Publ. Mat. **65** (2021), 129–140
DOI: 10.5565/PUBLMAT6512104

ACYCLIC 2-DIMENSIONAL COMPLEXES AND QUILLEN'S CONJECTURE

Kevin Iván Piterman, Iván Sadofschi Costa, and Antonio Viruel

Abstract: Let G be a finite group and $\mathcal{A}_p(G)$ be the poset of nontrivial elementary abelian p-subgroups of G. Quillen conjectured that $O_p(G)$ is nontrivial if $\mathcal{A}_p(G)$ is contractible. We prove that $O_p(G) \neq 1$ for any group G admitting a G-invariant acyclic p-subgroup complex of dimension 2. In particular, it follows that Quillen's conjecture holds for groups of p-rank 3. We also apply this result to establish Quillen's conjecture for some particular groups not considered in the seminal work of Aschbacher–Smith.

2010 Mathematics Subject Classification: 57S17, 20D05, 57M20, 55M20, 55M35, 57M60.

Key words: Quillen's conjecture, poset, p-subgroups.

1. Introduction

The study of the poset $\mathcal{S}_p(G)$ of nontrivial p-subgroups of a finite group G started when K. S. Brown proved that the Euler characteristic $\chi(\mathcal{K}(\mathcal{S}_p(G)))$ of its order complex is 1 modulo the greatest power of p dividing the order of G [**6**]. Recall that the order complex $\mathcal{K}(X)$ of a poset X is the simplicial complex whose simplices are the finite nonempty totally ordered subsets of X. Some years later, D. Quillen studied the homotopy properties of $\mathcal{K}(\mathcal{S}_p(G))$ [**11**]. In that article, Quillen considered the subposet $\mathcal{A}_p(G)$ of nontrivial elementary abelian p-subgroups and proved that its order complex is homotopy equivalent to $\mathcal{K}(\mathcal{S}_p(G))$ [**11**,

---

This work was partially done at the University of Málaga, during a research stay of the first two authors, supported by project MTM2016-78647-P.
The first author was supported by a CONICET doctoral fellowship and grants CONICET PIP 112201701 00357CO and UBACyT 20020160100081BA.
The second author was supported by a CONICET postdoctoral fellowship and grants ANPCyT PICT-2017-2806, CONICET PIP 11220170100357CO, and UBACyT 20020160100081BA.
The third author was partially supported by Ministerio de Economía y Competitividad (Spain), grant MTM2016-78647-P (AEI/FEDER, UE, support included).

Proposition 2.1]. Quillen also proved that, if the largest normal p-subgroup $O_p(G)$ of G is nontrivial, then $\mathcal{K}(\mathcal{A}_p(G))$ is contractible [**11**, Proposition 2.4] and conjectured that the converse should hold.

In this paper we study the following version of Quillen's conjecture. Recall that the homology of a poset is the homology of its order complex.

Quillen's conjecture. *If* $O_p(G) = 1$*, then* $\widetilde{H}_*(\mathcal{A}_p(G)) \neq 0$.

Aschbacher and Smith's formulation relates rational acyclicity of $\mathcal{K}(\mathcal{A}_p(G))$ with nontriviality of $O_p(G)$ [**3**]. Thus, our integral homology version is stronger than Quillen's original statement but weaker than the Aschbacher–Smith version.

Quillen proved the conjecture for solvable groups [**11**, Theorem 12.1]. In [**3**] M. Aschbacher and S. D. Smith made a huge progress on the study of this conjecture. By using the classification of finite simple groups, they proved that Quillen's conjecture holds if $p > 5$ and G does not contain certain unitary components. Previously, Aschbacher and Kleidman ([**1**]) had proved Quillen's conjecture for almost simple groups (i.e. finite groups G such that $L \leq G \leq \mathrm{Aut}(L)$ for some non-abelian simple group L).

The main result of our paper, which depends on the classification of the finite simple groups, is the following.

Theorem 3.2. *If* X *is an acyclic and* 2*-dimensional* G*-invariant subcomplex of* $\mathcal{K}(\mathcal{S}_p(G))$*, then* $O_p(G) \neq 1$.

Recall that the action of G on $\mathcal{S}_p(G)$ is by conjugation. The previous result provides then a convenient tool to prove that a group verifies Quillen's conjecture.

Corollary 3.3. *Let* G *be a finite group. Suppose that* $\mathcal{K}(\mathcal{S}_p(G))$ *admits a* 2*-dimensional and* G*-invariant subcomplex homotopy equivalent to itself. Then Quillen's conjecture holds for* G.

In particular, it follows that Quillen's conjecture holds for groups of p-rank 3. Recall that the *p-rank* of G, usually denoted by $m_p(G)$, is the maximum possible rank of an elementary abelian p-subgroup of G. The p-rank 2 case was considered by Quillen [**11**, Proposition 2.10] and is a consequence of Serre's result: an action of a finite group on a tree has a fixed point.

In Section 4 we make an extensive use of Corollary 3.3 to establish Quillen's conjecture for some particular groups (of p-ranks 3 and 4) for which the hypotheses of the results of Aschbacher–Smith ([**3**]) do not hold.

A related conjecture, due to C. Casacuberta and W. Dicks, is that a finite group acting on a contractible 2-complex has a fixed point [**7**]. This conjecture was studied by Aschbacher and Segev in [**2**]. Posteriorly, Oliver and Segev classified the groups which admit a fixed point free action on an acyclic (finite) 2-complex [**10**]. Our proof of Theorem 3.2 is built upon the results of [**10**], which depend on the classification of finite simple groups. Theorem 3.2 can also be seen as a special case of the Casacuberta–Dicks conjecture.

Acknowledgements. We are grateful to the anonymous referee for her or his suggestions which greatly improved the exposition of the paper and in particular for simplifying the proofs in Examples 4.10 and 4.11 by indicating Proposition 4.9.

2. The results of Oliver and Segev

In this section we review the results of [**10**] needed in the proof of Theorem 3.2. By a *G-complex* we mean a G-CW complex. Note that the order complex of a G-poset is always a G-complex.

Definition 2.1 ([**10**])**.** A G-complex X is *essential* if there is no normal subgroup $1 \neq N \triangleleft G$ such that for each $H \subseteq G$, the inclusion $X^{HN} \to X^H$ induces an isomorphism on integral homology.

The main results of [**10**] are the following two theorems.

Theorem 2.2 ([**10**, Theorem A])**.** *For any finite group G, there is an essential fixed point free 2-dimensional (finite) acyclic G-complex if and only if G is isomorphic to one of the simple groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Furthermore, the isotropy subgroups of any such G-complex are all solvable.*

Theorem 2.3 ([**10**, Theorem B])**.** *Let G be any finite group, and let X be any 2-dimensional acyclic G-complex. Let N be the subgroup generated by all normal subgroups $N' \triangleleft G$ such that $X^{N'} \neq \emptyset$. Then X^N is acyclic; X is essential if and only if $N = 1$; and the action of G/N on X^N is essential.*

The set of subgroups of G will be denoted by $\mathcal{S}(G)$.

Definition 2.4 ([**10**])**.** By a *family* of subgroups of G we mean any subset $\mathcal{F} \subseteq \mathcal{S}(G)$ which is closed under conjugation. A nonempty family is said to be *separating* if it has the following three properties: (a) $G \notin \mathcal{F}$; (b) if $H' \subseteq H$ and $H \in \mathcal{F}$, then $H' \in \mathcal{F}$; (c) for any $H \triangleleft K \subseteq G$ with K/H solvable, $K \in \mathcal{F}$ if $H \in \mathcal{F}$.

For any family $\mathcal{F}$ of subgroups of G, a $(G,\mathcal{F})$*-complex* will mean a G-complex all of whose isotropy subgroups lie in $\mathcal{F}$. A $(G,\mathcal{F})$-complex is H*-universal* if the fixed point set of each $H \in \mathcal{F}$ is acyclic.

Lemma 2.5 ([**10**, Lemma 1.2])**.** *Let X be any 2-dimensional acyclic G-complex without fixed points. Let $\mathcal{F}$ be the set of subgroups $H \subseteq G$ such that $X^H \neq \emptyset$. Then $\mathcal{F}$ is a separating family of subgroups of G, and X is an H-universal $(G,\mathcal{F})$-complex.*

If G is not solvable, the separating family of solvable subgroups of G is denoted by $\mathcal{SLV}$.

Proposition 2.6 ([**10**, Proposition 6.4])**.** *Assume that L is one of the simple groups $\mathrm{PSL}_2(q)$ or $\mathrm{Sz}(q)$, where $q = p^k$ and p is prime ($p = 2$ in the second case). Let $G \subseteq \mathrm{Aut}(L)$ be any subgroup containing L, and let $\mathcal{F}$ be a separating family for G. Then there is a 2-dimensional acyclic $(G,\mathcal{F})$-complex if and only if $G = L$, $\mathcal{F} = \mathcal{SLV}$, and q is a power of 2 or $q \equiv \pm 3 \pmod 8$.*

Definition 2.7 ([**10**, Definition 2.1])**.** For any family $\mathcal{F}$ of subgroups of G define

$$i_{\mathcal{F}}(H) = \frac{1}{[N_G(H) : H]}(1 - \chi(\mathcal{K}(\mathcal{F}_{>H}))).$$

Lemma 2.8 ([**10**, Lemma 2.3])**.** *Fix a separating family $\mathcal{F}$, a finite H-universal $(G,\mathcal{F})$-complex X, and a subgroup $H \subseteq G$. For each n, let $c_n(H)$ denote the number of orbits of n-cells of type G/H in X. Then $i_{\mathcal{F}}(H) = \sum_{n\geq 0}(-1)^n c_n(H)$.*

Proposition 2.9 ([**10**, Tables 2, 3, 4])**.** *Let G be one of the simple groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Then $i_{\mathcal{SLV}}(1) = 1$.*

3. The two-dimensional case

Using the results of Oliver and Segev stated in the previous section we prove the following.

Theorem 3.1. *Every acyclic 2-dimensional G-complex has an orbit with normal stabilizer.*

Proof: If $X^G \neq \emptyset$, we are done. Otherwise, G acts fixed point freely on X. Consider the subgroup N generated by the subgroups $N' \triangleleft G$ such that $X^{N'} \neq \emptyset$. Clearly N is normal in G. By Theorem 2.3, $Y = X^N$ is acyclic (in particular it is nonempty) and the action of G/N on Y is essential and fixed point free. By Lemma 2.5, $\mathcal{F} = \{H \leq G/N : Y^H \neq \emptyset\}$ is a separating family and Y is an H-universal $(G/N,\mathcal{F})$-complex. Thus, Theorem 2.2 asserts that G/N must be one of the groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. In any case, by Proposition 2.6 we must have $\mathcal{F} = \mathcal{SLV}$. By Proposition 2.9, $i_{\mathcal{SLV}}(1) = 1$. Finally, by Lemma 2.8, Y must have at least one free G/N-orbit. Therefore, X has a G-orbit of type G/N and we are done. □

Theorem 3.2. *If X is an acyclic and 2-dimensional G-invariant subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$, then $O_p(G) \neq 1$.*

Proof: By Theorem 3.1 there is a simplex $\sigma = (A_0 < \cdots < A_j)$ of X with stabilizer $N \triangleleft G$. Since $A_0 \triangleleft N$, we deduce that $O_p(N)$ is nontrivial. On the other hand, $N \triangleleft G$ and $O_p(N) \operatorname{char} N$ implies that $O_p(N) \triangleleft G$. Therefore, $O_p(N) \leq O_p(G)$ and $O_p(G)$ is thus nontrivial. □

From Theorem 3.2 we deduce

Corollary 3.3. *Let G be a finite group. Suppose that $\mathcal{K}(\mathcal{S}_p(G))$ admits a 2-dimensional and G-invariant subcomplex homotopy equivalent to itself. Then Quillen's conjecture holds for G.*

Since the p-rank of G is equal to $\dim \mathcal{K}(\mathcal{A}_p(G)) + 1$ we obtain:

Corollary 3.4. *Let G be a finite group of p-rank 3. If $\widetilde{H}_*(\mathcal{A}_p(G)) = 0$, then $O_p(G) \neq 1$.*

We now apply Corollary 3.3 to obtain results for some related p-subgroup complexes. Recall that a p-subgroup $Q \leq G$ is *radical* if $Q = O_p(N_G(Q))$. The *Bouc poset* $\mathcal{B}_p(G)$ is the poset of nontrivial radical p-subgroups of G. It is well-known that $\mathcal{K}(\mathcal{B}_p(G))$ is homotopy equivalent to $\mathcal{K}(\mathcal{S}_p(G))$ [**5**]. Then, by Corollary 3.3, we have

Corollary 3.5. *Let G be a finite group such that $\mathcal{B}_p(G)$ has height 2. If $\widetilde{H}_*(\mathcal{B}_p(G)) = 0$, then $O_p(G) \neq 1$.*

We say that a poset X is a *reduced lattice* if it is obtained from a finite lattice by removing its minimum and maximum. If X is a reduced lattice, $\mathfrak{i}(X)$ denotes the subposet of X given by the elements which can be written as the infimum of a set of maximal elements of X. It is a general fact that the order complex of $\mathfrak{i}(X)$ is homotopy equivalent to the order complex of X for any reduced lattice X [**4**, Subsection 9.1]. Hence, by Corollary 3.3, we have

Corollary 3.6. *Let G be a finite group. If either $\mathfrak{i}(\mathcal{S}_p(G))$ or $\mathfrak{i}(\mathcal{A}_p(G))$ has height 2, then G satisfies Quillen's conjecture.*

For a detailed account of the relations between the different p-subgroup complexes, see [**12**].

4. Some examples

In this section we apply the corollaries of Theorem 3.2 to establish Quillen's conjecture for some groups constructed so that the hypotheses of the results of [**3**] are not satisfied. The main result of [**3**] is the following.

Theorem 4.1 (Aschbacher–Smith [**3**, Main Theorem])**.** *Let G be a finite group and $p > 5$ a prime number. Assume that whenever G has a unitary component $U_n(q)$ with $q \equiv -1 \pmod{p}$ and q odd, then the Quillen dimension property at p holds for all p-extensions of $U_m(q^{p^e})$ with $m \leq n$ and $e \in \mathbb{Z}$. Then G satisfies Quillen's conjecture.*

Recall that a group H satisfies the *Quillen dimension property at p* if $\tilde{H}_{m_p(H)-1}(\mathcal{A}_p(H)) \neq 0$. The presence of simple components of G isomorphic to $L_2(2^3)$ or $U_3(2^3)$ (in the $p = 3$ case) and $\mathrm{Sz}(2^5)$ (in the $p = 5$ case) is an obstruction to extending Theorem 4.1 to $p = 3$ and $p = 5$. The case $p = 2$ is not considered in [**3**] and would require a much more detailed analysis. One of the first steps in the proof of Theorem 4.1 is the reduction to the case $O_{p'}(G) = 1$ (see [**3**, Proposition 1.6]). To do this, [**3**, Theorems 2.3 and 2.4] are needed and these theorems make a strong use of the hypothesis $p > 5$. Concretely, it is not possible to apply [**3**, Theorem 2.3] if a component of $C_G(O_{p'}(G))$ is isomorphic to $L_2(2^3)$, $U_3(2^3)$ (if $p = 3$), or $\mathrm{Sz}(2^5)$ (if $p = 5$).

Before presenting the examples for $p = 3$ and $p = 5$, we give some motivation. Most of the groups G in these examples satisfy the following conditions. First, $O_{p'}(G) \neq 1$ and $C_G(O_{p'}(G))$ contains a component isomorphic to $U_3(2^3)$ if $p = 3$ and to $\mathrm{Sz}(2^5)$ if $p = 5$. Thus, we cannot find nontrivial homology for $\mathcal{A}_p(G)$ in the same way it is done in the proof of [**3**, Proposition 1.6] since we are not able to invoke [**3**, Theorems 2.3 and 2.4]. Secondly, since there is an inclusion $\tilde{H}_*(\mathcal{A}_p(G/O_{p'}(G)) : \mathbb{Q}) \hookrightarrow \tilde{H}_*(\mathcal{A}_p(G); \mathbb{Q})$ (see [**3**, Lemma 0.12]), we require $O_p(G/O_{p'}(G)) \neq 1$ so that $\tilde{H}_*(\mathcal{A}_p(G/O_{p'}(G))) = 0$. Finally, we require $O_p(G) = 1$.

The groups presented in Examples 4.5 and 4.7 have p-rank 3. The groups presented in Examples 4.6 and 4.8 have p-rank 4 and are constructed in the following way. We take a direct product of a group N, consisting of one or more copies of a particular simple p'-group, by a group K consisting of one or more copies of $L = U_3(2^3)$ if $p = 3$ or $L = \mathrm{Sz}(2^5)$ if $p = 5$. Then we take two cyclic p-groups A and B and we let them act on the direct product $N \times K$ as follows. We take a faithful action of $A \times B$ on N, and we choose a representation $A \times B \to \mathrm{Aut}(K)$ such that $O_p(K \rtimes (A \times B)) \cong O_p(C_A(K)) \neq 1$. The group $G = (N \times K) \rtimes (A \times B)$ satisfies the conditions $O_p(G) = 1$, $O_{p'}(G) = N \neq 1$, $C_G(N) = K$, and

$O_p(G/N) = O_p(K \rtimes (A \times B)) \neq 1$. Moreover, since the p-rank of L is at most 2, we can construct G to have p-rank 4 by adjusting the number of copies of L in K.

For these groups we show that $\mathcal{K}(\mathcal{S}_p(G))$ has a 2-dimensional G-invariant subcomplex homotopy equivalent to itself, and thus Corollary 3.3 applies.

In Examples 4.10 and 4.11 we describe two groups of 2-rank 4 such that $\mathcal{K}(\mathcal{S}_2(G))$ admits a 2-dimensional G-invariant homotopy equivalent subcomplex.

For the claims on the structure of the automorphism group of the finite groups of Lie type we refer to [**8**] and [**9**].

Lemma 4.2. *Let $1 \to N \to G \to K \to 1$ be an extension of finite groups. Then*

$$m_p(G) = \max_{A \in \mathcal{S}} m_p(C_N(A)) + m_p(A),$$

where $\mathcal{S}$ is the set of elementary abelian p-subgroups $1 \leq A \leq G$ such that $A \cap N = 1$. In particular, we have $m_p(G) \leq m_p(N) + m_p(K)$.

Proof: If $A \in \mathcal{S}$, we have $C_N(A) \times A \cong C_N(A)A$ and hence $m_p(C_N(A)) + m_p(A) \leq m_p(C_N(A)A) \leq m_p(G)$. Taking maximum over $A \in \mathcal{S}$ gives the lower bound for $m_p(G)$. We now prove the other inequality. Let E be an elementary abelian p-subgroup of G and write $E = (E \cap N)A$ for some complement A of $E \cap N$ in E. Then $m_p(E \cap N) \leq m_p(C_N(A))$ and $A \in \mathcal{S}$. Now $m_p(E) = m_p(E \cap N) + m_p(A) \leq m_p(C_N(A)) + m_p(A)$, giving the upper bound for $m_p(G)$. For the last claim note that $C_N(A) \leq N$ and $m_p(A) \leq m_p(K)$ by the isomorphism theorems. □

The following lemma will be used to obtain proper subcomplexes of $\mathcal{K}(\mathcal{A}_p(G))$ without changing the homotopy type. We write $X \simeq Y$ if the order complexes $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are homotopy equivalent.

Lemma 4.3. *Let G be a finite group and let $H \leq G$. In addition, suppose that $O_p(C_H(E)) \neq 1$ for each $E \in \mathcal{A}_p(G)$ with $E \cap H = 1$. Then $\mathcal{A}_p(G) \simeq \mathcal{A}_p(H)$.*

Proof: Consider the subposet $\mathcal{N} = \{E \in \mathcal{A}_p(G) : E \cap H \neq 1\}$. We have order preserving maps $r\colon \mathcal{N} \to \mathcal{A}_p(H)$ and $i\colon \mathcal{A}_p(H) \hookrightarrow \mathcal{N}$, given by $r(E) = E \cap H$ and $i(E) = E$ such that $ir(E) \leq E$ and $ri(E) = E$. Therefore, $\mathcal{N} \simeq \mathcal{A}_p(H)$.

Let $\mathcal{S} = \{E \in \mathcal{A}_p(G) : E \cap H = 1\}$ be the complement of $\mathcal{N}$ in $\mathcal{A}_p(G)$. For any $E \in \mathcal{S}$ consider $\mathcal{A}_p(G)_{>E} \cap \mathcal{N} = \{A \in \mathcal{N} : A > E\}$. It is easy to see that $r\colon \mathcal{A}_p(G)_{>E} \cap \mathcal{N} \to \mathcal{A}_p(C_H(E))$ defined by $r(B) = B \cap H$ is a homotopy equivalence with inverse $i(B) = BE$. Then $\mathcal{A}_p(G)_{>E} \cap \mathcal{N} \simeq \mathcal{A}_p(C_H(E))$ is contractible since $O_p(C_H(E)) \neq 1$.

Now take a linear extension $E_1, \dots, E_r$ of $\mathcal{S}$ (i.e. enumerate the elements of $\mathcal{S}$ so that $E_i \le E_j$ implies $i \le j$) and let $X^i = \mathcal{N} \cup \{E_1, \dots, E_i\}$. Note that $X^i = X^{i-1} \cup \{E_i\}$ and by the linear extension $X^i_{>E_i} = \mathcal{A}_p(G)_{>E_i} \cap \mathcal{N}$, which is contractible. Now $X^i_{\ge E_i}$ is a cone over $X^i_{>E_i}$ with vertex E_i. Therefore, $X^{i-1} \hookrightarrow X^i$ is a homotopy equivalence for each $1 \le i \le r$. In consequence,

$$\mathcal{A}_p(G) = X^r \simeq X^0 = \mathcal{N} \simeq \mathcal{A}_p(H). \qquad \square$$

Remark 4.4. In the above result it can be shown that if $H \lhd G$, then the homotopy equivalence is G-equivariant.

Example 4.5. Let $p = 3$ and let $L = L_2(2^3) \times L_2(2^3) \times L_2(2^3)$. Let A be a cyclic group of order 3 acting on L by permuting the copies of $L_2(2^3)$. Take $G = L \rtimes A$. Since $m_3(L_2(2^3)) = 1$ and $C_L(A) \cong L_2(2^3)$, we see that $m_3(G) = 3$. By Corollary 3.4, G satisfies Quillen's conjecture.

Example 4.6. Let $p = 3$, $N = \mathrm{Sz}(2^3) \times \mathrm{Sz}(2^3) \times \mathrm{Sz}(2^3)$, and $U = U_3(2^3)$. Let $A = \langle a \rangle$ and $B = \langle b \rangle$ be cyclic groups of order 3. We construct a semidirect product $G = (N \times U) \rtimes (A \times B)$. To do this we need to define a map $A \times B \to \mathrm{Aut}(N \times U) = \mathrm{Aut}(N) \times \mathrm{Aut}(U)$.

Choose a field automorphism $\phi \in \mathrm{Aut}(U_3(2^3))$ of order 3. By the properties of the p-group actions, there exists an inner automorphism $x \in \mathrm{Inn}(U_3(2^3))$ of order 3 commuting with ϕ. Then $A \times B \to \mathrm{Aut}(U_3(2^3))$ is given by $a \mapsto x$ and $b \mapsto \phi$. Choose a field automorphism $\psi \in \mathrm{Aut}(\mathrm{Sz}(2^3))$ of order 3. Let A act on each coordinate of N as ψ and let B act on N by permuting its coordinates. This gives rise to a well defined map $A \times B \to \mathrm{Aut}(N)$.

The 3-rank of G is $m_3(G) = m_3(U_3(2^3)AB)$. We can take an elementary abelian subgroup $E \le C_U(\phi)$ of order 9 containing x since $C_U(\phi) \cong \mathrm{PGU}_3(2) \cong ((C_3 \times C_3) \rtimes Q_8) \rtimes C_3$ by [**9**, Chapter 4, Lemma 3.10] and $\mathcal{A}_3(\mathrm{PGU}_3(2))$ is connected of height 1. Then EAB is an elementary abelian subgroup of order 3^4. Hence, $m_3(UAB) \ge 4$. Since $m_3(U_3(2^3)) = 2$ and $m_3(AB) = 2$, by Lemma 4.2 we have $m_3(G) = 4$.

By Corollary 3.3, to show that Quillen's conjecture holds for G and $p = 3$, it is enough to find a 2-dimensional G-invariant subcomplex X of $\mathcal{K}(\mathcal{S}_3(G))$ homotopy equivalent to $\mathcal{K}(\mathcal{S}_3(G))$ (or, equivalently, to $\mathcal{K}(\mathcal{A}_3(G))$).

Let $H = (N \times U) \rtimes A$. Note that $H \lhd G$ and $m_3(H) = 3$. Therefore, $\mathcal{K}(\mathcal{A}_3(H))$ is a 2-dimensional G-invariant subcomplex of $\mathcal{K}(\mathcal{A}_3(G))$. Now the plan is to use Lemma 4.3 to show that $\mathcal{A}_3(H) \simeq \mathcal{A}_3(G)$. Let $E \in \mathcal{A}_3(G)$ be such that $E \cap H = 1$. Then $E \cong EH/H \le B \cong C_3$ and hence E is cyclic generated by some element $e \in E$. Write $e = nua^ib^j$ with

$n \in N$, $u \in U$, and $i, j \in \{0, 1, 2\}$. Note that $j \neq 0$ since $E \cap H = 1$. If $v \in U$, then

$$v^e = v^{nua^i b^j} = (v^{ua^i})^{b^j}.$$

Since $j \neq 0$ and e induces an automorphism of U of order 3 in $\mathrm{Inn}(U)\phi^j$, by [**8**, Proposition 4.9.1] and the definition of field automorphisms [**8**, Definition 2.5.13], e is $\mathrm{Inndiag}(U)$-conjugate to ϕ^j and acts as a field automorphism on U. In particular, $C_U(E) = C_U(e) \cong C_U(\phi^j) = C_U(\phi)$. Note that $O_3(C_U(E)) \cong O_3(C_U(\phi)) \cong C_3 \times C_3 \neq 1$. Since $C_U(E) \lhd C_H(E)$ and $O_3(C_U(E)) \neq 1$, we conclude that $O_3(C_H(E)) \neq 1$. By Lemma 4.3, $\mathcal{A}_3(G) \simeq \mathcal{A}_3(H)$, which is 2-dimensional and G-invariant. In conclusion, the subcomplex $\mathcal{K}(\mathcal{A}_3(H))$ satisfies the hypothesis of Corollary 3.3, and therefore Quillen's conjecture holds for G.

Note that $O_3(G) = 1$, $O_{3'}(G) = N$, $C_G(O_{3'}(G)) = U_3(2^3)$, and $O_3(G/O_{3'}(G)) = O_3(U_3(2^3)AB) = \langle ax^{-1} \rangle \cong C_3$.

Example 4.7. Let $p = 5$. Let r be a prime number such that $r \equiv 2$ or $3 \pmod 5$ and let $q = r^{5^n}$ with $n \geq 2$. Let N be one of the simple groups $L_2(q)$, $G_2(q)$, ${}^3D_4(q^3)$, or ${}^2G_2(3^{5^n})$ and let $A = \langle a \rangle$ be a cyclic group of order 5^n. Note that $5 \nmid |N|$. Let a act on N as a field automorphism of order 5^n. Choose a field automorphism $\phi \in \mathrm{Aut}(\mathrm{Sz}(2^5))$ of order 5 and let A act on $\mathrm{Sz}(2^5) \times \mathrm{Sz}(2^5)$ as $\phi \times \phi$. Now consider the semidirect product $G = (N \times \mathrm{Sz}(2^5) \times \mathrm{Sz}(2^5)) \rtimes A$ defined by this action.

Since the Sylow 5-subgroups of $\mathrm{Sz}(2^5)$ are cyclic of order 25, by Lemma 4.2 we have that $m_5(G) = 3$. By Corollary 3.4, Quillen's conjecture holds for G.

Moreover, $O_5(G) = 1$, $O_{5'}(G) = N$, $C_G(O_{5'}(G)) = \mathrm{Sz}(2^5)^2$, and $O_5(G/O_{5'}(G)) = C_A(\mathrm{Sz}(2^5)^2) = \langle a^5 \rangle \neq 1$.

Example 4.8. Let $p = 5$ and let $N = L^5$, where L is one of the simple $5'$-groups of the previous example. Let $A = \langle a \rangle \cong C_{5^n}$ and $B = \langle b \rangle \cong C_5$. Let $G = (N \times \mathrm{Sz}(2^5)^2) \rtimes (A \times B)$, where a acts on each copy of L as a field automorphism of order 5^n and trivially on $\mathrm{Sz}(2^5)^2$, and b permutes the copies of L and acts as a field automorphism of order 5 on each copy of $\mathrm{Sz}(2^5)$.

To compute the 5-rank of G we use Lemma 4.2:

$$\begin{aligned} m_5(G) &= m_5(\mathrm{Sz}(2^5)^2 \rtimes (A \times B)) \\ &= m_5(A \times (\mathrm{Sz}(2^5)^2 \rtimes B)) \\ &= m_5(A) + m_5(\mathrm{Sz}(2^5)^2 \rtimes B) \\ &= 1 + 3 \\ &= 4. \end{aligned}$$

Now the aim is to apply Corollary 3.3 on G by finding a 2-dimensional G-invariant homotopy equivalent subcomplex X of $\mathcal{K}(\mathcal{S}_5(G))$.

Let $H = (N \times \mathrm{Sz}(2^5)^2) \rtimes A = NA \times \mathrm{Sz}(2^5)^2$. Note that $H \lhd G$ and $m_5(H) = 3$. Hence $\mathcal{K}(\mathcal{A}_5(H))$ is 2-dimensional and G-invariant. We will show that $\mathcal{A}_5(H) \simeq \mathcal{A}_5(G)$ by applying Lemma 4.3.

Let $E \in \mathcal{A}_5(G)$ be such that $E \cap H = 1$. Then E is cyclic generated by an element e of order 5 and $e = lsa^i b^j$ with $l \in N$, $s \in \mathrm{Sz}(2^5)^2$, $0 \leq i \leq 5^n - 1$, and $j \in \{1, 2, 3, 4\}$. Thus E acts by field automorphisms on each copy of the Suzuki group and e is Inndiag(Sz(2^5))-conjugate to the field automorphism induced by b^j on Sz(2^5) (see [**8**, Proposition 4.9.1] and Example 4.6). Hence, $C_H(E) = C_{NA}(E) \times C_{\mathrm{Sz}(2^5)^2}(E)$. Note that $C_{\mathrm{Sz}(2^5)^2}(E) \lhd C_H(E)$ and $C_{\mathrm{Sz}(2^5)^2}(E) \cong C_{\mathrm{Sz}(2^5)}(E)^2 \cong (C_5 \rtimes C_4)^2$ has a nontrivial normal 5-subgroup. Therefore, $\mathcal{A}_5(G) \simeq \mathcal{A}_5(H)$ by Lemma 4.3 and Quillen's conjecture holds for G by Corollary 3.3 applied to the subcomplex $\mathcal{K}(\mathcal{A}_5(H))$.

Note that $O_{5'}(G) = N$ and $C_G(O_{5'}(G)) = \mathrm{Sz}(2^5)^2$. On the other hand, $O_5(G) = 1$ and $O_5(G/O_{5'}(G)) = A \neq 1$.

We conclude with two examples of groups satisfying Quillen's conjecture for $p = 2$. We say that a finite group G has the *trivial intersection property at* p if any two different Sylow p-subgroups of G have trivial intersection.

Proposition 4.9. *Let L_1 and L_2 be two finite groups with the trivial intersection property at p. Let $L = L_1 \times L_2$ and take an extension G of L such that $|G : L| = p$. Then $\mathfrak{i}(\mathcal{S}_p(G))$ and $\mathcal{B}_p(G)$ are at most 2-dimensional. If in addition the Sylow p-subgroups of L_1 and L_2 have abelian Ω_1, then $\mathfrak{i}(\mathcal{A}_p(G))$ is at most 2-dimensional.*

Proof: The elements of $\mathfrak{i}(\mathcal{S}_p(L))$ are of the form $P_1 \times P_2$, $1 \times P_2$, or $P_1 \times 1$, where $P_i \leq L_i$ are Sylow p-subgroups. Hence, $\mathfrak{i}(\mathcal{S}_p(L))$ is 1-dimensional.

Now suppose that $Q_0 < Q_1 < \cdots < Q_n$ is a chain in $\mathfrak{i}(\mathcal{S}_p(G))$. Then

$$Q_0 \cap L \leq Q_1 \cap L \leq \cdots \leq Q_n \cap L$$

is a chain in $\mathfrak{i}(\mathcal{S}_p(L))$. We claim that there is at most one index i such that $Q_i \cap L = Q_{i+1} \cap L$. To see this note that

$$|Q_j : Q_j \cap L| = \begin{cases} 1 & \text{if } Q_j \subseteq L, \\ p & \text{if } Q_j \not\subseteq L. \end{cases}$$

We have $|Q_{i+1} : Q_i| \cdot |Q_i : Q_i \cap L| = |Q_{i+1} : Q_{i+1} \cap L| \cdot |Q_{i+1} \cap L : Q_i \cap L|$. Then, if $Q_i \cap L = Q_{i+1} \cap L$, since $|Q_{i+1} : Q_i| \geq p$, we must have $|Q_i : Q_i \cap L| = 1$ and $|Q_{i+1} : Q_{i+1} \cap L| = p$. Then $i = \max\{j : Q_j \subseteq L\}$.

From this we conclude that $\dim \mathfrak{i}(\mathcal{S}_p(G)) \leq 1 + \dim \mathfrak{i}(\mathcal{S}_p(L)) = 2$. It is well-known that $\mathcal{B}_p(G)$ is a subposet of $\mathfrak{i}(\mathcal{S}_p(G))$ (i.e. every radical p-subgroup is an intersection of Sylow p-subgroups). Then $\mathcal{B}_p(G)$ is at most 2-dimensional also. The same proof can be easily adapted to prove that if the Sylow p-subgroups of L_1 and L_2 have abelian Ω_1, $\mathfrak{i}(\mathcal{A}_p(G))$ is at most 2-dimensional. □

In the following examples we use the fact that the groups A_5 and $U_3(2^2)$ have the trivial intersection property at 2 and that $\Omega_1(P)$ is abelian for P a Sylow 2-subgroup of either A_5 or $U_3(2^2)$.

Example 4.10. Let G be the group extension $(A_5 \times A_5) \rtimes C_2$, where the generator of C_2 acts on each coordinate as conjugation by the transposition (1 2). Since $m_2(A_5) = 2 = m_2(\mathrm{Aut}(A_5))$, by Lemma 4.2, G has 2-rank 4. By Proposition 4.9, $\mathfrak{i}(\mathcal{A}_2(G))$, $\mathfrak{i}(\mathcal{S}_2(G))$, and $\mathcal{B}_2(G)$ are 2-dimensional and then Quillen's conjecture holds for G since Corollaries 3.5 and 3.6 apply.

Example 4.11. Let $G = (U_3(2^2) \times A_5) \rtimes C_2$ be the semidirect product constructed in the following way. Let $H = U_3(2^2) \times A_5$. Then $\mathrm{Out}(H) \cong \mathrm{Aut}(U_3(2^2))/\mathrm{Inn}(U_3(2^2)) \times \mathrm{Aut}(A_5)/\mathrm{Inn}(A_5) \cong C_4 \times C_2$. Take $t \in \mathrm{Out}(H)$ to be the involution which acts nontrivially on both factors. Therefore, $G = H \rtimes \langle t \rangle$. Since $m_2(U_3(2^2)) = 2 = m_2(A_5) = m_2(\mathrm{Aut}(A_5))$ and $m_2(\mathrm{Aut}(U_3(2^2))) = 3$, by Lemma 4.2, G has 2-rank 4. Just as before, Quillen's conjecture holds for G.

References

[1] M. Aschbacher and P. B. Kleidman, On a conjecture of Quillen and a lemma of Robinson, *Arch. Math. (Basel)* **55(3)** (1990), 209–217. DOI: 10.1007/BF01191159.

[2] M. Aschbacher and Y. Segev, A fixed point theorem for groups acting on finite 2-dimensional acyclic simplicial complexes, *Proc. London Math. Soc. (3)* **67(2)** (1993), 329–354. DOI: 10.1112/plms/s3-67.2.329.

[3] M. Aschbacher and S. D. Smith, On Quillen's conjecture for the p-groups complex, *Ann. of Math. (2)* **137(3)** (1993), 473–529. DOI: 10.2307/2946530.

[4] J. A. Barmak, *"Algebraic Topology of Finite Topological Spaces and Applications"*, Lecture Notes in Mathematics **2032**, Springer, Heidelberg, 2011. DOI: 10.1007/978-3-642-22003-6.

[5] S. Bouc, Homologie de certains ensembles ordonnés, *C. R. Acad. Sci. Paris Sér. I Math.* **299(2)** (1984), 49–52.

[6] K. S. Brown, Euler characteristics of groups: the p-fractional part, *Invent. Math.* **29(1)** (1975), 1–5. DOI: 10.1007/BF01405170.

[7] C. Casacuberta and W. Dicks, On finite groups acting on acyclic complexes of dimension two, *Publ. Mat.* **36(2A)** (1992), 463–466. DOI: 10.5565/PUBLMAT_362A92_10.

[8] D. Gorenstein, R. Lyons, and R. Solomon, *"The Classification of the Finite Simple Groups, Number 3, Part I, Chapter A: Almost Simple $\mathcal{K}$-Groups"*, Mathematical Surveys and Monographs **40.3**, American Mathematical Society, Providence, RI, 1998. DOI: 10.1090/surv/040.3.

[9] D. Gorenstein, R. Lyons, and R. Solomon, *"The Classification of the Finite Simple Groups, Number 4, Part II, Chapters 1–4: Uniqueness Theorems"*, Mathematical Surveys and Monographs **40.4**, American Mathematical Society, Providence, RI, 1999. DOI: 10.1090/surv/040.4.

[10] B. Oliver and Y. Segev, Fixed point free actions on **Z**-acyclic 2-complexes, *Acta Math.* **189(2)** (2002), 203–285. DOI: 10.1007/BF02392843.

[11] D. Quillen, Homotopy properties of the poset of nontrivial p-subgroups of a group, *Adv. in Math.* **28(2)** (1978), 101–128. DOI: 10.1016/0001-8708(78)90058-0.

[12] S. D. Smith, *"Subgroup Complexes"*, Mathematical Surveys and Monographs **179**, American Mathematical Society, Providence, RI, 2011. DOI: 10.1090/surv/179.

Kevin Iván Piterman
Universidad de Buenos Aires, Facultad de Ciencias Exactas y Naturales, Departamento de Matemática, Buenos Aires, Argentina
E-mail address: kpiterman@dm.uba.ar

Iván Sadofschi Costa
CONICET-Universidad de Buenos Aires, Instituto de Investigaciones Matemáticas Luis A. Santaló (IMAS), Buenos Aires, Argentina
E-mail address: isadofschi@dm.uba.ar

Antonio Viruel
Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain
E-mail address: viruel@uma.es

Primera versió rebuda el 24 de juliol de 2019,
darrera versió rebuda el 25 d'octubre de 2019.