y)$ is a local biholomorphism at p.

Case (B): $\phi(p) \notin \mathrm{tang}(\mathcal{F}_0, \mathcal{G}_0)\cap C_0$. We have

$$\Phi_{F,G}(x,y) = (x, \xi - x + h(\xi(x,y))).$$

Writing $P(\Phi_{F,G}(x,y)) = yv(x,y)$, we see that $v(x,0) = B(x,0)+\cdots$ and

$$\Phi^*_{F,G}\,\omega = (x+b)(v\,dy + y\,dv) - n.yv\,dx.$$

We conclude that

$$Z(\mathcal{L}^*, C, p) = \mathrm{mult}_0(v(x,0))_0 = \mathrm{mult}_0(B(x,0)).$$

Again, $Z(\mathcal{L}^*, C, p) = 0$ implies that $B(0,0) \neq 0$, that is, $\Phi_{F,G}(x, y)$ is a local biholomorphism at the point p. $\square$

We intend now to see the implications of having two maps $\Phi_{F,G}$ and $\Phi_{F,H}$ simultaneously. The fibrations $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are associated to pencil submersions $\mathcal{F}_0$, $\mathcal{G}_0$, and $\mathcal{H}_0$. Let us call $B = \operatorname{tang}(\mathcal{F}, \mathcal{H}) \cap C$ and $B_0 = \operatorname{tang}(\mathcal{F}_0, \mathcal{H}_0) \cap C_0$. We consider two foliations $\mathcal{I}$ and $\mathcal{L}$ on $\mathbb{P}^2$ as before. We remark that $Z(\mathcal{I}, C_0) = Z(\mathcal{L}, C_0) = n$. We will assume: 1) all singularities of $\mathcal{I}$ and $\mathcal{L}$ lie outside the set $K = A_0 \cup \phi(A) \cup B_0 \cup \phi(B)$; 2) all curves of tangencies between $\mathcal{I}$ and $\mathcal{L}$ cross C_0 outside the set K. We denote $\mathcal{I}^* = \Phi^*_{F,G}(\mathcal{I})$ and $\mathcal{L}^* = \Phi^*_{F,H}(\mathcal{L})$. We will use again the formulae from [**2**, Section 2.2] to compute numerical invariants associated to tangent lines between two foliations. We have:

$$\operatorname{tang}(\mathcal{I}, \mathcal{L}) \cdot C_0 = N_{\mathcal{I}} \cdot C_0 + N_{\mathcal{L}} \cdot C_0 + K_{\mathbb{P}^2} \cdot C_0 = 2n^2 - n,$$

since $\mathcal{I}$ and $\mathcal{H}$ have degree $n - 1$ and $K_{\mathbb{P}^2} \cdot C_0 = -3n$.

Let us call $\mathcal{Z}_1(\mathcal{I}^*, C)$ (resp. $\mathcal{Z}_1(\mathcal{L}^*, C)$) the set of points where $\Phi_{F,G}$ is not a local biholomorphism (resp. $\Phi_{F,H}$ is not a local biholomorphism). We define $Z_1(\mathcal{I}^*, C)$ as the sum of all indexes $Z(\mathcal{I}^*, C, p)$ at points of $\mathcal{Z}_1(\mathcal{I}^*, C)$ (we put $Z_1(\mathcal{L}^*, C)$ for the corresponding sum at points of $\mathcal{Z}_1(\mathcal{L}^*, C)$).

As for the foliations $\mathcal{I}^*$ and $\mathcal{L}^*$, we have that

$$\begin{aligned} \operatorname{tang}(\mathcal{I}^*, \mathcal{L}^*) \cdot C &= N_{\mathcal{I}^*} \cdot C + N_{\mathcal{L}^*} \cdot C + K_S \cdot C \\ &= Z(\mathcal{I}^*, C) + Z(\mathcal{L}^*, C) + 2n^2 - 3n \\ &= Z_1(\mathcal{I}^*, C) + Z_1(\mathcal{L}^*, C) + 2n^2 - n. \end{aligned}$$

We conclude therefore that

$$\operatorname{tang}(\mathcal{I}^*, \mathcal{L}^*) \cdot C = Z_1(\mathcal{I}^*, C) + Z_1(\mathcal{L}^*, C) + \operatorname{tang}(\mathcal{I}, \mathcal{L}) \cdot C_0.$$

Observe that the curves C and C_0 appear as components of the tangency locus in both sides of the last equation. Thus we cancel n from the equation and consider, from now on, tangency loci besides C and C_0. This formula suggests that $\operatorname{tang}(\mathcal{I}^*, \mathcal{L}^*) \cap C$ may be also computed looking at the points of $\phi^{-1}(\operatorname{tang}(\mathcal{I}, \mathcal{L}) \cap C) \cup \mathcal{Z}_1(\mathcal{I}^*, C) \cup \mathcal{Z}_1(\mathcal{L}^*, C)$.

Our aim is to prove that $\Phi_{F,G}$ and $\Phi_{F,H}$ are everywhere local biholomorphisms. First of all, we have to associate the tangencies between $\mathcal{I}$ and $\mathcal{L}$ to tangencies between $\mathcal{I}^*$ and $\mathcal{L}^*$. There is a little difficulty here because $\mathcal{I}^*$ and $\mathcal{L}^*$ are obtained from $\mathcal{I}$ and $\mathcal{L}$ using different pull-backs; the pre-image by ϕ of a point of tangency between $\mathcal{I}$ and $\mathcal{L}$ might not be a point of tangency between $\mathcal{I}^*$ and $\mathcal{L}^*$. We take the foliations $\mathcal{I}$ and $\mathcal{L}$ defined by the equations $L\,dP - n.P\,dL = 0$ and $(L + a)\,dP + n.P\,dL = 0$. Their curve of tangencies is defined by

$dL \wedge dP = 0$ (besides the curve C_0). When intersecting with C_0, these are the points of tangency of $\mathcal{F}_0$ with C_0.

Proposition 4.2. *Let $\phi(p)$ be a point of tangency between $\mathcal{F}_0$ and C_0. Then* $(\operatorname{tang}(\mathcal{I}^*, \mathcal{L}^*), C)_p = 1$.

Proof: We may take local coordinates (x, y) around p and (r, s) around $\phi(p)$ such that

- $C = \{y = 0\}$ and $C_0 = \{s = 0\}$.
- $\mathcal{F}$ and $\mathcal{F}_0$ are defined by $d(y - x^2) = 0$ and $d(s - r^2) = 0$ respectively.

The foliations $\mathcal{I}$ and $\mathcal{L}$ are defined as $su\, d(s - r^2) - (s - r^2 + \delta)\, d(su) = 0$ and $su\, d(s - r^2) - (s - r^2 + a + \delta)\, d(su) = 0$, where u is a holomorphic function such that $u(0,0) \neq 0$ and $\delta \neq 0$. Let us write $\Phi_{F,G}(x, y) = (f(x, y), yA(x, y))$ and $\Phi_{F,H}(x, y) = (g(x, y), yB(x, y))$. We have $A(0,0) \neq 0$, $B(0,0) \neq 0$, and $(f(x,0), 0) = (g(x,0), 0) = \phi(x)$.

The foliations $\mathcal{I}^* = \Phi^*_{F,G}\mathcal{I}$ and $\mathcal{L}^* = \Phi^*_{F,H}\mathcal{L}$ are defined as

$$yAu\, d(yA - f^2) - (yA - f^2 + \delta)\, d(yAu) = 0,$$
$$yBu\, d(yB - g^2) - (yB - g^2 + a + \delta)\, d(yBu) = 0.$$

We see easily that the curve of tangencies is given by $ABau^2\phi\phi' y + y^2(\dots) = 0$, so that the component different from $C = \{y = 0\}$ crosses C at p transversely. $\square$

We proceed now to examine the points of tangency between $\mathcal{I}^*$ and $\mathcal{L}^*$ that possibly appear at $\mathcal{Z}_1(\mathcal{I}^*, C) \cup \mathcal{Z}_1(\mathcal{L}^*, C)$. If we denote their number as $\operatorname{tang}_1(\mathcal{I}^*, \mathcal{L}^*)$, we have seen that

$$\operatorname{tang}_1(\mathcal{I}^*, \mathcal{L}^*) = Z_1(\mathcal{I}^*, C) + Z_1(\mathcal{L}^*, C).$$

In fact, we have seen that, out of $\mathcal{Z}_1(\mathcal{I}^*, C) \cup \mathcal{Z}_1(\mathcal{L}^*, C)$, tangency curves correspond to each other when restricted to C and C_0.

We claim that this equality holds at each point of $\mathcal{Z}_1(\mathcal{I}^*, C) \cup \mathcal{Z}_1(\mathcal{L}^*, C)$. Let us consider some point $p \in \mathcal{Z}_1(\mathcal{I}^*, C)$. Since $\Phi_{F,G}$ is not a local biholomorphism, we have as explained before the possibilities (A1), (A2), and (B), the first two occurring when $\phi(p) \in \operatorname{tang}(\mathcal{F}_0, \mathcal{G}_0) \cap C_0$. If p satisfies (A1) or (A2) for $\Phi_{F,G}$, then p satisfies (B) for $\Phi_{F,H}$. (In the same way, when $q \in \mathcal{Z}_1(\mathcal{L}^*, C)$ satisfies (A1) or (A2) for $\Phi_{F,H}$, then q satisfies (B) for $\Phi_{F,G}$.) It may happen also that p satisfies (B) for $\Phi_{F,G}$ and $\Phi_{F,H}$. The reason is that we are supposing the submersions F, G, and H to be independent so that we are in case (A) for the maps $\Phi_{F,G}$ and $\Phi_{F,H}$ simultaneously.

Case 1: $p \in \mathcal{Z}_1(\mathcal{I}^*, C)$ satisfies (A1) for $\Phi_{F,G}$ and (B) for $\Phi_{F,H}$. The local equations for $\mathcal{I}^*$ and $\mathcal{L}^*$ at p are

$$(x+b)\xi(y\,dv + v\,dy) - n.yv\{(x+b)\,d\xi + \xi\,dx\} = 0,$$
$$(x+b')(v'\,dy + y\,dv') - n.yv'\,dx = 0.$$

The line of tangencies has equation

$$(b-b')\xi vv' - (x+b)(x+b')[vv'\xi_x + \xi(v'v_x - vv'_x)] = 0.$$

We observe that $\mathrm{mult}_0(\xi vv') = \mathrm{mult}_0(v) + 1 + \mathrm{mult}_0(v') = Z(\mathcal{I}^*, C, p) + Z(\mathcal{L}^*, C, p)$ (it may happen $Z(\mathcal{L}^*, C, p) = 0$).

Case 2: $p \in \mathcal{Z}_1(\mathcal{I}^*, C)$ satisfies (A2) for $\Phi_{F,G}$ and (B) for $\Phi_{F,H}$. The local equations are

$$(x+b)(y\,dv + v\,dy) - n.yv\,dx = 0,$$
$$(x+b')(v'\,dy + y\,dv') - n.yv'\,dx = 0.$$

The line of tangencies has equation

$$(b-b')vv' - (x+b)(x+b')[v'v_x - vv'_x] = 0.$$

We remark that $\mathrm{mult}_0(vv') = \mathrm{mult}_0(v) + \mathrm{mult}_0(v') = Z(\mathcal{I}^*, C, p) + Z(\mathcal{L}^*, C, p)$ (it may happen that $Z(\mathcal{L}^*, C, p) = 0$).

Case 3: $p \in \mathcal{Z}_1(\mathcal{I}^*, C)$ satisfies (B) for $\Phi_{F,G}$ and $\Phi_{F,H}$. The conclusion is the same as above: $\mathrm{mult}_0(vv') = \mathrm{mult}_0(v) + \mathrm{mult}_0(v') = Z(\mathcal{I}^*, C, p) + Z(\mathcal{L}^*, C, p)$ (it may happen $Z(\mathcal{L}^*, C, p) = 0$).

The remaining cases (when $p \in \mathcal{Z}_1(\mathcal{L}^*, C, p)$): p satisfies (A1) for $\Phi_{F,H}$ and (B) for $\Phi_{F,G}$; p satisfies (A2) for $\Phi_{F,H}$ and (B) for $\Phi_{F,G}$; p satisfies (B) for both $\Phi_{F,H}$ and $\Phi_{F,G}$ are entirely similar.

We conclude from $\mathrm{tang}_1(\mathcal{I}^*, \mathcal{L}^*) = Z_1(\mathcal{I}^*, C) + Z_1(\mathcal{L}^*, C)$ (and the fact that b, b' are generic) that the terms $[vv'\xi_x + \xi(v'v_x - vv'_x)]$ (first case) and $[v'v_x - vv'_x]$ (second and third cases) have the same multiplicities at 0 as $\xi vv'$ and vv' respectively. Thus, the claim is proved.

Let us make explicit the relations between the several multiplicities involved before.

Case 1: We write $v(x,0) = ax^l + \cdots$ and $v'(x,0) = cx^m + \cdots$. It follows that $[vv'\xi_x + \xi(v'v_x - vv'_x)] = ac(1-m+l)x^{m+l} + \cdots$. Since $\mathrm{mult}_0(\xi vv') = m + l + 1$, necessarily $m = l + 1$. Using $v(x,0) = x^{n-1}A_1(x,0)$ (and $v'(x,0) = B'(x,0)$) we get

$$\mathrm{mult}_0(A_1(x,0)) + n = \mathrm{mult}_0(B'(x,0)).$$

Case 2: We write again $v(x,0) = ax^l + \cdots$ and $v'(x,0) = cx^m + \cdots$. Then $[v'v_x - vv'_x)] = ac(l-m)x^{m+l-1} + \cdots$. Since $\mathrm{mult}_0(vv') = m + l$, we see that $l = m$. Using $v(x,0) = A_2(x,0)$ and $v'(x,0) = B'(x,0)$, we obtain

$$\mathrm{mult}_0(A_2(x,0)) = \mathrm{mult}_0(B'(x,0)).$$

Case 3: It is analogous to Case 2 and we find

$$\mathrm{mult}_0(B(x,0)) = \mathrm{mult}_0(B'(x,0)).$$

There are correspondent equalities when p satisfies (A1) for $\Phi_{F,H}$ and (B) for $\Phi_{F,G}$ ($\mathrm{mult}_0(A'_1(x,0)) + n = \mathrm{mult}_0(B(x,0))$), or p satisfies (A2) for $\Phi_{F,H}$ and (B) for $\Phi_{F,G}$ ($\mathrm{mult}_0(A'_2(x,0)) = \mathrm{mult}_0(B(x,0))$).

5. Proof of Theorem A

Let us take some C^∞ perturbation $\tilde{C}$ of C and look at the curves $\Phi_{F,G}(\tilde{C})$ and $\Phi_{F,H}(\tilde{C})$, which are C^∞ perturbations of C_0. We ask $\tilde{C}$ to be a holomorphic smooth curve with $(\tilde{C}.C)_p = 1$ when passing through each $p \in \mathcal{Z}_1(\mathcal{I}^*, C) \cup \mathcal{Z}_1(\mathcal{L}^*, C)$ and ask also that $\Phi_{F,G}$ and $\Phi_{F,H}$ be holomorphic along these (local) holomorphic curves. Let us observe again that $\Phi_{F,G}$ is not a local biholomorphism at a point $p \in \mathcal{Z}_1(\mathcal{I}^*, C)$ (and $\Phi_{F,H}$ is not a local biholomorphism at a point $p \in \mathcal{Z}_1(\mathcal{L}^*, C)$ either).

We proceed now to prove that for any $p \in \mathcal{Z}_1(\mathcal{I}^*, C) \cup \mathcal{Z}_1(\mathcal{L}^*, C)$ one has

$$\sum_q (\Phi_{F,G}(\tilde{C}).C_0)_q + (\Phi_{F,H}(\tilde{C}).C_0)_q \geq 2,$$

for q close to $\phi(p)$. Observe that in principle this number should be equal to $(\tilde{C}.C)_p + (\tilde{C}.C)_P = 2$. Let us go back to the cases we discussed in the last section.

Case 1: $p \in \mathcal{Z}_1(\mathcal{I}^*, C)$ satisfies (A1) for $\Phi_{F,G}$ and (B) for $\Phi_{F,H}$. The perturbation $\tilde{C}$ near p has to be contained in some small sector around C, where $\Phi_{F,G}$ is holomorphic. Since

$$\Phi_{F,G} = \left(x, \frac{yA_1(x,y) + xh(\xi(x,y))}{x + yA_1(x,y)} \right)$$

when we put $y = \epsilon x$ (for $\tilde{C}$), we see that $\sum_q (\Phi_{F,G}(\tilde{C}).C_0)_q$ is the number of solutions (near $\phi(p)$) to the equation

$$\frac{\epsilon\, x\, A_1(x, \epsilon\, x) + xh(\xi(x, \epsilon\, x))}{x + \epsilon\, x\, A_1(x, \epsilon\, x)} = h(x),$$

which is $= \operatorname{mult}_0(A_1(x,0))$. In order to estimate $\sum_q (\Phi_{F,H}(\tilde{C}).C_0)_q$ we use

$$\Phi_{F,H}(x,y) = (x, yB'(x,y) + h(\xi(x,y)))$$

and we have to find the number of solutions of

$$\epsilon\, x\, B'(x, \epsilon\, x) + h(\xi(x, \epsilon\, x)) = h(x)$$

(remember that now p satisfies (B) for $\Phi_{F,H}$), which is readily seen to be $1 + \operatorname{mult}_0(B'(x,0))$. We have seen before that $\operatorname{mult}_0(B'(x,0)) = \operatorname{mult}_0(A_1(x,0)) + n$, so for q close to $\phi(p)$,

$$\sum_q (\Phi_{F,G}(\tilde{C}).C_0)_q + (\Phi_{F,H}(\tilde{C}).C_0)_q = 2\operatorname{mult}_0(A_1(x,0)) + n + 1,$$

which is strictly bigger than 2 when $n > 1$.

Case 2: $p \in \mathcal{Z}_1(\mathcal{I}^*, C)$ satisfies (A2) for $\Phi_{F,G}$ and (B) for $\Phi_{F,H}$. We have

$$\Phi_{F,G}(x,y) = \left(x, \frac{yA_2(x,y) + h(\xi(x,y))}{1 + yA_2(x,y)}\right)$$

and

$$\Phi_{F,H}(x,y) = (x, yB'(x,y) + h(\xi(x,y))).$$

Using again $y = \epsilon x$, we get $\sum_q (\Phi_{F,G}(\tilde{C}).C_0)_q = 1 + \operatorname{mult}_0 A_2(x,0)$ and $\sum_q (\Phi_{F,H}(\tilde{C}).C_0)_{\phi(q)} = 1 + \operatorname{mult}_0(B'(x,0))$. Therefore, for q close to $\phi(p)$,

$$\sum_q (\Phi_{F,G}(\tilde{C}).C_0)_q + (\Phi_{F,H}(\tilde{C}).C_0)_q = 2 + 2\operatorname{mult}_0(A_2(x,0)).$$

Case 3: $p \in \mathcal{Z}_1(\mathcal{I}^*, C)$ satisfies (B) for $\Phi_{F,G}$ and (B) for $\Phi_{F,H}$. Similarly, we find for q close to $\phi(p)$:

$$\sum_q (\Phi_{F,G}(\tilde{C}).C_0)_q + (\Phi_{F,H}(\tilde{C}).C_0)_q = 2 + 2\operatorname{mult}_0(B(x,0)).$$

The remaining cases are analogous. We conclude that Case (A1) never appears and that Cases (A2) and (B) are present only at points p where $\Phi_{F,G}$ is a local biholomorphisms and give us the desired conjugation.

Appendix A. Automorphisms of a plane curve

We present a proof of the following theorem which was explained to us by J. F. Voloch.

Theorem A.1. *Let C be a smooth plane curve of degree $n \geq 4$. Then every automorphism of C is linear, i.e. it comes from an element of* $\mathrm{Aut}(\mathbb{P}^2)$.

Before proving the theorem we give some useful remarks based on exercises 17 and 18 of [**1**, Chapter 1]. We say that the set $S = \{p_1, \dots, p_k\} \subseteq \mathbb{P}^2$ of distinct points *impose independent conditions on curves of degree d* if $h^0(\mathbb{P}^2, \mathcal{I}_S(d)) = h^0(\mathbb{P}^2, \mathcal{O}(d)) - k$.

Lemma A.2. *Any set of $d+1$ points impose independent conditions on curves of degree d. On the other hand, $d+2$ points impose independent conditions if and only if they are not aligned.*

Proof: Take first $S = \{p_1, \dots, p_{d+1}\}$ and denote $S_k = \{p_1, \dots, p_k\}$. Taking the product of d lines through another point we see that $H^0(\mathbb{P}^2, \mathcal{I}_{S_{i+1}}(d))$ is strictly contained in $H^0(\mathbb{P}^2, \mathcal{I}_{S_i}(d))$. Therefore,

$$h^0(\mathbb{P}^2, \mathcal{I}_S(d)) = h^0(\mathbb{P}^2, \mathcal{O}(d)) - (d+1).$$

Consider now a set $S = \{p_1, \dots, p_{d+1}, p_{d+2}\}$. If they are on a line L and E is a curve of degree d passing through $d+1$ of them, Bezout's Theorem implies $L \subseteq E$. This shows that S fails to impose independent conditions on curves of degree d. Suppose now that every curve of degree d passing by $d+1$ points contains also the other point of S. If they are not aligned, we can take for example the curve E formed by lines joining p_{d+1} with points $p_1, \dots, p_d$. Thus p_{d+2} must be on this curve and we can assume that p_d, p_{d+1}, and p_{d+2} are aligned. If some p_j, $j = 1, \dots, d-1$, is not on this line, we consider E' obtained from E replacing $\overline{p_{d+1}, p_j}$ by a generic line passing by p_{d+1}. Thus E' contains $(d+1)$ points but not S, a contradiction. □

Let D be an effective divisor on C of degree m. We use the previous lemma in order to study meromorphic functions on C having D as polar divisor. Changing the fiber if necessary we will assume from now on that D has not multiple points. We recall that $l(D)$ is the dimension of the space of meromorphic functions f such that $(f) + D \geq 0$ and $i(D)$ is the dimension of the space of holomorphic forms ω such that $(\omega) \geq D$.

Proposition A.3. *If $m \leq n-2$, then $l(D) = 1$.*

Proof: Recall (see [**6**, Chapter VII, Section 4]) that holomorphic 1-forms on $C = \{P = 0\}$ are generated by elements $\frac{x^i y^j}{P_y}\, dx$ with $i + j \leq n-3$. By the previous lemma the dimension of the space of polynomials vanishing at D is $g(C) - m$. Thus $i(D) = g(C) - m$ and Riemann–Roch's Theorem gives $l(D) = 1$. □

Proposition A.4. *If $m = n - 1$ and $l(D) \geq 2$, then $D = E - p$ where $p \in C$ and $E \in |\mathcal{O}_C(1)|$.*

Proof: Once again Riemann–Roch's Theorem gives $i(D) = g - n + l(D) \geq g - (n-1) + 1$. Then the points of D do not impose independent conditions and they must be aligned. We conclude by noting that the intersection of a line with C is a divisor of degree n. □

Finally, we have

Proposition A.5. *$|\mathcal{O}_C(1)|$ is the only linear system of degree n and dimension* 3*.*

Proof: Let $D \in |\mathcal{O}_C(1)|$ be an aligned divisor of degree n on C. Then the points of D fail to impose independent conditions on curves of degree $n-3$ and $i(D) = g(C) - (n-2)$ or, equivalently, $l(D) = 3$. If A is another effective divisor of degree n and $l(A) = 3$, then any subset of $n-1$ points are aligned. We conclude that $A \in |\mathcal{O}_C(1)|$ and is linearly equivalent to D. □

Proof of Theorem A.1: Let $\phi\colon C \to C$ be an automorphism of C. The last proposition implies that for any line L on $\mathbb{P}^2$, points of $\phi(L \cap C)$ determine a line $L' \subseteq \mathbb{P}^2$, and thus ϕ comes from an automorphism of $\check{\mathbb{P}}^2$ which corresponds to an element of $\mathrm{Aut}(\mathbb{P}^2)$. □

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