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ON THE STRONG CONVERGENCE OF MULTIPLE ORDINARY INTEGRALS TO MULTIPLE STRATONOVICH INTEGRALS

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Abstract: Given $\{W^{(m)}(t),\, t\in[0,T]\}_{m\geq 1}$, a sequence of approximations to a standard Brownian motion W in $[0,T]$ such that $W^{(m)}(t)$ converges almost surely to $W(t)$, we show that, under regular conditions on the approximations, the multiple ordinary integrals with respect to $dW^{(m)}$ converge to the multiple Stratonovich integral. We are integrating functions of the type

$$f(t_1,\dots,t_n)=f_1(t_1)\cdots f_n(t_n)I_{\{t_1\leq\cdots\leq t_n\}},$$

where for each $i\in\{1,\dots,n\}$, f_i has continuous derivatives in $[0,T]$. We apply this result to approximations obtained from uniform transport processes.

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1. Introduction

Consider $W:=\{W(t),\, 0\leq t\leq T\}$ a standard Brownian motion and let $\{W^{(m)}(t),\, t\in[0,T]\}_{m\geq 1}$ be a sequence of approximations to W in $[0,T]$ such that $W^{(m)}(t)$ converges almost surely to $W(t)$ as m tends to ∞. It is well known that if the approximations are of bounded variation and continuous, then almost surely

$$\begin{aligned}\lim_{m\to\infty}\int_0^t W^{(m)}(s)\,dW^{(m)}(s)&=\lim_{m\to\infty}\frac{(W^{(m)}(t))^2}{2}=\frac{(W(t))^2}{2}\\&=\int_0^t W(s)\,dW(s)+\frac{t}{2}=\int_0^t W(s)\,d^\circ W(s),\end{aligned}$$

for $t\in[0,T]$, where dW denotes the Itô integral and $d^\circ W$ is the Stratonovich one.

Wong and Zakai ([**16**]) extended this result obtaining a relationship between limits of integrals of the type $\int_0^t\psi(W^{(m)}(s),s)\,dW^{(m)}(s)$ and $\int_0^t\psi(W(s),s)\,d^\circ W(s)$. More precisely, assuming that the approximations are of bounded variation, continuous, that for each m can be bounded by a finite random variable, and that $\psi(\eta, s)$ has continuous partial derivatives with respect to η and s, they proved that, for any $t \in [0,T]$, almost surely

$$\begin{aligned}\lim_{m\to\infty} \int_0^t \psi(W^{(m)}(s), s)\, dW^{(m)}(s) &= \int_0^t \psi(W(s), s)\, dW(s) \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial \psi}{\partial \eta}(W(s), s)\, ds \\ &= \int_0^t \psi(W(s), s)\, d^\circ W(s).\end{aligned}$$

Gorostiza and Griego ([**10**]) applied this result to get strong approximations to stochastic integrals using uniform transport processes as approximations to Brownian motion. They also obtained the corresponding rate of convergence.

Our aim is to extend this result to study the strong convergence for all $t \in [0,T]$, when m goes to ∞, of the multiple integral

$$\int_0^t \int_0^{s_n} \cdots \int_0^{s_2} f_n(s_n) \cdots f_1(s_1)\, dW^{(m)}(s_1) \dots dW^{(m)}(s_n),$$

where for each $i \in \{1, \dots, n\}$, f_i has continuous derivatives in $[0,T]$. We will obtain the almost sure convergence to the multiple Stratonovich integral

$$\int_0^t \int_0^{s_n} \cdots \int_0^{s_2} f_n(s_n) \cdots f_1(s_1)\, d^\circ W(s_1) \dots d^\circ W(s_n),$$

$t \in [0,T]$. We will also apply this convergence to approximations obtained from uniform transport processes. This kind of approximations of multiple integrals has been studied by Bardina and Jolis [**3**] where they obtained a weak convergence result. Multiple stochastic integrals are useful to represent functionals of Wiener processes and they appear in stochastic Taylor expansions. For a detailed explanation about multiple Wiener and Stratonovich integrals we refer the reader to Chapter 5 in Hu [**12**].

The processes usually called uniform transport processes can be presented as

$$W^{(m)}(t) = m^{\frac{1}{2}} (-1)^B \int_0^t (-1)^{N(um)}\, du, \tag{1}$$

where $\{N(t),\, t \geq 0\}$ is a Poisson process and $B \sim$ Bernoulli $\left(\frac{1}{2}\right)$ independent of the Poisson process N.

There is some literature related with strong convergence using approximations based on realizations of these processes. Griego, Heath, and Ruiz-Moncayo ([**11**]) proved that these processes converge almost surely and uniformly on $[0,T]$ to Brownian motion. In [**9**] Gorostiza and Griego studied the case of diffusions. Also, Gorostiza and Griego ([**10**]) and Csörgő and Horváth ([**4**]) got a rate of convergence. Garzón, Gorostiza, and León ([**5**]) presented a family of processes that converges almost surely to fractional Brownian motion of Hurst parameter H uniformly on $[0,T]$ for any $H \in (0,1)$. In [**6**] and [**7**] the same authors studied fractional stochastic differential equations and subfractional Brownian motion. The Rosenblatt process is studied by Garzón, Torres, and Tudor in [**8**]. In [**2**] we get strong approximations to Brownian sheet and in [**1**] to complex Brownian motion.

On the other hand, there exist other methods of strong approximation to Stratonovich integrals, as Milstein method or the method of the multiple Fourier series. We refer the reader to Kuznetsov [**14**] and the references therein for a complete survey on these methods. Notice that our method is different from the existing ones, as far as we know, and can be seen as an extension of the initial work of Wong and Zakai [**16**] for non multiple integrals.

The proof of our results follows the ideas in [**11**], using Itô's formula as a main tool. It is important to get an expression of the multiple Stratonovich integral and the relationship with the multiple Itô integral well-posed for our problem. We give a detailed expression of this relationship, inspired by the well-known Hu–Meyer formula, the work of Kloeden and Platen [**13**], and Kuznetsov [**15**].

The paper is organized as follows. In Section 2 we define the processes and we state the main results. Section 3 is devoted to study the relationship between the Itô integral and the Stratonovich integral. In Section 4 we give the proof of the main theorem.

2. Notations and main result

Consider

$$f(t_1,\dots,t_n) = f_1(t_1)\cdots f_n(t_n) I_{\{t_1\le\cdots\le t_n\}},$$

where $f_i \in L^2[0,T]$ for all $i \in \{1,\dots,n\}$. From Lemma A.1 and A.2 of [**3**] it is known that f is Stratonovich integrable and the process

$$\begin{aligned} I_n^S(t) &:= I_n^S(f_1,\dots,f_n)(t) \\ &= \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} f_n(s_n)\cdots f_1(s_1)\, d^\circ W(s_1)\dots d^\circ W(s_n) \end{aligned}$$

has a version with continuous paths. Moreover, the iterated simple integrals

$$Y_k(t) = \int_0^t f_k(s) Y_{k-1}(s)\, d^\circ W(s) \tag{2}$$

exist for $k \in \{2, \dots, n\}$, where $Y_1(t) = \int_0^t f_1(s)\, d^\circ W(s)$, and all these integrals have a continuous version and Y_n coincides with I_n^S.

Set, for each m, $W^{(m)}(t)$ a sequence of approximations to $W(t)$ such that $W^{(m)}(t)$ is of bounded variation, continuous, and converges a.s. to $W(t)$ as $m \to \infty$. Then, we define the ordinary multiple integral

$$\begin{aligned} J_n^{(m)}(t) &:= J_n^{(m)}(f_1, \dots, f_n)(t) \\ &= \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} f_n(s_n) \cdots f_1(s_1)\, dW^{(m)}(s_1) \dots dW^{(m)}(s_n). \end{aligned}$$

Obviously, we can consider the iterated integrals

$$J_k^{(m)}(f_1, \dots, f_k)(t) = \int_0^t f_k(s) J_{k-1}^{(m)}(f_1, \dots, f_{k-1})(s)\, dW^{(m)}(s), \tag{3}$$

for $k \in \{2, \dots, n\}$ and where

$$J_1^{(m)}(f_1)(t) = \int_0^t f_1(s)\, dW^{(m)}(s).$$

Following the ideas of Wong and Zakai [**16**], we state the set of hypothesis (H) on $W^{(m)}$ as follows:

(H1) Almost surely, $W^{(m)}$ are continuous and of bounded variation.
(H2) Almost surely, $\lim_{m\to\infty} W^{(m)}(s) = W(s)$ for all $s \in [0, T]$.
(H3) For almost all ω, there exists $m_0(\omega)$ and $K(\omega)$, both finite, such that $W^{(m)}(s, \omega) \le K(\omega)$ for all $m > m_0$ and all $s \in [0, T]$.
(H4) $\lim_{m\to\infty} \sup_{s\in[0,T]} |W^{(m)}(s) - W(s)| = 0$ almost surely.

Then, the main theorem reads as follows:

Theorem 2.1. *Assume that $f\colon [0, T]^n \to \mathbb{R}$ is a function*

$$f(t_1, \dots, t_n) = f_1(t_1) \cdots f_n(t_n) I_{\{t_1 \le \cdots \le t_n\}},$$

such that for each $i \in \{1, \dots, n\}$, f_i has continuous derivatives in $[0, T]$. Let $(W^{(m)})$ be a family of approximations of W that satisfies hypotheses (H1), (H2), and (H3). Then, almost surely

$$\lim_{m\to\infty} J_n^{(m)}(t) = I_n^S(t)$$

for all $t \in [0, T]$.

If (H4) also holds, then

$$\lim_{m\to\infty} \sup_{s\in[0,T]} |J_n^{(m)}(s) - I_n^S(s)| = 0$$

almost surely.

We can give now our theorem using approximations based on uniform transport processes.

Theorem 2.2. *Consider* $W^{(m)} := \{W^{(m)}(t),\, t \in [0,T]\}$, *versions of the uniform transport processes* (1) *on the same probability space as a Brownian motion* $\{W(t),\, t \in [0,T]\}$, *such that*

$$\lim_{m\to\infty} \sup_{s\in[0,T]} |W^{(m)}(s) - W(s)| = 0$$

almost surely. Assume that $f\colon [0,T]^n \to \mathbb{R}$ *is a function*

$$f(t_1,\ldots,t_n) = f_1(t_1)\cdots f_n(t_n) I_{\{t_1\le\cdots\le t_n\}},$$

such that for each $i \in \{1,\ldots,n\}$, f_i *has continuous derivatives in* $[0,T]$. *Then*

$$\lim_{m\to\infty} \sup_{s\in[0,T]} |J_n^{(m)}(s) - I_n^S(s)| = 0$$

almost surely.

Remark 2.3. Our result can be used to simulate multiple Stratonovich stochastic integrals I_n^S using $J_n^{(m)}(t)$ as approximation. In order to compute $J_n^{(m)}(t)$ we can use the following decomposition that we prove in Lemma 4.1:

$$\begin{aligned} J_n^{(m)}(t) &= \sum_{k=1}^{n} (-1)^{k+1} J_{n-k}^{(m)}(t) \frac{(W^{(m)}(t))^k}{k!} \left(\prod_{l=1}^{k} f_{n+1-l}\right)(t) \\ &\quad + \sum_{k=1}^{n} (-1)^k \int_0^t J_{n-k}^{(m)}(s) \frac{(W^{(m)}(s))^k}{k!} \left(\prod_{l=1}^{k} f_{n+1-l}\right)'(s)\, ds, \end{aligned}$$

for any $t \in [0,T]$ with $J_0^{(m)}(t) = 1$.

Remark 2.4. In [**3**] the authors prove the weak convergence, when m tends to infinity, of the multiple integral processes $\{J_n^{(m)}(t),\, t \in [0,T]\}$ in the space of the real continuous functions when $f_i \in L^2[0,T]$ for all $i \in \{1,\ldots,n\}$. To obtain the almost sure convergence we need to assume stronger conditions on functions f_i in order to be able to use Itô's formula. The derivability hypothesis that we assume is similar to the one used in the paper of Wong and Zakai [**16**].

Bardina and Jolis ([**3**]) also study the weak limit for other class of integrals. Particularly, they consider the limit of integrals

$$\left\{\int_0^t \int_0^{s_n} \cdots \int_0^{s_2} f(s_1,\dots,s_n)\, dW^{(m)}(s_1)\dots dW^{(m)}(s_n),\, t\in[0,T]\right\},$$

when f is given by a multimeasure. Notice that in this case the almost sure convergence is obtained easily from Theorem 3.1 in [**3**] and the almost sure convergence in [**11**].

3. Multiple Itô and Stratonovich integrals

The relationship between simple Itô and Stratonovich integrals has been studied deeply. For instance, it is well known that if $X := \{X(t),\, t\geq 0\}$ is an adapted continuous process, then

$$\int_0^t X(s)\, d^\circ W(s) = \int_0^t X(s)\, dW(s) + \frac{1}{2}[X,W]_t,$$

where [,] denotes the quadratic covariation. Particularly, if

$$X(t) = X(0) + \int_0^t u(s)\, dW(s) + \int_0^t v(s)\, ds,$$

where $\{u(t),\, t\geq 0\}$ and $\{v(t),\, t\geq 0\}$ are adapted processes, and g is a differentiable real function, we have that

$$g(t)X(t) = g(0)X(0) + \int_0^t g(s)\, dX(s) + \int_0^t X(s)g'(s)\, ds.$$

Then, we obtain that

$$\int_0^t g(s)X(s)\, d^\circ W(s) = \int_0^t g(s)X(s)\, dW(s) + \frac{1}{2}\int_0^t g(s)u(s)\, ds. \tag{4}$$

There exists also literature for the multiple stochastic integral, beginning with the well-known Hu–Meyer formula. Here we obtain some results well adapted to our problem following the notation presented by Kloeden and Platen [**13**].

Let us introduce some notation in order to deal with multiple integrals. We assume again that $f_i \in L^2[0,T]$ for all $i\in\{1,\dots,n\}$. Set

$$\mathcal{G}_n := \left\{(\alpha_1,\dots,\alpha_m) \text{ such that } \alpha_i\in\{1,2\}\,\forall i,\, \sum_{i=1}^m \alpha_i = n\right\}.$$

Notice that, if $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathcal{G}_n$, then $\frac{n}{2} \le m \le n$. For any $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathcal{G}_n$, let us define

$$I_{(\alpha_1,\dots,\alpha_m)}(f_1, \dots, f_n)(t) = \begin{cases} \int_0^t f_n(s) I_{(\alpha_1,\dots,\alpha_{m-1})}(f_1, \dots, f_{n-1})(s)\, dW(s) & \text{if } \alpha_m = 1, \\ \int_0^t f_n(s) f_{n-1}(s) I_{(\alpha_1,\dots,\alpha_{m-1})}(f_1, \dots, f_{n-2})(s)\, ds & \text{if } \alpha_m = 2, \end{cases}$$

with

$$I_{(1)}(f_1)(t) = \int_0^t f_1(s)\, dW(s),$$
$$I_{(2)}(f_1, f_2)(t) = \int_0^t f_2(s) f_1(s)\, ds.$$

Clearly, under our assumptions, all these integrals are well-defined. Moreover, for $\alpha = (1, \dots, 1)$ we have the iterated Itô integral that coincides with the classical multiple Itô integral:

$$I_{(1,\dots,1)}(f_1, \dots, f_n)(t) = \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} f_n(s_n) \cdots f_1(s_1)\, dW(s_1) \dots dW(s_n).$$

We will also use the notation

$$I_n(t) = I_n(f_1, \dots, f_n)(t) = I_{(1,\dots,1)}(f_1, \dots, f_n)(t).$$

We finish this section with three propositions that give us three different ways to express the Stratonovich integral $I_n^S(t) = I_n^S(f_1, \dots, f_n)(t)$.

The first one can be found in [**15**]. We present a version using the notation we have introduced above. We give a sketch of the proof for the sake of completeness.

Proposition 3.1. *Assume that $f_i \in L^2[0,T]$ for all $i \in \{1, \dots, n\}$. Then*

$$I_n^S(f_1, \dots, f_n)(t) = \sum_{(\alpha_1,\dots,\alpha_m) \in \mathcal{G}_n} \frac{1}{2^{n-m}} I_{(\alpha_1,\dots,\alpha_m)}(f_1, \dots, f_n)(t),$$

for any $t \in [0,T]$.

Proof: We will check the equality by induction on n. For $n = 1$ it is easy to see that

$$I_1^S(f_1)(t) = \int_0^t f_1(s)\, d^\circ W(s) = \int_0^t f_1(s)\, dW(s) = I_{(1)}(f_1)(t)$$

and that $\mathcal{G}_1 = \{(1)\}$.

Let us assume that the statement is true until n and we will check what happens for $n+1$. Using (2) and the induction hypothesis we have that

$$
\begin{aligned}
&I^S_{n+1}(f_1,\dots,f_n,f_{n+1})(t)\\
&\quad=\int_0^t f_{n+1}(s) I^S_n(f_1,\dots,f_n)(s)\, d^\circ W(s)\\
&\quad=\int_0^t f_{n+1}(s)\Bigg(\sum_{(\alpha_1,\dots,\alpha_m)\in\mathcal{G}_n}\frac{1}{2^{n-m}} I_{(\alpha_1,\dots,\alpha_m)}(f_1,\dots,f_n)(s)\Bigg)\, d^\circ W(s) \qquad (5)\\
&\quad=\sum_{(\alpha_1,\dots,\alpha_m)\in\mathcal{G}_n}\frac{1}{2^{n-m}}\int_0^t f_{n+1}(s) I_{(\alpha_1,\dots,\alpha_m)}(f_1,\dots,f_n)(s)\, d^\circ W(s).
\end{aligned}
$$

Notice that, if $\alpha_m = 1$, using (4) we can write

$$
\begin{aligned}
&\int_0^t f_{n+1}(s) I_{(\alpha_1,\dots,\alpha_m)}(f_1,\dots,f_n)(s)\, d^\circ W(s)\\
&\qquad=\int_0^t f_{n+1}(s) I_{(\alpha_1,\dots,\alpha_m)}(f_1,\dots,f_n)(s)\, dW(s)\\
&\qquad\quad+\frac{1}{2}\int_0^t f_{n+1}(s) f_n(s) I_{(\alpha_1,\dots,\alpha_{m-1})}(f_1,\dots,f_{n-1})(s)\, ds \qquad (6)\\
&\qquad= I_{(\alpha_1,\dots,\alpha_m,1)}(f_1,\dots,f_n,f_{n+1})(t)\\
&\qquad\quad+\frac{1}{2} I_{(\alpha_1,\dots,\alpha_{m-1},2)}(f_1,\dots,f_{n-1},f_n,f_{n+1})(t).
\end{aligned}
$$

On the other hand, using (4), when $\alpha_m = 2$ we have

$$
\begin{aligned}
&\int_0^t f_{n+1}(s) I_{(\alpha_1,\dots,\alpha_m)}(f_1,\dots,f_n)(s)\, d^\circ W(s)\\
&\qquad=\int_0^t f_{n+1}(s) I_{(\alpha_1,\dots,\alpha_m)}(f_1,\dots,f_n)(s)\, dW(s) \qquad (7)\\
&\qquad= I_{(\alpha_1,\dots,\alpha_m,1)}(f_1,\dots,f_n,f_{n+1})(t).
\end{aligned}
$$

We finish the proof putting together (5), (6), (7), and using that

$$
\begin{aligned}
&\{(\alpha_1,\dots,\alpha_m,1);(\alpha_1,\dots,\alpha_m)\in\mathcal{G}_n\}\\
&\qquad\cup\{(\alpha_1,\dots,\alpha_{m-1},2);(\alpha_1,\dots,\alpha_m)\in\mathcal{G}_n \text{ with } \alpha_m=1\}\\
&\quad=\left\{(\alpha_1,\dots,\alpha_{m+1}) \text{ with } \alpha_i\in\{1,2\}\,\forall i,\ \alpha_{m+1}=1,\ \sum_{i=1}^{m+1}\alpha_i=n+1\right\}\\
&\qquad\cup\left\{(\alpha_1,\dots,\alpha_m) \text{ with } \alpha_i\in\{1,2\}\,\forall i,\ \alpha_m=2,\ \sum_{i=1}^{m}\alpha_i=n+1\right\}\\
&\quad=\mathcal{G}_{n+1}.
\end{aligned}
$$

□

Now, applying Proposition 3.1 we get an iterative definition of I_n^S in terms of I_{n-1}^S and I_{n-2}^S.

Proposition 3.2. *Assume that $f_i\in L^2[0,T]$ for all $i\in\{1,\dots,n\}$. Then*

$$I_n^S(t)=\int_0^t f_n(s)I_{n-1}^S(s)\,dW_s+\frac{1}{2}\int_0^t f_n(s)f_{n-1}(s)I_{n-2}^S(s)\,ds,$$

for any $t\in[0,T]$.

Proof: This follows easily from the fact that

$$
\begin{aligned}
&I_n^S(f_1,\dots,f_n)(t)=\sum_{(\alpha_1,\dots,\alpha_m)\in\mathcal{G}_n}\frac{1}{2^{n-m}}I_{(\alpha_1,\dots,\alpha_m)}(f_1,\dots,f_n)(t)\\
&=\sum_{(\alpha_1,\dots,\alpha_m)\in\mathcal{G}_n,\alpha_m=1}\frac{1}{2^{n-m}}\int_0^t f_n(s)I_{(\alpha_1,\dots,\alpha_{m-1})}(f_1,\dots,f_{n-1})(s)\,dW(s)\\
&\quad+\sum_{(\alpha_1,\dots,\alpha_m)\in\mathcal{G}_n,\alpha_m=2}\frac{1}{2^{n-m}}\\
&\qquad\qquad\times\int_0^t f_n(s)f_{(n-1)}(s)I_{(\alpha_1,\dots,\alpha_{m-1})}(f_1,\dots,f_{n-2})(s)\,ds\\
&=\int_0^t f_n(s)\\
&\quad\times\Bigg(\sum_{(\alpha_1,\dots,\alpha_{m-1})\in\mathcal{G}_{n-1}}\frac{1}{2^{n-1-(m-1)}}I_{(\alpha_1,\dots,\alpha_{m-1})}(f_1,\dots,f_{n-1})(s)\Bigg)dW(s)\\
&\quad+\frac{1}{2}\int_0^t f_n(s)f_{(n-1)}(s)\\
&\quad\times\Bigg(\sum_{(\alpha_1,\dots,\alpha_{m-1})\in\mathcal{G}_{n-2},}\frac{1}{2^{n-2-(m-1)}}I_{(\alpha_1,\dots,\alpha_{m-1})}(f_1,\dots,f_{n-2})(s)\Bigg)\,ds.
\end{aligned}
$$

□

In the next proposition we express I_n^S using I_k^S for all $k \in \{1, \dots, n-1\}$. This proposition is inspired by the work of Wong and Zakai [**16**] and it is the key of the proof of our main theorem.

Proposition 3.3. *Assume that, for each $i \in \{1, \dots, n\}$, f_i has continuous derivatives in $[0,T]$. Then*

$$I_n^S(t) = \sum_{k=1}^{n} (-1)^{k+1} I_{n-k}^S(t) \frac{W(t)^k}{k!} \left(\prod_{l=1}^{k} f_{n+1-l} \right)(t)$$

$$- \sum_{k=1}^{n} (-1)^k \int_0^t I_{n-k}^S(s) \frac{W(s)^k}{k!} \left(\prod_{l=1}^{k} f_{n+1-l} \right)'(s)\, ds,$$

for any $t \in [0,T]$ and where $I_0^S = 1$.

Proof: Let us consider the function $F(x,y,u) = xyf_n(u)$. By Itô's formula we can write

$$\begin{aligned} F(I_{n-1}^S(t), W(t), t) &= \int_0^t W(s) f_n(s)\, dI_{n-1}^S(s) \\ &\quad + \int_0^t I_{n-1}^S(s) f_n(s)\, dW(s) \\ &\quad + \int_0^t I_{n-1}^S(s) W(s) f_n'(s)\, ds \\ &\quad + \frac{1}{2} \int_0^t f_n(s)\, d[I_{n-1}^S(s), W(s)]. \end{aligned} \tag{8}$$

Let us check first that

$$\int_0^t I_{n-1}^S(s) f_n(s)\, dW(s) + \frac{1}{2} \int_0^t f_n(s)\, d[I_{n-1}^S(s), W(s)] = I_n^S(t). \tag{9}$$

Using Proposition 3.2 it suffices to check that

$$\int_0^t f_n(s)\, d[I_{n-1}^S(s), W(s)] = \int_0^t f_n(s) f_{n-1}(s) I_{n-2}^S(s)\, ds,$$

but this is an obvious consequence again of Proposition 3.2. So it is clear that (9) holds.

Then, from (8), (9), and the definition of F it follows that

$$I_n^S(t) = I_{n-1}^S(t)W(t)f_n(t) - \int_0^t W(s)f_n(s)\,dI_{n-1}^S(s) - \int_0^t I_{n-1}^S(s)W(s)f_n'(s)\,ds. \tag{10}$$

Let us now study the term

$$\int_0^t W(s)f_n(s)\,dI_{n-1}^S(s).$$

Actually, we will study the more general term

$$H_k^S(t) := \int_0^t \frac{(W(s))^k}{k!}\left(\prod_{l=1}^k f_{n+1-l}\right)(s)\,dI_{n-k}^S(s).$$

From Proposition 3.2 it follows that

$$H_k^S(t) = \int_0^t \frac{(W(s))^k}{k!}\left(\prod_{l=1}^{k+1} f_{n+1-l}\right)(s)I_{n-k-1}^S(s)\,dW(s) + \frac{1}{2}\int_0^t \frac{(W(s))^k}{k!}\left(\prod_{l=1}^{k+2} f_{n+1-l}\right)(s)I_{n-k-2}^S(s)\,ds. \tag{11}$$

Consider now the function $F(x,y,u) = x\frac{y^{k+1}}{(k+1)!}\left(\prod_{l=1}^{k+1} f_{n+1-l}\right)(u)$. By Itô's formula we can write

$$\begin{aligned} F(I_{n-k-1}^S(t), W(t), t) &= \int_0^t \frac{(W(s))^{k+1}}{(k+1)!}\left(\prod_{l=1}^{k+1} f_{n+1-l}\right)(s)\,dI_{n-k-1}^S(s) \\ &+ \int_0^t I_{n-k-1}^S(s)\frac{(W(s))^k}{k!}\left(\prod_{l=1}^{k+1} f_{n+1-l}\right)(s)\,dW(s) \\ &+ \int_0^t I_{n-k-1}^S(s)\frac{(W(s))^{k+1}}{(k+1)!}\left(\prod_{l=1}^{k+1} f_{n+1-l}\right)'(s)\,ds \\ &+ \frac{1}{2}\int_0^t \frac{(W(s))^k}{k!}\left(\prod_{l=1}^{k+1} f_{n+1-l}\right)(s)\,d[I_{n-k-1}^S(s), W(s)]. \end{aligned} \tag{12}$$

Putting together (11), (12), and using that from Proposition 3.2

$$\frac{1}{2}\int_0^t \frac{(W(s))^k}{k!}\Big(\prod_{l=1}^{k+1} f_{n+1-l}\Big)(s)\, d[I^S_{n-k-1}(s), W(s)]$$
$$= \frac{1}{2}\int_0^t \frac{W(s)^k}{k!}\Big(\prod_{l=1}^{k+1} f_{n+1-l}\Big)(s) f_{n-k-1}(s) I^S_{n-k-2}(s)\, ds,$$

we get that

$$\begin{aligned} H^S_k(t) &= I^S_{n-k-1}(t)\frac{(W(t))^{k+1}}{(k+1)!}\Big(\prod_{l=1}^{k+1} f_{n+1-l}\Big)(t) \\ &\quad - \int_0^t I^S_{n-k-1}(s)\frac{(W(s))^{k+1}}{(k+1)!}\Big(\prod_{l=1}^{k+1} f_{n+1-l}\Big)'(s)\, ds - H^S_{k+1}(t). \end{aligned} \tag{13}$$

So, from (10) we can write

$$\begin{aligned} I^S_n(t) &= I^S_{n-1}(t)W(t)f_n(t) - \int_0^t I^S_{n-1}(s)W(s)f'_n(s)\, ds - H^S_1(t) \\ &= I^S_{n-1}(t)W(t)f_n(t) - \int_0^t I^S_{n-1}(s)W(s)f'_n(s)\, ds \\ &\quad - I^S_{n-2}(t)\frac{(W(t))^2}{2!} f_n(t) f_{n-1}(t) \\ &\quad + \int_0^t I^S_{n-2}(s)\frac{(W(s))^2}{2!}(f_n f_{n-1})'(s)\, ds + H^S_2(t), \end{aligned}$$

and the proof finishes iterating (13) $n-1$ times. □

4. Proof of the main result

We begin with a technical lemma where we obtain an expression for $J^{(m)}_n = J^{(m)}_n(f_1, \dots, f_n)(t)$ similar to the expression given in Proposition 3.3 for the multiple Stratonovich integral. This result is also inspired by [**16**].

Lemma 4.1. *Assume that, for each $i \in \{1,\dots,n\}$, f_i has continuous derivatives in $[0,T]$. Then*

$$J_n^{(m)}(t) = \sum_{k=1}^{n} (-1)^{k+1} J_{n-k}^{(m)}(t) \frac{(W^{(m)}(t))^k}{k!} \left(\prod_{l=1}^{k} f_{n+1-l} \right)(t) + \sum_{k=1}^{n} (-1)^k \int_0^t J_{n-k}^{(m)}(s) \frac{(W^{(m)}(s))^k}{k!} \left(\prod_{l=1}^{k} f_{n+1-l} \right)'(s)\, ds,$$

for any $t \in [0,T]$ and with $J_0^{(m)}(t) = 1$.

Proof: Consider the function $F(x,y,u) = xyf_n(u)$. Since $J_{n-1}^{(m)}$, $W^{(m)}$, and f_n are functions of bounded variation, we have

$$F(J_{n-1}^{(m)}(t), W^{(m)}(t), t) = \int_0^t W^{(m)}(s) f_n(s)\, dJ_{n-1}^{(m)}(s) + \int_0^t J_{n-1}^{(m)}(s) f_n(s)\, dW^{(m)}(s) + \int_0^t J_{n-1}^{(m)}(s) W^{(m)}(s) f_n'(s)\, ds. \tag{14}$$

From (3) and (14) we can get

$$J_n^{(m)}(t) = J_{n-1}^{(m)}(t) W^{(m)}(t) f_n(t) - \int_0^t W^{(m)}(s) f_n(s)\, dJ_{n-1}^{(m)}(s) - \int_0^t J_{n-1}^{(m)}(s) W^{(m)}(s) f_n'(s)\, ds.$$

Thus, now we need to study

$$\int_0^t W^{(m)}(s) f_n(s)\, dJ_{n-1}^{(m)}(s). \tag{15}$$

Actually, we will study the more general integral

$$H_k(t) := \int_0^t \frac{(W^{(m)}(s))^k}{k!} \left(\prod_{l=1}^{k} f_{n+1-l} \right)(s)\, dJ_{n-k}^{(m)}(s).$$

Consider the function $F(x,y,u) = x\frac{y^{k+1}}{(k+1)!}\big(\prod_{l=1}^{k+1} f_{n+1-l}\big)(u)$. Since the functions $J_{n-k-1}^{(m)}$, $W^{(m)}$, and $\prod_{l=1}^{k+1} f_{n+1-l}$ are of bounded variation we can write

$$
\begin{aligned}
&F(J_{n-k-1}^{(m)}(t), W^{(m)}(t), t)\\
&\quad = \int_0^t \frac{(W^{(m)}(s))^{k+1}}{(k+1)!}\Big(\prod_{l=1}^{k+1} f_{n+1-l}\Big)(s)\, dJ_{n-k-1}^{(m)}(s)\\
&\qquad + \int_0^t J_{n-(k+1)}^{(m)}(s)\frac{(W^{(m)}(s))^{k}}{k!}\Big(\prod_{l=1}^{k+1} f_{n+1-l}\Big)(s)\, dW^{(m)}(s)\\
&\qquad + \int_0^t J_{n-(k+1)}^{(m)}(s)\frac{(W^{(m)}(s))^{k+1}}{(k+1)!}\Big(\prod_{l=1}^{k+1} f_{n+1-l}\Big)'(s)\, ds.
\end{aligned}
\tag{16}
$$

Then, using the definition of F and (16) we have that

$$
\begin{aligned}
H_k(t) &= \int_0^t J_{n-(k+1)}^{(m)}(s)\frac{(W^{(m)}(s))^{k}}{k!}\Big(\prod_{l=1}^{k+1} f_{n+1-l}\Big)(s)\, dW^{(m)}(s)\\
&= J_{n-(k+1)}^{(m)}(t)\frac{(W^{(m)}(t))^{k+1}}{(k+1)!}\Big(\prod_{l=1}^{k+1} f_{n+1-l}\Big)(t)\\
&\quad - \int_0^t \frac{(W^{(m)}(s))^{k+1}}{(k+1)!}\Big(\prod_{l=1}^{k+1} f_{n+1-l}\Big)(s)\, dJ_{n-(k+1)}^{(m)}(s)\\
&\quad - \int_0^t J_{n-(k+1)}^{(m)}(s)\frac{(W^{(m)}(s))^{k+1}}{(k+1)!}\Big(\prod_{l=1}^{k+1} f_{n+1-l}\Big)'(s)\, ds\\
&= J_{n-(k+1)}^{(m)}(t)\frac{(W^{(m)}(t))^{k+1}}{(k+1)!}\Big(\prod_{l=1}^{k+1} f_{n+1-l}\Big)(t) - H_{k+1}(t)\\
&\quad - \int_0^t J_{n-(k+1)}^{(m)}(s)\frac{(W^{(m)}(s))^{k+1}}{(k+1)!}\Big(\prod_{l=1}^{k+1} f_{n+1-l}\Big)'(s)\, ds.
\end{aligned}
$$

Iterating the same argument $n-1$ times we finish the proof. □

We can give now the proof of the main theorem.

Proof of Theorem 2.1: We have to check that, almost surely

$$\lim_{m\to\infty} J_n^{(m)}(f_1,\dots,f_n)(t) = I_n^S(f_1,\dots,f_n)(t),$$

for any $t \in [0,T]$. We use an induction argument. Let us introduce our induction hypothesis:

$(\bar{H}_j)$ For any $l \le j$, almost surely

$$\lim_{m\to\infty} J_l^{(m)}(f_1,\dots,f_l)(t) = I_l^S(f_1,\dots,f_l)(t),$$

for all $t \in [0,T]$. Moreover, for almost all ω there exists $m_j(\omega)$ and $K_j(\omega)$, both finite, such that $J_l^{(m)}(s,\omega) \le K_j(\omega)$ for all $m > m_j$, all $s \in [0,T]$, and all $l \le j$.

Let us study $(\bar{H}_1)$ first. It is well-known that almost surely

$$\lim_{m\to\infty} J_1^{(m)}(t) = \lim_{m\to\infty} \int_0^t W^{(m)}(s)\, dW^{(m)}(s) = \int_0^t W(s)\, d^\circ W(s) = I_1^{(S)}(t),$$

for any $t \in [0,T]$. Moreover, from Lemma 4.1 we have that

$$J_1^{(m)}(t) = W^{(m)}(t) f_1(t) - \int_0^t W^{(m)}(s) f_1'(s)\, ds.$$

On the other hand, the boundedness of $J_1^{(m)}$ is a consequence of the boundedness of $W^{(m)}$ and the continuity of f_1 and f_1'. So, $(\bar{H}_1)$ is clearly true.

Consider now $j > 1$. Let us assume now that $(\bar{H}_{j-1})$ is true and we will check that $(\bar{H}_j)$ holds. Using Proposition 3.3 and Lemma 4.1 to get expressions of I_j^S and $J_j^{(m)}$, to prove the almost sure convergence it is enough to check that, for any $k \in \{1,\dots,j\}$, almost surely

$$\begin{aligned}\lim_{m\to\infty} J_{j-k}^{(m)}(t) \frac{(W^{(m)}(t))^k}{k!} \left(\prod_{l=1}^k f_{j+1-l}\right)(t) \\ = I_{j-k}^S(t) \frac{W(t)^k}{k!} \left(\prod_{l=1}^k f_{j+1-l}\right)(t),\end{aligned} \tag{17}$$

$$\begin{aligned}\lim_{m\to\infty} \int_0^t J_{j-k}^{(m)}(s) \frac{(W^{(m)}(s))^k}{k!} \left(\prod_{l=1}^k f_{j+1-l}\right)'(s)\, ds \\ = \int_0^t I_{j-k}^S(s) \frac{W(s)^k}{k!} \left(\prod_{l=1}^k f_{j+1-l}\right)'(s)\, ds,\end{aligned} \tag{18}$$

for any $t \in [0,T]$. The limit (17) is an evident consequence of $(\bar{H}_{j-1})$ and (18) follows easily using again $(\bar{H}_{j-1})$ and dominated convergence. Notice that the boundedness of $J_j^{(m)}$ is obtained from $(\bar{H}_{j-1})$ and Lemma 4.1.

When (H4) is also true, from Proposition 3.3 and Lemma 4.1 we can write

$$
\begin{aligned}
&\sup_{s\in[0,T]} |J_n^{(m)}(s) - I_n^S(s)| \\
&\leq \sum_{k=1}^{n} \sup_{s\in[0,T]} \left| \left(J_{n-k}^{(m)}(s) \frac{W^{(m)}(s)^k}{k!} - I_{n-k}^S(s) \frac{W(s)^k}{k!} \right) \left(\prod_{l=1}^{k} f_{n+1-l} \right)(s) \right| \\
&\quad + \sum_{k=1}^{n} \int_0^T \left| \left(J_{n-k}^{(m)}(s) \frac{W^{(m)}(s)^k}{k!} - I_{n-k}^S(s) \frac{W(s)^k}{k!} \right) \left(\prod_{l=1}^{k} f_{n+1-l} \right)'(s) \right| ds \\
&\leq \sum_{k=1}^{n} \sup_{s\in[0,T]} |J_{n-k}^{(m)}(s)| \sup_{s\in[0,T]} \left| \frac{W^{(m)}(s)^k}{k!} - \frac{W(s)^k}{k!} \right| \sup_{s\in[0,T]} \left| \left(\prod_{l=1}^{k} f_{n+1-l} \right)(s) \right| \\
&\quad + \sum_{k=1}^{n} \sup_{s\in[0,T]} |(J_{n-k}^{(m)}(s) - I_{n-k}^S(s)| \sup_{s\in[0,T]} \left| \frac{W(s)^k}{k!} \right| \sup_{s\in[0,T]} \left| \left(\prod_{l=1}^{k} f_{n+1-l} \right)(s) \right| \\
&\quad + \sum_{k=1}^{n} \int_0^T |J_{n-k}^{(m)}(s)| \times \left| \frac{W^{(m)}(s)^k}{k!} - \frac{W(s)^k}{k!} \right| \times \left| \left(\prod_{l=1}^{k} f_{n+1-l} \right)'(s) \right| ds \\
&\quad + \sum_{k=1}^{n} \int_0^T |(J_{n-k}^{(m)}(s) - I_{n-k}^S(s)| \times \left| \frac{W(s)^k}{k!} \right| \times \left| \left(\prod_{l=1}^{k} f_{n+1-l} \right)'(s) \right| ds.
\end{aligned}
$$

Using the equality $(x^k - y^k) = (x-y)\sum_{k=1}^{n} x^{n-k}y^{k-1}$ we have that

$$
|W^{(m)}(s)^k - W(s)^k| \leq |W^{(m)}(s) - W(s)| \sum_{k=1}^{n} |W^{(m)}(s)|^{n-k} |W(s)|^{k-1}.
$$

Then, using that

- $\sup_{s\in[0.T]} |W(s)| < +\infty$ almost surely,
- $\sup_{s\in[0,T]} \big(\big| \big(\prod_{l=1}^{k} f_{n+1-l}\big)(s) \big| + \big| \big(\prod_{l=1}^{k} f_{n+1-l}\big)'(s) \big| \big) < \infty$,
- for almost all ω there exists $m_n(\omega)$ and $K_n(\omega)$, both finite, such that $J_l^{(m)}(s,\omega) \leq K_j(\omega)$ for all $m > m_n$, all $s \in [0,T]$, and all $l \leq n$,
- $\lim_{m\to\infty} \sup_{s\in[0,T]} |W^{(m)}(s) - W(s)| = 0$ almost surely,

and doing an iteration procedure we can easily finish the proof. $\square$

Proof of Theorem 2.2: The proof follows easily from Theorem 2.1 and the strong convergence result in [**11**]. □

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References

[1] X. BARDINA, G. BINOTTO, AND C. ROVIRA, The complex Brownian motion as a strong limit of processes constructed from a Poisson process, *J. Math. Anal. Appl.* **444(1)** (2016), 700–720. DOI: 10.1016/j.jmaa.2016.06.061.

[2] X. BARDINA, M. FERRANTE, AND C. ROVIRA, Strong approximations of Brownian sheet by uniform transport processes, *Collect. Math.* **71(2)** (2020), 319–329. DOI: 10.1007/s13348-019-00263-4.

[3] X. BARDINA AND M. JOLIS, Weak convergence to the multiple Stratonovich integral, *Stochastic Process. Appl.* **90(2)** (2000), 277–300. DOI: 10.1016/S0304-4149(00)00045-4.

[4] M. CSÖRGŐ AND L. HORVÁTH, Rate of convergence of transport processes with an application to stochastic differential equations, *Probab. Theory Related Fields* **78(3)** (1988), 379–387. DOI: 10.1007/BF00334201.

[5] J. GARZÓN, L. G. GOROSTIZA, AND J. A. LEÓN, A strong uniform approximation of fractional Brownian motion by means of transport processes, *Stochastic Process. Appl.* **119(10)** (2009), 3435–3452. DOI: 10.1016/j.spa.2009.06.003.

[6] J. GARZÓN, L. G. GOROSTIZA, AND J. A. LEÓN, Approximations of fractional stochastic differential equations by means of transport processes, *Commun. Stoch. Anal.* **5(3)** (2011), 433–456. DOI: 10.31390/cosa.5.3.01.

[7] J. GARZÓN, L. G. GOROSTIZA, AND J. A. LEÓN, A strong approximation of subfractional Brownian motion by means of transport processes, in: *"Malliavin Calculus and Stochastic Analysis"*, Springer Proc. Math. Stat. **34**, Springer, New York, 2013, pp. 335–360. DOI: 10.1007/978-1-4614-5906-4_15.

[8] J. GARZÓN, S. TORRES, AND C. A. TUDOR, A strong convergence to the Rosenblatt process, *J. Math. Anal. Appl.* **391(2)** (2012), 630–647. DOI: 10.1016/j.jmaa.2012.02.040.

[9] L. G. GOROSTIZA AND R. J. GRIEGO, Strong approximation of diffusion processes by transport processes, *J. Math. Kyoto Univ.* **19(1)** (1979), 91–103. DOI: 10.1215/kjm/1250522470.

[10] L. G. GOROSTIZA AND R. J. GRIEGO, Rate of convergence of uniform transport processes to Brownian motion and application to stochastic integrals, *Stochastics* **3(4)** (1980), 291–303. DOI: 10.1080/17442508008833152.

[11] R. J. GRIEGO, D. HEATH, AND A. RUIZ-MONCAYO, Almost sure convergence of uniform transport processes to Brownian motion, *Ann. Math. Statist.* **42(3)** (1971), 1129–1131. DOI: 10.1214/aoms/1177693346.

[12] Y. HU, *"Analysis on Gaussian Spaces"*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. DOI: 10.1142/10094.

[13] P. E. KLOEDEN AND E. PLATEN, Relations between multiple Itô and Stratonovich integrals, *Stochastic Anal. Appl.* **9(3)** (1991), 311–321. DOI: 10.1080/07362999108809243.

[14] D. F. Kuznetsov, *“Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach”*, Lambert Academic Publishing, Saarbrücken, 2012.
[15] D. F. Kuznetsov, Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise, Preprint (2018). arXiv:1801.00784v6.
[16] E. Wong and M. Zakai, On the convergence of ordinary integrals to stochastic integrals, *Ann. Math. Statist.* **36(5)** (1965), 1560–1564. DOI: 10.1214/aoms/1177699916.

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