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CONVEX FOLIATIONS OF DEGREE 5 ON THE COMPLEX PROJECTIVE PLANE

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Abstract: We show that, up to automorphisms of $\mathbb{P}^2_{\mathbb{C}}$, there are fourteen homogeneous convex foliations of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$. We establish some properties of the Fermat foliation $\mathcal{F}_0^d$ of degree $d \geq 2$ and of the Hilbert modular foliation $\mathcal{F}_H^5$ of degree 5. As a consequence, we obtain that every reduced convex foliation of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$ is linearly conjugated to one of the two foliations $\mathcal{F}_0^5$ or $\mathcal{F}_H^5$, which is a partial answer to a question posed in 2013 by D. Marín and J. V. Pereira. We end with two conjectures about the Camacho–Sad indices along the line at infinity at the non radial singularities of the homogeneous convex foliations of degree $d \geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$.

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1. Introduction and statements of results

This article is part of a series of works by the authors ([**1**, **2**, **3**]) on holomorphic foliations on the complex projective plane. For the definitions and notations used (radial singularities, Camacho–Sad index $\mathrm{CS}(\mathcal{F}, \ell, s)$, homogeneous foliations, etc.) we refer to [**2**, Sections 1 and 2].

Following [**9**], a foliation on the complex projective plane is said to be *convex* if its leaves other than straight lines have no inflection points. Notice (see [**12**]) that if $\mathcal{F}$ is a foliation of degree $d \geq 1$ on $\mathbb{P}^2_{\mathbb{C}}$, then $\mathcal{F}$ cannot have more than $3d$ (distinct) invariant lines. Moreover, if this bound is reached, then $\mathcal{F}$ is necessarily convex; in this case $\mathcal{F}$ is said to be *reduced convex*.

To our knowledge the only reduced convex foliations known in the literature are those presented in [**9**, Table 1.1]: the Fermat foliation $\mathcal{F}_0^d$ of degree d, the Hesse pencil $\mathcal{F}_H^4$ of degree 4, the Hilbert modular foliation $\mathcal{F}_H^5$ of degree 5, and a foliation $\mathcal{F}_H^7$ of degree 7 related to the

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extended Hesse arrangement defined in affine chart respectively by the 1-forms

$$\begin{aligned}\bar{\omega}_0^d &= (x^d - x)\mathrm{d}y - (y^d - y)\mathrm{d}x,\\ \omega_H^4 &= (2x^3 - y^3 - 1)y\mathrm{d}x + (2y^3 - x^3 - 1)x\mathrm{d}y,\\ \omega_H^5 &= (y^2 - 1)(y^2 - (\sqrt{5} - 2)^2)(y + \sqrt{5}x)\mathrm{d}x\\ &\quad - (x^2 - 1)(x^2 - (\sqrt{5} - 2)^2)(x + \sqrt{5}y)\mathrm{d}y,\\ \omega_H^7 &= (y^3 - 1)(y^3 + 7x^3 + 1)y\mathrm{d}x - (x^3 - 1)(x^3 + 7y^3 + 1)x\mathrm{d}y.\end{aligned}$$

D. Marín and J. V. Pereira ([**9**, Problem 9.1]) asked the following question: are there other reduced convex foliations? The answer in degree 2, resp. 3, resp. 4, to this question is negative, thanks to [**8**, Proposition 7.4], resp. [**2**, Corollary 6.9], resp. [**3**, Theorem B]. In this paper we show that the answer in degree 5 to [**9**, Problem 9.1] is also negative. To do this, we follow the same approach as that described in degree 4 in [**3**]. It mainly consists of using Proposition 3.2 of [**3**] which allows us to associate to every pair $(\mathcal{F}, \ell)$, where $\mathcal{F}$ is a reduced convex foliation of degree d on $\mathbb{P}^2_{\mathbb{C}}$ and ℓ an invariant line of $\mathcal{F}$, a homogeneous convex foliation $\mathcal{H}^{\ell}_{\mathcal{F}}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$ belonging to the Zariski closure of the $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$-orbit of $\mathcal{F}$, and then to study for $d = 5$ the set of foliations $\mathcal{H}^{\ell}_{\mathcal{F}}$ where ℓ runs through the invariant lines of $\mathcal{F}$.

A homogeneous foliation $\mathcal{H}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$ is given, for a suitable choice of affine coordinates (x, y), by a homogeneous 1-form $\omega = A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y$, where A, B are complex homogeneous polynomials of degree d with $\gcd(A, B) = 1$. By [**2**] one associates to such a foliation the rational map $\underline{\mathcal{G}}_{\mathcal{H}} \colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ defined by

$$\underline{\mathcal{G}}_{\mathcal{H}}([x : y]) = [-A(x, y) : B(x, y)].$$

Notice (see [**2**]) that a homogeneous foliation $\mathcal{H}$ on $\mathbb{P}^2_{\mathbb{C}}$ is convex if and only if its associated map $\underline{\mathcal{G}}_{\mathcal{H}}$ is critically fixed, i.e. every critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$ is a fixed point of $\underline{\mathcal{G}}_{\mathcal{H}}$. More precisely, a homogeneous foliation $\mathcal{H}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$ is convex of type $\mathcal{T}_{\mathcal{H}} = \sum_{k=1}^{d-1} r_k \cdot \mathrm{R}_k$ (i.e. having r_1, resp. $r_2, \ldots$, resp. r_{d-1} radial singularities of order 1, resp. $2, \ldots$, resp. $d-1$, the R_k's being just symbols) if and only if the map $\underline{\mathcal{G}}_{\mathcal{H}}$ possesses r_1, resp. $r_2, \ldots$, resp. r_{d-1} fixed critical points of multiplicity 1, resp. $2 \ldots$, resp. $d-1$, with $\sum_{k=1}^{d-1} k r_k = 2d - 2$.

Using results of [**6**, pp. 79–80] on critically fixed rational maps of degree 5 from $\mathbb{P}^1_{\mathbb{C}}$ to itself and studying the convexity of a homogeneous foliation $\mathcal{H}$ of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$ according to the shape of its type $\mathcal{T}_{\mathcal{H}}$, we obtain the classification, up to automorphisms of $\mathbb{P}^2_{\mathbb{C}}$, of homogeneous convex foliations of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$.

Theorem A. *Up to automorphisms of $\mathbb{P}^2_{\mathbb{C}}$ there are fourteen homogeneous convex foliations $\mathcal{H}_1,\ldots,\mathcal{H}_{14}$ of degree 5 on the complex projective plane. They are respectively described in affine chart by the following 1-forms:*

$$\omega_1 = y^5\mathrm{d}x - x^5\mathrm{d}y;$$
$$\omega_2 = y^2(10x^3 + 10x^2y + 5xy^2 + y^3)\mathrm{d}x - x^4(x + 5y)\mathrm{d}y;$$
$$\omega_3 = y^3(10x^2 + 5xy + y^2)\mathrm{d}x - x^3(x^2 + 5xy + 10y^2)\mathrm{d}y;$$
$$\omega_4 = y^4(5x - 3y)\mathrm{d}x + x^4(3x - 5y)\mathrm{d}y;$$
$$\omega_5 = y^3(5x^2 - 3y^2)\mathrm{d}x - 2x^5\mathrm{d}y;$$
$$\omega_6 = y^3(220x^2 - 165xy + 36y^2)\mathrm{d}x - 121x^5\mathrm{d}y;$$
$$\omega_7 = y^4((5-\sqrt{5})x - 2y)\mathrm{d}x + x^4((7-3\sqrt{5})x - 2(5-2\sqrt{5})y)\mathrm{d}y;$$
$$\omega_8 = y^4(5(3-\sqrt{21})x + 6y)\mathrm{d}x + x^4(3(23-5\sqrt{21})x - 10(9-2\sqrt{21})y)\mathrm{d}y;$$
$$\omega_9 = y^3(2(5+a)x^2 - (15+a)xy + 6y^2)\mathrm{d}x - x^4((1-a)x + 2ay)\mathrm{d}y,$$

where $a = \sqrt{5(4\sqrt{61} - 31)}$;

$$\omega_{10} = y^3(2(5+\mathrm{i}b)x^2 - (15+\mathrm{i}b)xy + 6y^2)\mathrm{d}x - x^4((1-\mathrm{i}b)x + 2\mathrm{i}by)\mathrm{d}y,$$

where $b = \sqrt{5(4\sqrt{61} + 31)}$;

$$\omega_{11} = y^3(5x^2 - y^2)\mathrm{d}x + x^3(x^2 - 5y^2)\mathrm{d}y;$$
$$\omega_{12} = y^3(20x^2 - 5xy - y^2)\mathrm{d}x + x^3(x^2 + 5xy - 20y^2)\mathrm{d}y;$$
$$\omega_{13} = y^2(5x^3 - 10x^2y + 10xy^2 - 4y^3)\mathrm{d}x - x^5\mathrm{d}y;$$
$$\begin{aligned}\omega_{14} = {} & y^3(u(\sigma)x^2 + v(\sigma)xy + w(\sigma)y^2)\mathrm{d}x \\ & + \sigma x^4(2\sigma(\sigma^2 - \sigma + 1)x - (\sigma+1)(3\sigma^2 - 5\sigma + 3)y)\mathrm{d}y,\end{aligned}$$

where $u(\sigma) = (\sigma^2 - 3\sigma + 1)(\sigma^2 + 5\sigma + 1)$, $v(\sigma) = -2(\sigma+1)(\sigma^2 - 5\sigma + 1)$, $w(\sigma) = (\sigma^2 - 7\sigma + 1)$, $\sigma = \rho + \mathrm{i}\sqrt{\frac{1}{6} - \frac{4}{3}\rho - \frac{1}{3}\rho^2}$, *and* ρ *is the unique real number satisfying* $8\rho^3 - 52\rho^2 + 134\rho - 15 = 0$.

In the course of the proof of Theorem A we also obtain the following dual result (see §2).

Theorem B. *Up to conjugation by a Möbius transformation there are fourteen critically fixed rational maps of degree 5 from the Riemann sphere to itself, namely the maps* $\underline{\mathcal{G}}_{\mathcal{H}_1},\ldots,\underline{\mathcal{G}}_{\mathcal{H}_{14}}$.

To every foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ and to every integer $d \geq 2$, we associate respectively the following two subsets of $\mathbb{C} \setminus \{0,1\}$:

- $\mathcal{CS}(\mathcal{F})$ is, by definition, the set of $\lambda \in \mathbb{C} \setminus \{0,1\}$ for which there is a line ℓ invariant by $\mathcal{F}$ and a non-degenerate singular point $s \in \ell$ of $\mathcal{F}$ such that $\mathrm{CS}(\mathcal{F}, \ell, s) = \lambda$;
- $\mathcal{HCS}_d$ is defined as the set of $\lambda \in \mathbb{C} \setminus \{0,1\}$ for which there exist two homogeneous convex foliations $\mathcal{H}$ and $\mathcal{H}'$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$ having respective singular points s and s' on the line at infinity ℓ_∞ such that $\mathrm{CS}(\mathcal{H}, \ell_\infty, s) = \lambda$ and $\mathrm{CS}(\mathcal{H}', \ell_\infty, s') = \frac{1}{\lambda}$.

The following proposition, which will be proved in §2, motivates the introduction of the sets $\mathcal{CS}(\mathcal{F})$ and $\mathcal{HCS}_d$.

Proposition C. *Let $\mathcal{F}$ be a reduced convex foliation of degree $d \geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$. Then*

(i) $\emptyset \neq \mathcal{CS}(\mathcal{F}) \subset \mathcal{HCS}_d$;
(ii) $\forall \, \lambda \in \mathcal{CS}(\mathcal{F})$, $\frac{1}{\lambda} \in \mathcal{CS}(\mathcal{F})$.

Remark 1.1. In particular, for the foliations $\mathcal{F}_H^5$ and $\mathcal{F}_0^d$, we have

- $\{-\frac{3}{2} \pm \frac{\sqrt{5}}{2}\} = \mathcal{CS}(\mathcal{F}_H^5) \subset \mathcal{HCS}_5$, cf. [**10**, Theorem 2];
- $\{(1-d)^{\pm 1}\} = \mathcal{CS}(\mathcal{F}_0^d) \subset \mathcal{HCS}_d$ for any $d \geq 2$, cf. [**2**, Example 6.5].

The following theorem gives equivalent conditions for a foliation of degree $d \geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$ to be conjugated to the Fermat foliation $\mathcal{F}_0^d$.

Theorem D. *Let $\mathcal{F}$ be a foliation of degree $d \geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$. The following assertions are equivalent:*

(i) *$\mathcal{F}$ is linearly conjugated to the Fermat foliation $\mathcal{F}_0^d$;*
(ii) *$\mathcal{F}$ is reduced convex and $\mathcal{CS}(\mathcal{F}) = \{(1-d)^{\pm 1}\}$;*
(iii) *$\mathcal{F}$ possesses three radial singularities of maximal order $d-1$, necessarily non-aligned.*

In Theorem D, the implication (iii) $\Rightarrow$ (i) is a slight generalization of our previous result [**2**, Proposition 6.3], where we had obtained the same conclusion but with the additional hypothesis that the three radial singularities of $\mathcal{F}$ are not aligned.

Corollary E. *If $\mathcal{HCS}_d = \{(1-d)^{\pm 1}\}$, then, up to automorphisms of $\mathbb{P}^2_{\mathbb{C}}$, the Fermat foliation $\mathcal{F}_0^d$ is the unique reduced convex foliation in degree d.*

The following theorem gives equivalent conditions for a foliation of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$ to be conjugated to the Hilbert modular foliation $\mathcal{F}_H^5$.

Theorem F. *Let $\mathcal{F}$ be a foliation of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$. The following assertions are equivalent:*

(i) *$\mathcal{F}$ is linearly conjugated to the Hilbert modular foliation $\mathcal{F}^5_H$;*

(ii) *$\mathcal{F}$ is reduced convex and $\mathcal{CS}(\mathcal{F}) = \{-\frac{3}{2} \pm \frac{\sqrt{5}}{2}\}$;*

(iii) *$\mathcal{F}$ possesses three radial singularities m_1, m_2, m_3 of order 3 (necessarily non-aligned) and two radial singularities of order 1 on each invariant line $(m_j m_l)$, $1 \leq j < l \leq 3$ (see Figure 1).*

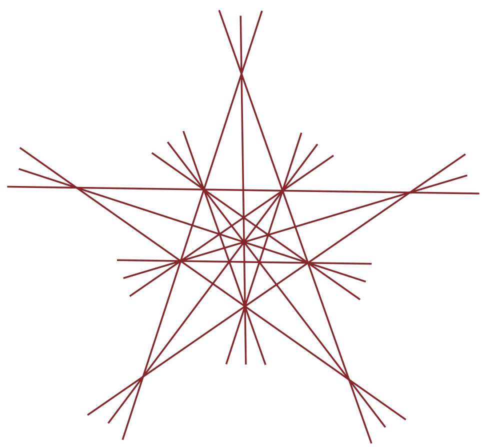

Figure 1. Arrangement of invariant lines of the Hilbert modular foliation $\mathcal{F}^5_H$ which possesses six radial singularities of order 3, ten radial singularities of order 1, and fifteen non-radial singularities with Baum–Bott invariant -1. Through each radial singularity of order $k \geq 1$ pass $k+2$ invariant lines.

Using essentially Theorems A, D, F, and Proposition C, we establish the following theorem.

Theorem G. *Up to automorphisms of $\mathbb{P}^2_{\mathbb{C}}$ the Fermat foliation $\mathcal{F}^5_0$ and the Hilbert modular foliation $\mathcal{F}^5_H$ are the only reduced convex foliations of degree five on $\mathbb{P}^2_{\mathbb{C}}$.*

2. Proof of the main results

We need to know the numbers r_{ij} of radial singularities of order j of the homogeneous foliations $\mathcal{H}_i$, $i = 1, \ldots, 14$, $j = 1, \ldots, 4$, and the values of the Camacho–Sad indices $\mathrm{CS}(\mathcal{H}_i, \ell_\infty, s)$, $s \in \mathrm{Sing}(\mathcal{H}_i) \cap \ell_\infty$, $i = 1, \ldots, 14$. For this reason, we have computed, for each $i = 1, \ldots, 14$, the type $\mathcal{T}_{\mathcal{H}_i}$ of $\mathcal{H}_i$ and the following polynomial (called *Camacho–Sad polynomial of the homogeneous foliation* $\mathcal{H}_i$):

$$\mathrm{CS}_{\mathcal{H}_i}(\lambda) = \prod_{s\in \mathrm{Sing}(\mathcal{H}_i)\cap\ell_\infty} (\lambda - \mathrm{CS}(\mathcal{H}_i,\ell_\infty,s)).$$

Table 1 below summarizes the types and Camacho–Sad polynomials of the foliations $\mathcal{H}_i$, $i=1,\ldots,14$.

i	$\mathcal{T}_{\mathcal{H}_i}$	$\mathrm{CS}_{\mathcal{H}_i}(\lambda)$
1	$2\cdot\mathrm{R}_4$	$(\lambda-1)^2\big(\lambda+\frac{1}{4}\big)^4$
2	$1\cdot\mathrm{R}_1+1\cdot\mathrm{R}_3+1\cdot\mathrm{R}_4$	$\frac{1}{491}(\lambda-1)^3(491\lambda^3+982\lambda^2+463\lambda+64)$
3	$2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4$	$(\lambda-1)^3\big(\lambda+\frac{3}{7}\big)^2\big(\lambda+\frac{8}{7}\big)$
4	$1\cdot\mathrm{R}_2+2\cdot\mathrm{R}_3$	$(\lambda-1)^3\big(\lambda+\frac{9}{11}\big)^2\big(\lambda+\frac{4}{11}\big)$
5	$2\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4$	$(\lambda-1)^4\big(\lambda+\frac{3}{2}\big)^2$
6	$2\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4$	$\frac{1}{59}(\lambda-1)^4(59\lambda^2+177\lambda+64)$
7	$2\cdot\mathrm{R}_1+2\cdot\mathrm{R}_3$	$(\lambda-1)^4(\lambda^2+3\lambda+1)$
8	$2\cdot\mathrm{R}_1+2\cdot\mathrm{R}_3$	$(\lambda-1)^4\big(\lambda+\frac{3}{2}\big)^2$
9	$1\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$	$\frac{1}{197}(\lambda-1)^4(197\lambda^2+591\lambda+302-10\sqrt{61})$
10	$1\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$	$\frac{1}{197}(\lambda-1)^4(197\lambda^2+591\lambda+302+10\sqrt{61})$
11	$4\cdot\mathrm{R}_2$	$(\lambda-1)^4\big(\lambda+\frac{3}{2}\big)^2$
12	$2\cdot\mathrm{R}_1+3\cdot\mathrm{R}_2$	$(\lambda-1)^5(\lambda+4)$
13	$4\cdot\mathrm{R}_1+1\cdot\mathrm{R}_4$	$(\lambda-1)^5(\lambda+4)$
14	$3\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$	$(\lambda-1)^5(\lambda+4)$

TABLE 1. Types and Camacho–Sad polynomials of the homogeneous foliations $\mathcal{H}_1,\ldots,\mathcal{H}_{14}$.

Proof of Theorem A: Let $\mathcal{H}$ be a homogeneous convex foliation of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$, defined in the affine chart (x,y), by the 1-form

$$\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y, \quad A,B\in\mathbb{C}[x,y]_5, \quad \gcd(A,B)=1.$$

By [**1**, Remark 2.5] the foliation $\mathcal{H}$ cannot have $5+1=6$ distinct radial singularities; in other words, it cannot be of one of the two types $5\cdot\mathrm{R}_1+1\cdot\mathrm{R}_3$ or $4\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2$. We are then in one of the following situations:

$$\begin{array}{ll}
\mathcal{T}_{\mathcal{H}}=2\cdot\mathrm{R}_4; & \mathcal{T}_{\mathcal{H}}=1\cdot\mathrm{R}_1+1\cdot\mathrm{R}_3+1\cdot\mathrm{R}_4;\\
\mathcal{T}_{\mathcal{H}}=2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4; & \mathcal{T}_{\mathcal{H}}=1\cdot\mathrm{R}_2+2\cdot\mathrm{R}_3;\\
\mathcal{T}_{\mathcal{H}}=2\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4; & \mathcal{T}_{\mathcal{H}}=2\cdot\mathrm{R}_1+2\cdot\mathrm{R}_3;\\
\mathcal{T}_{\mathcal{H}}=1\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3; & \mathcal{T}_{\mathcal{H}}=4\cdot\mathrm{R}_2;\\
\mathcal{T}_{\mathcal{H}}=2\cdot\mathrm{R}_1+3\cdot\mathrm{R}_2; & \mathcal{T}_{\mathcal{H}}=4\cdot\mathrm{R}_1+1\cdot\mathrm{R}_4;\\
\mathcal{T}_{\mathcal{H}}=3\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3. &
\end{array}$$

The proof consists of analyzing these eleven possibilities, either by applying some results in [**2**], or else by appealing to a specific classification taken from [**6**].

(1) We know from [**2**, Propositions 4.1 and 4.2] that if a homogeneous convex foliation of degree $d\geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$ is of type $2\cdot\mathrm{R}_{d-1}$, resp. $1\cdot\mathrm{R}_{\nu}+1\cdot\mathrm{R}_{d-\nu-1}+1\cdot\mathrm{R}_{d-1}$ with $\nu\in\{1,2,\ldots,d-2\}$, then it is linearly conjugated to the foliation $\mathcal{H}_1^d$, resp. $\mathcal{H}_3^{d,\nu}$, given by

$$\omega_1^d=y^d\mathrm{d}x-x^d\mathrm{d}y,\quad \text{resp. } \omega_3^{d,\nu}=\sum_{i=\nu+1}^{d}\binom{d}{i}x^{d-i}y^i\mathrm{d}x-\sum_{i=0}^{\nu}\binom{d}{i}x^{d-i}y^i\mathrm{d}y.$$

It follows that if the foliation $\mathcal{H}$ is of type $\mathcal{T}_{\mathcal{H}}=2\cdot\mathrm{R}_4$, resp. $\mathcal{T}_{\mathcal{H}}=1\cdot\mathrm{R}_1+1\cdot\mathrm{R}_3+1\cdot\mathrm{R}_4$, resp. $\mathcal{T}_{\mathcal{H}}=2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4$, then the 1-form ω is linearly conjugated to

$$\omega_1^5=y^5\mathrm{d}x-x^5\mathrm{d}y=\omega_1,$$

$$\text{resp. } \omega_3^{5,1}=\sum_{i=2}^{5}\binom{5}{i}x^{5-i}y^i\mathrm{d}x-\sum_{i=0}^{1}\binom{5}{i}x^{5-i}y^i\mathrm{d}y=\omega_2,$$

$$\text{resp. } \omega_3^{5,2}=\sum_{i=3}^{5}\binom{5}{i}x^{5-i}y^i\mathrm{d}x-\sum_{i=0}^{2}\binom{5}{i}x^{5-i}y^i\mathrm{d}y=\omega_3.$$

(2) Assume that $\mathcal{T}_{\mathcal{H}}=1\cdot\mathrm{R}_2+2\cdot\mathrm{R}_3$. This means that the rational map $\underline{\mathcal{G}}_{\mathcal{H}}\colon\mathbb{P}^1_{\mathbb{C}}\to\mathbb{P}^1_{\mathbb{C}}$, $\underline{\mathcal{G}}_{\mathcal{H}}(z)=-\frac{A(1,z)}{B(1,z)}$, possesses three fixed critical points, one of multiplicity 2 and two of multiplicity 3. By [**6**, p. 79], $\underline{\mathcal{G}}_{\mathcal{H}}$ is conjugated by a Möbius transformation to $z\mapsto-\frac{z^4(3z-5)}{5z-3}$. As a result, ω is linearly conjugated to ω_4.

(3) Let us study the possibility $\mathcal{T}_{\mathcal{H}} = 2\cdot \mathrm{R}_1 + 1\cdot \mathrm{R}_2 + 1\cdot \mathrm{R}_4$. Up to linear conjugation we can assume that, for some $\alpha \in \mathbb{C}\setminus\{0,1\}$, the points $[1:0:0],[0:1:0],[1:1:0],[1:\alpha:0]\in\mathbb{P}^2_{\mathbb{C}}$ are radial singularities of $\mathcal{H}$ with respective orders 4, 2, 1, 1 or, equivalently, that the points $\infty=[1:0],[0:1],[1:1],[1:\alpha]\in\mathbb{P}^1_{\mathbb{C}}$ are fixed and critical for $\underline{\mathcal{G}}_{\mathcal{H}}$ with respective multiplicities 4, 2, 1, 1. By [**2**, Lemma 3.9], there exist constants $a_0,a_2,b\in\mathbb{C}^*$, $a_1\in\mathbb{C}$, such that

$$B(x,y)=bx^5,\quad A(x,y)=(a_0x^2+a_1xy+a_2y^2)y^3,$$
$$(z-1)^2 \text{ divides } P(z),\quad (z-\alpha)^2 \text{ divides } Q(z),$$

where $P(z):=A(1,z)+B(1,z)$ and $Q(z):=A(1,z)+\alpha B(1,z)$. A straightforward computation leads to

$$a_0=\frac{5a_2\alpha}{3},\quad a_1=-\frac{5a_2(\alpha+1)}{4},\quad b=-\frac{a_2(5\alpha-3)}{12},\quad (\alpha+1)(3\alpha^2-5\alpha+3)=0.$$

Replacing ω by $\frac{12}{a_2}\omega$, we reduce it to

$$\omega=y^3(20\alpha x^2-15(\alpha+1)xy+12y^2)\mathrm{d}x-(5\alpha-3)x^5\mathrm{d}y,\qquad (\alpha+1)(3\alpha^2-5\alpha+3)=0.$$

This 1-form is linearly conjugated to one of the two 1-forms ω_5 or ω_6. Indeed, on the one hand, if $\alpha=-1$, then $\omega_5=-\frac{1}{4}\omega$. On the other hand, if $3\alpha^2-5\alpha+3=0$, then

$$\omega_6=\frac{121(15\alpha-16)}{81(3\alpha-8)^5}\varphi^*\omega,\quad \text{where}\quad \varphi=((3\alpha-8)x,-3y).$$

(4) Assume that $\mathcal{T}_{\mathcal{H}}=2\cdot\mathrm{R}_1+2\cdot\mathrm{R}_3$. Then the rational map $\underline{\mathcal{G}}_{\mathcal{H}}$ admits four fixed critical points, two of multiplicity 1 and two of multiplicity 3. This implies, by [**6**, p. 79], that up to conjugation by a Möbius transformation, $\underline{\mathcal{G}}_{\mathcal{H}}$ can be written as

$$z\mapsto -\frac{z^4(3z+4cz-5c-4)}{z+c},$$

where $c=-1/2\pm\sqrt{5}/10$ or $c=-3/10\pm\sqrt{21}/10$. Thus, up to linear conjugation,

$$\omega=y^4(3y+4cy-5cx-4x)\mathrm{d}x+x^4(y+cx)\mathrm{d}y,\quad c\in\left\{-\frac{1}{2}\pm\frac{\sqrt{5}}{10},-\frac{3}{10}\pm\frac{\sqrt{21}}{10}\right\}.$$

In the case where $c = -1/2 \pm \sqrt{5}/10$, resp. $c = -3/10 \pm \sqrt{21}/10$, the 1-form ω is linearly conjugated to ω_7, resp. ω_8. Indeed, on the one hand, if $c = -1/2+\sqrt{5}/10$, resp. $c = -3/10+\sqrt{21}/10$, then $\omega_7 = -2(5-2\sqrt{5})\omega$, resp. $\omega_8 = -10(9-2\sqrt{21})\omega$. On the other hand, if $c = -1/2-\sqrt{5}/10$, resp. $c = -3/10-\sqrt{21}/10$, then

$$\omega_7 = -(25+11\sqrt{5})\varphi^*\omega, \quad \text{where} \quad \varphi = \left(\tfrac{3-\sqrt{5}}{2}x, y\right),$$

$$\text{resp. } \omega_8 = 5(87+19\sqrt{21})\psi^*\omega, \quad \text{where} \quad \psi = \left(\tfrac{\sqrt{21}-5}{2}x, y\right).$$

(5) We know from [**6**, p. 79] that, up to Möbius transformation, there are two rational maps of degree 5 from the Riemann sphere to itself having four distinct fixed critical points, one of multiplicity 1, two of multiplicity 2, and one of multiplicity 3. Thus, up to automorphisms of $\mathbb{P}^2_{\mathbb{C}}$, there are two homogeneous convex foliations of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$ having type $1\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$. Now, by Table 1, we have on the one hand $\mathrm{CS}_{\mathcal{H}_9} \neq \mathrm{CS}_{\mathcal{H}_{10}}$, so that the foliations $\mathcal{H}_9$ and $\mathcal{H}_{10}$ are not linearly conjugated, and on the other hand $\mathcal{T}_{\mathcal{H}_9} = \mathcal{T}_{\mathcal{H}_{10}} = 1\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$. It follows that if the foliation $\mathcal{H}$ is of type $\mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$, then $\mathcal{H}$ is linearly conjugated to one of the two foliations $\mathcal{H}_9$ or $\mathcal{H}_{10}$.

(6) Assume that $\mathcal{T}_{\mathcal{H}} = 4\cdot\mathrm{R}_2$. The rational map $\underline{\mathcal{G}}_{\mathcal{H}}$ has therefore four different fixed critical points of multiplicity 2. By [**6**, p. 80], up to conjugation by a Möbius transformation, $\underline{\mathcal{G}}_{\mathcal{H}}$ can be written as

$$z \mapsto -\frac{z^3(z^2-5z+5)}{5z^2-10z+4}.$$

As a consequence, up to linear conjugation

$$\omega = y^3(5x^2-5xy+y^2)\mathrm{d}x + x^3(4x^2-10xy+5y^2)\mathrm{d}y.$$

This 1-form is linearly conjugated to

$$\omega_{11} = \frac{1}{8}\varphi^*\omega, \quad \text{where} \quad \varphi = (x+y, 2y).$$

(7) Assume that $\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{R}_1+3\cdot\mathrm{R}_2$. Then the rational map $\underline{\mathcal{G}}_{\mathcal{H}}$ possesses five fixed critical points, two of multiplicity 1 and three of multiplicity 2. By [**6**, p. 80], $\underline{\mathcal{G}}_{\mathcal{H}}$ is conjugated by a Möbius transformation to $z \mapsto -\frac{z^3(z^2+5z-20)}{20z^2-5z-1}$, which implies that ω is linearly conjugated to ω_{12}.

(8) Let us consider the eventuality $\mathcal{T}_{\mathcal{H}} = 4\cdot\mathrm{R}_1+1\cdot\mathrm{R}_4$. Up to isomorphism, we can assume that, for some $\alpha, \beta \in \mathbb{C}\setminus\{0,1\}$ with $\alpha \neq \beta$, the points $\infty = [1:0], [0:1], [1:1], [1:\alpha], [1:\beta] \in \mathbb{P}^1_{\mathbb{C}}$ are fixed and critical for $\underline{\mathcal{G}}_{\mathcal{H}}$,

with respective multiplicities 4, 1, 1, 1, 1. By [**2**, Lemma 3.9], there exist constants $a_0, a_3, b \in \mathbb{C}^*$, $a_1, a_2 \in \mathbb{C}$, such that

$$B(x,y) = bx^5, \quad A(x,y) = (a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3)y^2,$$
$$(z-1)^2 \text{ divides } P(z), \quad (z-\alpha)^2 \text{ divides } Q(z), \quad (z-\beta)^2 \text{ divides } R(z),$$

where $P(z) := A(1,z) + B(1,z)$, $Q(z) := A(1,z) + \alpha B(1,z)$, and $R(z) := A(1,z) + \beta B(1,z)$. A straightforward computation gives us

$$b = \frac{a_3\alpha^2(\alpha-1)^2}{2(\alpha^2-\alpha+1)}, \qquad a_0 = -\frac{a_3\alpha(\alpha+1)(3\alpha^2-5\alpha+3)}{2(\alpha^2-\alpha+1)},$$

$$a_1 = \frac{a_3(\alpha^4+2\alpha^3-3\alpha^2+2\alpha+1)}{\alpha^2-\alpha+1}, \quad \beta = \frac{(\alpha+1)(3\alpha^2-5\alpha+3)}{5(\alpha^2-\alpha+1)},$$

$$a_2 = -\frac{a_3(\alpha+1)(4\alpha^2-5\alpha+4)}{2(\alpha^2-\alpha+1)},$$
$$(\alpha^2-2\alpha+2)(2\alpha^2-2\alpha+1)(\alpha^2+1) = 0.$$

Multiplying ω by $\frac{2}{a_3}(\alpha^2-\alpha+1)$, we reduce it to

$$\begin{aligned}\omega = -y^2\big(\alpha(\alpha+1)(3\alpha^2-5\alpha+3)x^3 + (\alpha+1)(4\alpha^2-5\alpha+4)xy^2 \\ - 2(\alpha^2-\alpha+1)y^3\big)\mathrm{d}x \\ + 2(\alpha^4+2\alpha^3-3\alpha^2+2\alpha+1)x^2y^3\mathrm{d}x + \alpha^2(\alpha-1)^2x^5\mathrm{d}y,\end{aligned}$$

with $(\alpha^2-2\alpha+2)(2\alpha^2-2\alpha+1)(\alpha^2+1) = 0$. This 1-form ω is linearly conjugated to

$$\omega_{13} = -\frac{(\alpha+1)(3\alpha^2-5\alpha+3)}{5\alpha^3(\alpha-1)^4}\varphi^*\omega,$$
$$\text{where} \quad \varphi = \left(x, \frac{5\alpha(\alpha-1)^2}{(\alpha+1)(3\alpha^2-5\alpha+3)}y\right).$$

(9) Finally, let us examine the case $\mathcal{T}_{\mathcal{H}} = 3\cdot\mathrm{R}_1 + 1\cdot\mathrm{R}_2 + 1\cdot\mathrm{R}_3$. Up to linear conjugation we can assume that the points $\infty = [1:0], [0:1], [1:1], [1:\alpha], [1:\beta] \in \mathbb{P}^1_{\mathbb{C}}$, where $\alpha\beta \in \mathbb{C}\setminus\{0,1\}$ and $\alpha \neq \beta$, are fixed and critical for $\underline{\mathcal{G}}_{\mathcal{H}}$, with respective multiplicities 3, 2, 1, 1, 1. A similar reasoning as in the previous case leads to

$$\begin{aligned}\omega = \omega(\alpha) = y^3\big((\alpha^2-3\alpha+1)(\alpha^2+5\alpha+1)x^2 \\ - 2(\alpha+1)(\alpha^2-5\alpha+1)xy + (\alpha^2-7\alpha+1)y^2\big)\mathrm{d}x \\ + \alpha x^4\big(2\alpha(\alpha^2-\alpha+1)x - (\alpha+1)(3\alpha^2-5\alpha+3)y\big)\mathrm{d}y,\end{aligned}$$

with $P(\alpha)=0$ where $P(z):=3z^6-39z^5+194z^4-203z^3+194z^2-39z+3$. The 1-form ω is linearly conjugated to

$$\begin{aligned}\omega_{14}=y^3\big((\sigma^2-3\sigma+1)(\sigma^2+5\sigma+1)x^2\\ -2(\sigma+1)(\sigma^2-5\sigma+1)xy+(\sigma^2-7\sigma+1)y^2\big)\mathrm{d}x\\ +\sigma x^4\big(2\sigma(\sigma^2-\sigma+1)x-(\sigma+1)(3\sigma^2-5\sigma+3)y\big)\mathrm{d}y,\end{aligned}$$

where $\sigma=\rho+\mathrm{i}\sqrt{\frac{1}{6}-\frac{4}{3}\rho-\frac{1}{3}\rho^2}$ and ρ is the unique real number satisfying $8\rho^3-52\rho^2+134\rho-15=0$. Indeed, on the one hand, it is easy to see that σ is a root of the polynomial P, so that $\omega_{14}=\omega(\sigma)$. On the other hand, a straightforward computation shows that, if α_1 and α_2 are any two roots of P, then

$$\begin{aligned}\omega(\alpha_2)=-\frac{\mu}{21600}(13035\alpha_1^5-167802\alpha_1^4+821633\alpha_1^3-777667\alpha_1^2\\ +743778\alpha_1-76185)\varphi^*(\omega(\alpha_1))\end{aligned}$$

with $\mu=195\alpha_2^4-202\alpha_2^3+233\alpha_2^2-42\alpha_2+3$, $\varphi=\big(x,-\frac{\lambda}{43200}y\big)$, where

$$\begin{aligned}\lambda=(39\alpha_2^5-501\alpha_2^4+2447\alpha_2^3-2293\alpha_2^2+2343\alpha_2-477)\\ \times(24\alpha_1^5-309\alpha_1^4+1510\alpha_1^3-1415\alpha_1^2+1446\alpha_1-21).\end{aligned}$$

The foliations $\mathcal{H}_1,\ldots,\mathcal{H}_{14}$ are not linearly conjugated because we have $\mathcal{T}_{\mathcal{H}_i}\neq\mathcal{T}_{\mathcal{H}_j}$ or $\mathrm{CS}_{\mathcal{H}_i}\neq\mathrm{CS}_{\mathcal{H}_j}$ for all $i,j\in\{1,\ldots,14\}$ with $i\neq j$ (see Table 1). This ends the proof Theorem A. $\square$

Let $\mathcal{F}$ be a reduced convex foliation of degree $d\geq 1$ on $\mathbb{P}^2_{\mathbb{C}}$ and let ℓ be one of its $3d$ invariant lines. To the pair $(\mathcal{F},\ell)$ we can associate, thanks to [**3**], a homogeneous convex foliation $\mathcal{H}_{\mathcal{F}}^{\ell}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$, called *homogeneous degeneration of $\mathcal{F}$ along ℓ*, as follows. Let us fix homogeneous coordinates $[x:y:z]\in\mathbb{P}^2_{\mathbb{C}}$ such that $\ell=(z=0)$. Since ℓ is $\mathcal{F}$-invariant, $\mathcal{F}$ is described in the affine chart $z=1$ by a 1-form ω of type

$$\omega=\sum_{i=0}^{d}(A_i(x,y)\mathrm{d}x+B_i(x,y)\mathrm{d}y),$$

where A_i, B_i are homogeneous polynomials of degree i. By [**3**, Proposition 3.2] we have $\gcd(A_d,B_d)=1$, which allows us to define the foliation $\mathcal{H}_{\mathcal{F}}^{\ell}$ by the 1-form

$$\omega_d=A_d(x,y)\mathrm{d}x+B_d(x,y)\mathrm{d}y.$$

It is easy to check that this definition is intrinsic, i.e. it does not depend on the choice of the homogeneous coordinates $[x:y:z]$ nor on the choice of the 1-form ω describing $\mathcal{F}$.

The following result, taken from [**3**, Proposition 3.2], will be very useful to us.

Proposition 2.1 ([**3**]). *With the previous notations, the foliation $\mathcal{H}_{\mathcal{F}}^{\ell}$ has the following properties:*

(i) *$\mathcal{H}_{\mathcal{F}}^{\ell}$ belongs to the Zariski closure of the* $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$*-orbit of $\mathcal{F}$;*
(ii) *ℓ is invariant by $\mathcal{H}_{\mathcal{F}}^{\ell}$;*
(iii) $\mathrm{Sing}(\mathcal{H}_{\mathcal{F}}^{\ell}) \cap \ell = \mathrm{Sing}(\mathcal{F}) \cap \ell$*;*
(iv) *every singular point of $\mathcal{H}_{\mathcal{F}}^{\ell}$ on ℓ is non-degenerate;*
(v) *a point $s \in \ell$ is a radial singularity of order $k \leq d-1$ for $\mathcal{H}_{\mathcal{F}}^{\ell}$ if and only if it is for $\mathcal{F}$;*
(vi) $\forall\, s \in \mathrm{Sing}(\mathcal{H}_{\mathcal{F}}^{\ell}) \cap \ell$, $\mathrm{CS}(\mathcal{H}_{\mathcal{F}}^{\ell}, \ell, s) = \mathrm{CS}(\mathcal{F}, \ell, s)$.

Proof of Proposition C: Since by hypothesis $\mathcal{F}$ is reduced convex, all its singularities are non-degenerate ([**2**, Lemma 6.8]). Let ℓ be an invariant line of $\mathcal{F}$. By [**4**, Proposition 2.3] it follows that $\mathcal{F}$ possesses exactly $d+1$ singularities on ℓ. The Camacho–Sad formula (see [**5**]) $\sum\limits_{s \in \mathrm{Sing}(\mathcal{F}) \cap \ell} \mathrm{CS}(\mathcal{F}, \ell, s) = 1$ then implies the existence of $s \in \mathrm{Sing}(\mathcal{F}) \cap \ell$ such that $\mathrm{CS}(\mathcal{F}, \ell, s) \in \mathbb{C} \setminus \{0,1\}$; as a result $\mathcal{CS}(\mathcal{F}) \neq \emptyset$.

Let $\lambda \in \mathcal{CS}(\mathcal{F}) \subset \mathbb{C} \setminus \{0,1\}$. There is a line ℓ_1 invariant by $\mathcal{F}$ and a singular point $s \in \ell_1$ of $\mathcal{F}$ such that $\mathrm{CS}(\mathcal{F}, \ell_1, s) = \lambda$. By [**3**, Lemma 3.1], through the point s passes a second $\mathcal{F}$-invariant line ℓ_2. Since $\mathrm{CS}(\mathcal{F}, \ell_1, s)\mathrm{CS}(\mathcal{F}, \ell_2, s) = 1$, we have $\mathrm{CS}(\mathcal{F}, \ell_2, s) = \frac{1}{\lambda}$; thus $\frac{1}{\lambda} \in \mathcal{CS}(\mathcal{F})$. Moreover, by [**3**, Proposition 3.2] (cf. assertion (vi) of Proposition 2.1 above), we have

$$\mathrm{CS}(\mathcal{H}_{\mathcal{F}}^{\ell_1}, \ell_1, s) = \mathrm{CS}(\mathcal{F}, \ell_1, s) = \lambda \quad \text{and} \quad \mathrm{CS}(\mathcal{H}_{\mathcal{F}}^{\ell_2}, \ell_2, s) = \mathrm{CS}(\mathcal{F}, \ell_2, s) = \frac{1}{\lambda},$$

which shows that $\lambda \in \mathcal{HCS}_d$, and hence $\mathcal{CS}(\mathcal{F}) \subset \mathcal{HCS}_d$. $\square$

An immediate consequence of Table 1 is the following:

Corollary 2.2. $\mathcal{HCS}_5 = \left\{-4^{\pm 1}, -\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right\} = \mathcal{CS}(\mathcal{F}_0^5) \cup \mathcal{CS}(\mathcal{F}_H^5)$.

The proof of Theorem D uses Lemmas 2.3 and 2.4 stated below.

Lemma 2.3. *Let $\mathcal{F}$ be a foliation of degree $d \geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$ having two radial singularities m_1, m_2 of maximal order $d-1$. Then the line (m_1m_2) cannot contain a third radial singularity of $\mathcal{F}$.*

Proof: Let us choose homogeneous coordinates $[x:y:z] \in \mathbb{P}^2_{\mathbb{C}}$ such that $m_1 = [0:1:0]$ and $m_2 = [1:0:0]$. Thanks to [**4**, Proposition 2.2] (cf. [**1**, Remark 1.2]), the line $\ell = (m_1m_2)$ must be invariant by $\mathcal{F}$. Then the foliation $\mathcal{F}$ is given in the affine chart $z=1$ by a 1-form ω of type $\omega = \omega_0 + \omega_1 + \cdots + \omega_d$, where, for $0 \leq i \leq d$, $\omega_i = A_i(x,y)\mathrm{d}x + B_i(x,y)\mathrm{d}y$, with A_i, B_i homogeneous polynomials of degree i.

Writing explicitly that the points m_j, $j = 1, 2$, are radial singularities of maximal order $d-1$ of $\mathcal{F}$ (see [**2**, Proposition 6.3]), we obtain that the highest degree homogeneous part ω_d of ω is of the form $\omega_d = ay^d\mathrm{d}x + bx^d\mathrm{d}y$, with $a, b \in \mathbb{C}^*$. Thus, ω_d defines a homogeneous convex foliation $\mathcal{H}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$ of type $\mathcal{T}_{\mathcal{H}} = 2 \cdot \mathrm{R}_{d-1}$. If we would know that $\mathcal{F}$ was a convex reduced foliation, then $\mathcal{H} = \mathcal{H}^{\ell}_{\mathcal{F}}$ for the invariant line $\ell = (m_1m_2)$ and we could apply Proposition 2.1 to conclude. Anyway, reasoning as in the proof of [**2**, Proposition 6.4], we see that $\mathcal{F}$ and $\mathcal{H}$ have the same singularities on the line (m_1m_2) and that every singularity s of $\mathcal{F}$ on (m_1m_2) distinct from m_1 and m_2 is non-degenerate and has Camacho–Sad index $\mathrm{CS}(\mathcal{F}, (m_1m_2), s) = \mathrm{CS}(\mathcal{H}, (m_1m_2), s) = \frac{1}{1-d} \neq 1$, hence the lemma follows. □

Lemma 2.4. *Let $\mathcal{H}$ be a homogeneous convex foliation of degree $d \geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$. Assume that every non radial singularity s of $\mathcal{H}$ on ℓ_∞ has Camacho–Sad index $\mathrm{CS}(\mathcal{H}, \ell_\infty, s) \in \{(1-d)^{\pm 1}\}$. Denote by κ_0 the number of (distinct) radial singularities of $\mathcal{H}$ and by κ_1 (resp. κ_2) the number of singularities $s \in \ell_\infty$ of $\mathcal{H}$ such that $\mathrm{CS}(\mathcal{H}, \ell_\infty, s) = 1-d$ (resp. $\mathrm{CS}(\mathcal{H}, \ell_\infty, s) = \frac{1}{1-d}$). Then*

- *either $(\kappa_0, \kappa_1, \kappa_2) = (d, 1, 0)$;*
- *or $(\kappa_0, \kappa_1, \kappa_2) = (2, 0, d-1)$, in which case $\mathcal{T}_{\mathcal{H}} = 2 \cdot \mathrm{R}_{d-1}$.*

Before proving this lemma let us make two remarks:

Remark 2.5. By [**7**, Theorem 4.3], every homogeneous convex foliation of degree d on the complex projective plane has exactly $d+1$ singularities on the line at infinity, necessarily non-degenerate.

Remark 2.6. A straightforward computation shows that, if a homogeneous foliation $\mathcal{H}$ on $\mathbb{P}^2_{\mathbb{C}}$ possesses a non-degenerate singularity $s \in \ell_\infty$ such that $\mathrm{CS}(\mathcal{H}, \ell_\infty, s) = 1$, then s is necessarily radial. In particular, when $\mathcal{H}$ is convex, a singularity $s \in \ell_\infty$ of $\mathcal{H}$ is radial if and only if it has Camacho–Sad index $\mathrm{CS}(\mathcal{H}, \ell_\infty, s) = 1$.

Proof of Lemma 2.4: The Camacho–Sad formula

$$\sum_{s \in \mathrm{Sing}(\mathcal{H}) \cap \ell_\infty} \mathrm{CS}(\mathcal{H}, \ell_\infty, s) = 1$$

(see [**5**]) and Remarks 2.5 and 2.6 imply that

$$\kappa_0 + \kappa_1 + \kappa_2 = d + 1 \quad \text{and} \quad \kappa_0 + (1-d)\kappa_1 + \frac{\kappa_2}{1-d} = 1.$$

From these two equations we obtain $\kappa_0 = 2 + \kappa_1(d-2)$ and $\kappa_2 = (d-1)(1-\kappa_1) \geq 0$, so that $\kappa_1 \in \{0, 1\}$, as required. □

Proof of Theorem D: The implication (iii) $\Rightarrow$ (i) follows from [**2**, Proposition 6.3] and from Lemma 2.3.

The fact that (i) implies (ii) follows from the reduced convexity of the foliation $\mathcal{F}_0^d$ and from the equality $\mathcal{CS}(\mathcal{F}_0^d) = \{(1-d)^{\pm 1}\}$ (Remark 1.1).

Let us show that (ii) implies (iii). Assume that $\mathcal{F}$ is reduced convex and that $\mathcal{CS}(\mathcal{F}) = \{(1-d)^{\pm 1}\}$. Let m be a non radial singular point of $\mathcal{F}$; through m pass exactly two $\mathcal{F}$-invariant lines $\ell_m^{(1)}$ and $\ell_m^{(2)}$ ([**3**, Lemma 3.1]). It follows that $\mathrm{CS}(\mathcal{F}, \ell_m^{(i)}, m) = (1-d)^{\pm 1}$ for $i = 1, 2$. Up to renumbering the $\ell_m^{(i)}$, we can assume that $\mathrm{CS}(\mathcal{F}, \ell_m^{(1)}, m) = \frac{1}{1-d}$ and $\mathrm{CS}(\mathcal{F}, \ell_m^{(2)}, m) = 1 - d$ for any choice of the non radial singularity $m \in \operatorname{Sing}\mathcal{F}$. Moreover, according to Proposition 2.1, for any invariant line ℓ of $\mathcal{F}$ and for any non radial singularity $s \in \ell$ of the homogeneous degeneration $\mathcal{H}_{\mathcal{F}}^{\ell}$ of $\mathcal{F}$ along ℓ, we have $\mathrm{CS}(\mathcal{H}_{\mathcal{F}}^{\ell}, \ell, s) = \mathrm{CS}(\mathcal{F}, \ell, s) \in \mathbb{C} \setminus \{0, 1\}$ and therefore $\mathrm{CS}(\mathcal{H}_{\mathcal{F}}^{\ell}, \ell, s) \in \{(1-d)^{\pm 1}\}$. It follows by Lemma 2.4 that $\mathcal{H}_{\mathcal{F}}^{\ell_m^{(1)}}$ is of type $2\cdot\mathrm{R}_{d-1}$. This implies, according to assertion (v) of Proposition 2.1, that $\mathcal{F}$ possesses two radial singularities m_1, m_2 of maximal order $d-1$ on the line $\ell_m^{(1)}$. Let m' be another non radial singular point of $\mathcal{F}$ not belonging to the line $\ell_m^{(1)}$. As in [**2**, Section 1], for any $s \in \operatorname{Sing}\mathcal{F}$ let us denote by $\tau(\mathcal{F}, s)$ the tangency order of $\mathcal{F}$ with a generic line passing through s. For $i = 1, 2$ we have $\tau(\mathcal{F}, m') + \tau(\mathcal{F}, m_i) = 1 + d > \deg \mathcal{F}$, which implies (cf. [**4**, Proposition 2.2]) that the lines $(m'm_i)$ are invariant by $\mathcal{F}$. Thus, the line $\ell_{m'}^{(1)}$ is one of the lines $(m'm_1)$ or $(m'm_2)$ and it contains in turn another radial singularity m_3 of maximal order $d-1$ of $\mathcal{F}$. $\square$

The proof of Theorem F uses the following lemma for $d = 5$, which we state in arbitrary degree d as it could be used in other situations. It can be proved in the same way as in [**2**, Proposition 6.3].

Lemma 2.7. *Let $\mathcal{F}$ be a foliation of degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$. Assume that the points $m_1 = [0:0:1]$, $m_2 = [1:0:0]$, and $m_3 = [0:1:0]$ are radial singularities of order $d-2$ of $\mathcal{F}$. Let ω be a 1-form defining $\mathcal{F}$ in the affine chart $z = 1$. Then ω is of the form*

$$\begin{aligned} \omega = {} & (x\mathrm{d}y - y\mathrm{d}x)(\lambda_{0,0} + \lambda_{1,0}x + \lambda_{0,1}y + \lambda_{1,1}xy) \\ & + y^{d-2}(a_{1,0}x + a_{0,1}y + a_{1,1}xy + a_{0,2}y^2)\mathrm{d}x \\ & + x^{d-2}(b_{1,0}x + b_{0,1}y + b_{1,1}xy + b_{2,0}x^2)\mathrm{d}y, \end{aligned}$$

where $\lambda_{i,j}, a_{i,j}, b_{i,j} \in \mathbb{C}$ with $\lambda_{0,0} \neq 0$.

Proof of Theorem F: The implication (i) $\Rightarrow$ (ii) follows from the reduced convexity of the foliation $\mathcal{F}_H^5$ and from the equality $\mathcal{CS}(\mathcal{F}_H^5) = \{-\frac{3}{2} \pm \frac{\sqrt{5}}{2}\}$ (Remark 1.1).

Let us show that (ii) implies (iii). Assume that $\mathcal{F}$ is reduced convex and that $\mathcal{CS}(\mathcal{F}) = \{-\frac{3}{2} \pm \frac{\sqrt{5}}{2}\}$. Let ℓ be an invariant line of $\mathcal{F}$. The homogeneous foliation $\mathcal{H}_{\mathcal{F}}^{\ell}$ (homogeneous degeneration of $\mathcal{F}$ along ℓ), being convex of degree 5, must be linearly conjugated to one of the fourteen homogeneous foliations given by Theorem A. Moreover, let m be a non radial singular point of $\mathcal{F}$ on ℓ. Then we have $\mathrm{CS}(\mathcal{F}, \ell, m) = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}$. According to Proposition 2.1, the point m is also a non radial singularity for $\mathcal{H}_{\mathcal{F}}^{\ell}$ and we have $\mathrm{CS}(\mathcal{H}_{\mathcal{F}}^{\ell}, \ell, m) = \mathrm{CS}(\mathcal{F}, \ell, m) = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}$. It then follows from Table 1 that $\mathcal{H}_{\mathcal{F}}^{\ell}$ is of type $2 \cdot \mathrm{R}_1 + 2 \cdot \mathrm{R}_3$. This implies, according to assertion (v) of Proposition 2.1, that $\mathcal{F}$ has exactly four radial singularities on the line ℓ; two of them m_1, m_2 are of order 3 and the other two are of order 1. Let us consider another $\mathcal{F}$-invariant line $\ell' \neq \ell$ passing through m_1, whose existence is guaranteed by [**3**, Lemma 3.1]. Then ℓ' contains another radial singularity m_3 of order 3 of $\mathcal{F}$ and two radial singularities of order 1 of $\mathcal{F}$. By [**4**, Proposition 2.2], the fact that $\tau(\mathcal{F}, m_2) + \tau(\mathcal{F}, m_3) = 4 + 4 > \deg \mathcal{F}$ ensures the $\mathcal{F}$-invariance of the line $\ell'' = (m_2 m_3)$. Therefore ℓ'' in turn contains two radial singularities of order 1 of $\mathcal{F}$.

Finally, let us prove that (iii) implies (i). Assume that (iii) holds. Then there is a homogeneous coordinate system $[x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$ in which $m_1 = [0:0:1]$, $m_2 = [1:0:0]$, and $m_3 = [0:1:0]$. Moreover, in this coordinate system, the lines $x = 0$, $y = 0$, $z = 0$ must be invariant by $\mathcal{F}$ and there exist $x_0, y_0, z_0, x_1, y_1, z_1 \in \mathbb{C}^*$, $x_1 \neq x_0$, $y_1 \neq y_0$, $z_1 \neq z_0$, such that the points $m_4 = [x_0 : 0 : 1]$, $m_5 = [1 : y_0 : 0]$, $m_6 = [0 : 1 : z_0]$, $m_7 = [x_1 : 0 : 1]$, $m_8 = [1 : y_1 : 0]$, and $m_9 = [0 : 1 : z_1]$ are radial singularities of order 1 of $\mathcal{F}$. Let us set $\xi = \frac{x_1}{x_0}$, $\rho = \frac{y_1}{y_0}$, $\sigma = \frac{z_1}{z_0}$, $w_0 = x_0 y_0 z_0$. Then $w_0 \in \mathbb{C}^*$, $\xi, \rho, \sigma \in \mathbb{C} \setminus \{0, 1\}$, and, up to renumbering the x_i, y_i, z_i, we can assume that ξ, ρ, and σ are all of modulus greater than or equal to 1. Let ω be a 1-form defining $\mathcal{F}$ in the affine chart $z = 1$. By conjugating ω by the diagonal linear transformation $(x_0 x, x_0 y_0 y)$, we reduce ourselves to $m_4 = [1:0:1]$, $m_5 = [1:1:0]$, $m_6 = [0:1:w_0]$, $m_7 = [\xi:0:1]$, $m_8 = [1:\rho:0]$, and $m_9 = [0:1:\sigma w_0]$. Since m_1, m_2, and m_3 are radial singularities of order 3, ω can be written as in the expression given in Lemma 2.7 in the case $d = 5$. Then, as in the proof of [**3**, Theorem B], by writing explicitly that the points m_j, $4 \leq j \leq 9$, are radial singularities of order 1 of $\mathcal{F}$ we obtain that $w_0 = \pm(\sqrt{5}-2)$ and

$$\xi = \rho = \sigma = \frac{3}{2} + \frac{\sqrt{5}}{2}, \qquad a_{1,0} = (9+4\sqrt{5})(5w_0+5-2\sqrt{5})a_{0,2},$$
$$a_{0,1} = -\frac{25+11\sqrt{5}}{2}a_{0,2}w_0, \qquad a_{1,1} = -\frac{5+\sqrt{5}}{2}a_{0,2},$$
$$b_{1,0} = \frac{25+11\sqrt{5}}{2}a_{0,2}, \qquad b_{0,1} = -\frac{(65+29\sqrt{5})(w_0+5-2\sqrt{5})}{2}a_{0,2},$$
$$b_{1,1} = (5+2\sqrt{5})a_{0,2}, \qquad b_{2,0} = -\frac{7+3\sqrt{5}}{2}a_{0,2},$$
$$\lambda_{0,0} = \frac{47+21\sqrt{5}}{2}a_{0,2}, \qquad \lambda_{1,0} = -\frac{65+29\sqrt{5}}{2}a_{0,2},$$
$$\lambda_{0,1} = -(85+38\sqrt{5})a_{0,2}w_0, \qquad \lambda_{1,1} = \frac{(47+21\sqrt{5})(5w_0+5-2\sqrt{5})}{2}a_{0,2},$$

with $a_{0,2} \neq 0$. Thus ω is of the form

$$\begin{aligned}\omega = {}& \frac{a_{0,2}(47+21\sqrt{5})}{4}(x\mathrm{d}y - y\mathrm{d}x) \\ & \times \big(2-(5-\sqrt{5})x - w_0(5+\sqrt{5})y + (10w_0+10-4\sqrt{5})xy\big) \\ & + \frac{a_{0,2}}{2}y^3\big((9+4\sqrt{5})(10w_0+10-4\sqrt{5})x - w_0(25+11\sqrt{5})y \\ & \qquad - (5+\sqrt{5})xy + 2y^2\big)\mathrm{d}x \\ & + \frac{a_{0,2}}{2}x^3\big((25+11\sqrt{5})x - (65+29\sqrt{5})(w_0+5-2\sqrt{5})y \\ & \qquad - (7+3\sqrt{5})x^2 + (10+4\sqrt{5})xy\big)\mathrm{d}y.\end{aligned}$$

The 1-form ω is linearly conjugated to

$$\begin{aligned}\omega_H^5 = {}& (y^2-1)(y^2-(\sqrt{5}-2)^2)(y+\sqrt{5}x)\mathrm{d}x \\ & - (x^2-1)(x^2-(\sqrt{5}-2)^2)(x+\sqrt{5}y)\mathrm{d}y.\end{aligned}$$

Indeed, if $w_0 = \sqrt{5}-2$, resp. $w_0 = 2-\sqrt{5}$, then

$$\omega_H^5 = \frac{32(3571-1597\sqrt{5})}{a_{0,2}}\varphi_1^*\omega,$$

where $\varphi_1 = \Big(\frac{3+\sqrt{5}}{4}(x+1), -\frac{2+\sqrt{5}}{2}(y-1)\Big)$,

$$\text{resp. } \omega_H^5 = \frac{32(64079-28657\sqrt{5})}{a_{0,2}}\varphi_2^*\omega,$$

where $\varphi_2 = \Big(\frac{2+\sqrt{5}}{2}(x+\sqrt{5}-2), -\frac{7+3\sqrt{5}}{4}(y+\sqrt{5}-2)\Big)$. $\square$

Proof of Theorem G: Let $\mathcal{F}$ be a reduced convex foliation of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$. By assertion (i) of Proposition C and Corollary 2.2 we have $\emptyset \neq \mathcal{CS}(\mathcal{F}) \subset \mathcal{HCS}_5 = \left\{-4^{\pm 1}, -\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right\}$. Hence, according to assertion (ii) of Proposition C, one of the following three possibilities does occur:

(i) $\mathcal{CS}(\mathcal{F}) = \{-4^{\pm 1}\}$;

(ii) $\mathcal{CS}(\mathcal{F}) = \left\{-\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right\}$;

(iii) $\mathcal{CS}(\mathcal{F}) = \left\{-4^{\pm 1}, -\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right\}$.

In case (i) (resp. (ii)) the foliation $\mathcal{F}$ is linearly conjugated to $\mathcal{F}_0^5$ (resp. $\mathcal{F}_H^5$), thanks to Theorem D (resp. Theorem F). To establish the theorem, it therefore suffices to exclude the possibility (iii). Let us assume by contradiction that (iii) happens. Then $\mathcal{F}$ possesses two invariant lines ℓ, ℓ' and two non radial singularities $m \in \ell$, $m' \in \ell'$ such that $\mathrm{CS}(\mathcal{F}, \ell, m) = -\frac{1}{4}$ and $\mathrm{CS}(\mathcal{F}, \ell', m') = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}$. According to Proposition 2.1, the point m (resp. m') is also a non radial singularity for the homogeneous foliation $\mathcal{H}_{\mathcal{F}}^{\ell}$ (resp. $\mathcal{H}_{\mathcal{F}}^{\ell'}$) and we have

$$\mathrm{CS}(\mathcal{H}_{\mathcal{F}}^{\ell}, \ell, m) = \mathrm{CS}(\mathcal{F}, \ell, m) = -\tfrac{1}{4} \qquad \text{and}$$
$$\mathrm{CS}(\mathcal{H}_{\mathcal{F}}^{\ell'}, \ell', m') = \mathrm{CS}(\mathcal{F}, \ell', m') = -\tfrac{3}{2} \pm \tfrac{\sqrt{5}}{2}.$$

Moreover, as in the proof of Theorem F, each of the foliations $\mathcal{H}_{\mathcal{F}}^{\ell}$ and $\mathcal{H}_{\mathcal{F}}^{\ell'}$ is linearly conjugated to one of the fourteen homogeneous foliations given by Theorem A. It then follows from Table 1 that $\mathcal{H}_{\mathcal{F}}^{\ell}$ and $\mathcal{H}_{\mathcal{F}}^{\ell'}$ are respectively of types $2 \cdot \mathrm{R}_4$ and $2 \cdot \mathrm{R}_1 + 2 \cdot \mathrm{R}_3$. This implies, according to assertion (v) of Proposition 2.1, that $\mathcal{F}$ admits two radial singularities of order 4 on the line ℓ and four radial singularities on the line ℓ', two of order 1 and two of order 3. Let m_1 (resp. m_2) be a radial singularity of order 4 (resp. 3) of $\mathcal{F}$ on the line ℓ (resp. ℓ'). Since $\tau(\mathcal{F}, m_1) + \tau(\mathcal{F}, m_2) = 5 + 4 > \deg \mathcal{F}$, the line $\ell'' = (m_1 m_2)$ is invariant by $\mathcal{F}$ (cf. [**4**, Proposition 2.2]). The homogeneous foliation $\mathcal{H}_{\mathcal{F}}^{\ell''}$ being convex of degree 5, it must therefore be of type $1 \cdot \mathrm{R}_1 + 1 \cdot \mathrm{R}_3 + 1 \cdot \mathrm{R}_4$ so that it possesses a non radial singularity m'' on the line ℓ'' satisfying (see Table 1)

$$\mathrm{CS}(\mathcal{H}_{\mathcal{F}}^{\ell''}, \ell'', m'') = \mathrm{CS}(\mathcal{F}, \ell'', m'') = \lambda,$$

with $491\lambda^3 + 982\lambda^2 + 463\lambda + 64 = 0$ which is impossible. $\square$

3. Conjectures

The notion of convex reduced foliation has an interesting relation with certain line arrangements in $\mathbb{P}^2_{\mathbb{C}}$. Indeed, according to [**11**] we say that an arrangement $\mathcal{A}$ of $3d$ lines in $\mathbb{P}^2_{\mathbb{C}}$ has *Hirzebruch's property* if each line of $\mathcal{A}$ intersects the other lines of $\mathcal{A}$ in exactly $d+1$ points. The $3d$ invariant lines of a reduced convex foliation of degree d on $\mathbb{P}^2_{\mathbb{C}}$ form a line arrangement which satisfies Hirzebruch's property, thanks to [**2**, Lemma 6.8] and [**3**, Lemma 3.1]. The expected conjectural picture for the reduced convex foliations on $\mathbb{P}^2_{\mathbb{C}}$ is the following: besides the Fermat foliations $\mathcal{F}_0^d$, with $\mathcal{CS}(\mathcal{F}_0^d) = \{(1-d)^{\pm 1}\}$, there exist special reduced convex foliations only for $d = 4, 5$ and $d = 7$, namely, the Hesse pencil in degree 4, the Hilbert foliation of degree 5, and the foliation of degree 7 related to the extended Hesse arrangement presented in the introduction, for which

$$\mathcal{CS}(\mathcal{F}_H^d) = \begin{cases} \{-1\} & \text{for } d = 4, \\ \left\{-\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right\} & \text{for } d = 5, \\ \left\{-\left(\frac{3}{4}\right)^{\pm 1}\right\} & \text{for } d = 7, \end{cases}$$

i.e. we expect that there are no other convex reduced foliations on $\mathbb{P}^2_{\mathbb{C}}$ and for this reason we propose:

Conjecture 3.1. *We have*

$$\mathcal{HCS}_d = \begin{cases} \{(1-d)^{\pm 1}\} & \textit{for } 2 \leq d \neq 4, 5, 7, \\ \{(1-d)^{\pm 1}\} \cup \mathcal{CS}(\mathcal{F}_H^d) & \textit{for } d = 4, 5, 7. \end{cases}$$

This conjecture, combined with Corollary E, would imply a negative answer in degree $d \neq 7$ to [**9**, Problem 9.1] as we have already shown for $d \leq 5$.

To every rational map $f \colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ and to every integer $d \geq 2$, we associate respectively the following subsets of $\mathbb{C} \setminus \{0, 1\}$:

- $\mathcal{M}(f)$ is, by definition, the set of $\mu \in \mathbb{C} \setminus \{0, 1\}$ such that there is a fixed point p of f satisfying $f'(p) = \mu$;
- $\mathcal{M}_d$ is defined as the set of $\mu \in \mathbb{C} \setminus \{0, 1\}$ for which there exist critically fixed rational maps $f_1, f_2 \colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ of degree d having respective fixed points p_1 and p_2 such that $f_1'(p_1) = \mu$ and $f_2'(p_2) = \frac{\mu}{\mu-1}$.

The introduction of the sets $\mathcal{M}(f)$ and $\mathcal{M}_d$ is motivated by the following remark.

Remark 3.2. Let $\mathcal{H}$ be a homogeneous foliation of degree d on $\mathbb{P}^2_{\mathbb{C}}$. According to [**2**, Section 2], the point $s=[b:a:0]\in\ell_\infty$ is a non-degenerate singularity of $\mathcal{H}$ if and only if the point $p=[a:b]\in\mathbb{P}^1_{\mathbb{C}}$ is fixed by $\underline{\mathcal{G}}_{\mathcal{H}}$ with multiplier $\underline{\mathcal{G}}'_{\mathcal{H}}(p)\neq 1$, in which case the Camacho–Sad index $\mathrm{CS}(\mathcal{H},\ell_\infty,s)$ coincides with the index $\imath(\underline{\mathcal{G}}_{\mathcal{H}},p)$ of $\underline{\mathcal{G}}_{\mathcal{H}}$ at the fixed point p:

$$\mathrm{CS}(\mathcal{H},\ell_\infty,s)=\imath(\underline{\mathcal{G}}_{\mathcal{H}},p):=\frac{1}{2\mathrm{i}\pi}\int_{|z-p|=\varepsilon}\frac{\mathrm{d}z}{z-\underline{\mathcal{G}}_{\mathcal{H}}(z)}=\frac{1}{1-\underline{\mathcal{G}}'_{\mathcal{H}}(p)}.$$

Thus, the map $\mu\mapsto\frac{1}{1-\mu}$ sends $\mathcal{M}(\underline{\mathcal{G}}_{\mathcal{H}})$ (resp. $\mathcal{M}_d$) bijectively onto $\mathcal{CS}(\mathcal{H})$ (resp. $\mathcal{HCS}_d$).

Using the above definition of the sets $\mathcal{M}(f)$, Theorem 4.3 of [**7**] can be reformulated as follows:

Theorem 3.3 (Crane, [**7**])**.** *Let $f\colon\mathbb{P}^1_{\mathbb{C}}\to\mathbb{P}^1_{\mathbb{C}}$ be a critically fixed rational map of degree $d\geq 2$. Let $n\leq d$ denote the number of (distinct) critical points of f. Then*

(i) *f has exactly $d+1$ fixed points, of which $d+1-n$ are non-critical;*
(ii) *the set $\mathcal{M}(f)$ is contained in $\mathbb{C}\setminus\big(\overline{\mathbb{D}}\,(0,1)\cup\mathbb{D}\,(1+\rho,\rho)\big)$, where $\overline{\mathbb{D}}(0,1)$ denotes the closed unit disk of $\mathbb{C}$ and $\mathbb{D}(1+\rho,\rho)\subset\mathbb{C}$ the open disk of radius $\rho=\frac{1}{d+n-2}$ and center $1+\rho$. Moreover, $\mu\in\mathcal{M}(f)$ belongs to the boundary of the disk $\mathbb{D}(1+\rho,\rho)$ if and only if $n=d$, in which case $\mu=\frac{d}{d-1}$.*

This theorem translates in terms of homogeneous foliations as follows:

Corollary 3.4. *Let $\mathcal{H}$ be a homogeneous convex foliation of degree $d\geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$. Let $n=\deg\mathcal{T}_{\mathcal{H}}$ denote the number of (distinct) radial singularities of $\mathcal{H}$. Then*

(i) *$\mathcal{H}$ has exactly $d+1$ singularities on the line at infinity, of which $d+1-n$ are non radial;*
(ii) *for any non radial singularity $s\in\ell_\infty$ of $\mathcal{H}$, we have*
$$-\frac{1}{2}<-\mathrm{Re}(\mathrm{CS}(\mathcal{H},\ell_\infty,s))\leq\frac{d+n}{2}-1.$$
This last inequality is an equality if and only if $n=d$, in which case $\mathrm{CS}(\mathcal{H},\ell_\infty,s)=1-d$.

With the notations of Corollary 3.4, since $n\leq d$ we have in particular $-\frac{1}{2}<-\mathrm{Re}(\mathrm{CS}(\mathcal{H},\ell_\infty,s))\leq d-1$. According to Remark 1.1, the value d–1 is attained by $(\mathcal{H},s)\mapsto-\mathrm{Re}(\mathrm{CS}(\mathcal{H},\ell_\infty,s))$. However, after having checked many examples, we think that the lower bound $-\frac{1}{2}$ of $-\mathrm{Re}(\mathrm{CS}(\mathcal{H},\ell_\infty,s))$ is not optimal and we propose the following conjecture with the value $\frac{1}{d-1}$ which is also attained by $(\mathcal{H},s)\mapsto-\mathrm{Re}(\mathrm{CS}(\mathcal{H},\ell_\infty,s))$ (Remark 1.1).

Conjecture 3.5. *If $\mathcal{H}$ is a homogeneous convex foliation of degree $d \geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$, then for any non radial singularity $s \in \ell_\infty$ of $\mathcal{H}$ we have $\frac{1}{d-1} \leq -\mathrm{Re}\big(\mathrm{CS}(\mathcal{H}, \ell_\infty, s)\big)$. Alternatively, if $f\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is a critically fixed rational map of degree $d \geq 2$, then the set $\mathcal{M}(f)$ is contained in the closed disk $\overline{\mathbb{D}}\big(\frac{d+1}{2}, \frac{d-1}{2}\big) \subset \mathbb{C}$ of center $\frac{d+1}{2}$ and radius $\frac{d-1}{2}$ (see Figure 2).*

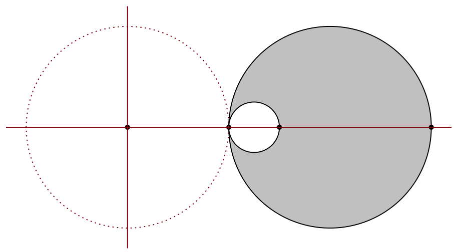

Figure 2. The set $\mathcal{M}(f)$ is conjectured to be contained in the grey region for any critically fixed rational map $f\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ of degree d. It is known that it is contained in the exterior of the union of the dashed circle and the inner white disk. The black points from left to right are 0, 1, $\frac{d}{d-1}$, and d. Conjecture 3.1 for $2 \leq d \neq 4, 5, 7$ is equivalent to the statement $\mathcal{M}_d = \big\{\frac{d}{d-1}, d\big\}$.

This conjecture is also motivated by the following remark:

Remark 3.6. If Conjecture 3.5 is true, Conjecture 3.1 claims that in degree $2 \leq d \neq 4, 5, 7$ the set $\mathcal{HCS}_d$ consists of the extreme values of $-\mathrm{Re}(\mathrm{CS}(\mathcal{H}, \ell_\infty, s))$ when $\mathcal{H}$ runs through the set of homogeneous convex foliations of degree d on $\mathbb{P}^2_{\mathbb{C}}$ and s runs through the set of non radial singularities of $\mathcal{H}$ on the line ℓ_∞.

Elementary computations, using the normal forms of homogeneous convex foliations of degree $d \in \{2, 3, 4, 5\}$ on $\mathbb{P}^2_{\mathbb{C}}$ presented in [**8**, Proposition 7.4], [**1**, Corollary C], [**3**, Theorem A], and in Theorem A, show the validity of Conjecture 3.1 for $d \in \{2, 3\}$ and Conjecture 3.5 for $d \in \{2, 3, 4, 5\}$. Moreover, very long computations carried out with `Maple` by the first author give forty nine normal forms for homogeneous convex foliations of degree 6 on $\mathbb{P}^2_{\mathbb{C}}$ and allow to verify the validity of Conjectures 3.1 and 3.5 for $d = 6$. The more difficult case $d = 7$ is out of reach at this moment.

References

[1] S. Bedrouni, Feuilletages de degré trois du plan projectif complexe ayant une transformée de Legendre plate, Thesis (Ph.D.)-University of Sciences and Technology Houari Boumediene (2017). Available on `arXiv:1712.03895`.

[2] S. Bedrouni and D. Marín, Tissus plats et feuilletages homogènes sur le plan projectif complexe, *Bull. Soc. Math. France* **146(3)** (2018), 479–516. DOI: 10.24033/bsmf.2764.

[3] S. Bedrouni and D. Marín, Convex foliations of degree 4 on the complex projective plane, *Math. Z.* **295(1–2)** (2020), 381–394. DOI: 10.1007/s00209-019-02356-z.

[4] M. Brunella, *"Birational Geometry of Foliations"*, IMPA Monographs **1**, Springer, Cham, 2015. DOI: 10.1007/978-3-319-14310-1.

[5] C. Camacho and P. Sad, Invariant varieties through singularities of holomorphic vector fields, *Ann. of Math. (2)* **115(3)** (1982), 579–595. DOI: 10.2307/2007013.

[6] K. Cordwell, S. Gilbertson, N. Nuechterlein, K. M. Pilgrim, and S. Pinella, On the classification of critically fixed rational maps, *Conform. Geom. Dyn.* **19** (2015), 51–94. DOI: 10.1090/S1088-4173-2015-00275-1.

[7] E. Crane, Mean value conjectures for rational maps, *Complex Var. Elliptic Equ.* **51(1)** (2006), 41–50. DOI: 10.1080/02781070500293380.

[8] C. Favre and J. V. Pereira, Webs invariant by rational maps on surfaces, *Rend. Circ. Mat. Palermo (2)* **64(3)** (2015), 403–431. DOI: 10.1007/s12215-015-0207-9.

[9] D. Marín and J. V. Pereira, Rigid flat webs on the projective plane, *Asian J. Math.* **17(1)** (2013), 163–191. DOI: 10.4310/AJM.2013.v17.n1.a7.

[10] L. G. Mendes and J. V. Pereira, Hilbert modular foliations on the projective plane, *Comment. Math. Helv.* **80(2)** (2005), 243–291. DOI: 10.4171/CMH/14.

[11] D. Panov and A. Petrunin, Ramification conjecture and Hirzebruch's property of line arrangements, *Compos. Math.* **152(12)** (2016), 2443–2460. DOI: 10.1112/S0010437X16007648.

[12] J. V. Pereira, Vector fields, invariant varieties and linear systems, *Ann. Inst. Fourier (Grenoble)* **51(5)** (2001), 1385–1405.

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