Publ. Mat. **65** (2021), 3–59
DOI: 10.5565/PUBLMAT6512101

UNIQUENESS PROPERTY FOR 2-DIMENSIONAL MINIMAL CONES IN $\mathbb{R}^3$

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Abstract: In this article we treat two closely related problems: 1) the upper semi-continuity property for Almgren minimal sets in regions with regular boundary; and 2) the uniqueness property for all the 2-dimensional minimal cones in $\mathbb{R}^3$.

Given an open set $\Omega \subset \mathbb{R}^n$, a closed set $E \subset \Omega$ is said to be Almgren minimal of dimension d in Ω if it minimizes the d-Hausdorff measure among all its Lipschitz deformations in Ω. We say that a d-dimensional minimal set E in an open set Ω admits upper semi-continuity if, whenever $\{f_n(E)\}_n$ is a sequence of deformations of E in Ω that converges to a set F, then we have $\mathcal{H}^d(F) \geq \limsup_n \mathcal{H}^d(f_n(E))$. This guarantees in particular that E minimizes the d-Hausdorff measure, not only among all its deformations, but also among limits of its deformations.

As proved in [**19**], when several 2-dimensional minimal cones are all translational and sliding stable, and admit the uniqueness property, then their almost orthogonal union stays minimal. As a consequence, the uniqueness property obtained in the present paper, together with the translational and sliding stability properties proved in [**18**] and [**20**] permit us to use all known 2-dimensional minimal cones in $\mathbb{R}^n$ to generate new families of minimal cones by taking their almost orthogonal unions.

The upper semi-continuity property is also helpful in various circumstances: when we have to carry on arguments using Hausdorff limits and some properties do not pass to the limit, the upper semi-continuity can serve as a link. As an example, it plays a very important role throughout [**19**].

2010 Mathematics Subject Classification: 28A75, 49Q20, 49K21.

Key words: minimal cones, uniqueness, Hausdorff measure, Plateau's problem.

1. Introduction

The notion of minimal sets in the sense of Almgren [**2**] and Reifenberg [**25**] (see David [**6**] and Liang [**15**] for other variants) is a way to try to solve Plateau's problem in the setting of sets. Plateau's problem, as one of the main interests in geometric measure theory, aims at understanding the existence, regularity, and local structure of physical objects that minimize the area while spanning a given boundary, such as soap films. The result of this article is closely linked to two important aspects of this problem: the local behavior and the local uniqueness of solutions. Here, the local uniqueness means that in a small ball with given Dirichlet value on the boundary of the ball, the solution to the problem is unique.

It is known (cf. Almgren [**2**], David and Semmes [**8**]) that a d-dimensional minimal set E admits a unique tangent plane at almost every point x. In this case the local structure around such a point is very clear: the set E is locally a minimal surface (and hence a real analytic surface) around the point, due to the famous regularity result of Allard [**1**].

So we are mostly interested in what happens around points that admit no tangent plane, namely, the singular points.

In [**6**] David proved that the blow-up limits ("tangent objects") of d-dimensional minimal sets at a point are d-dimensional minimal cones (minimal sets that are cones in the means time). Blow-up limits of a set at a point reflect the asymptotic behavior of the set at infinitesimal scales around this point. As a consequence, a first step to study the local structures of minimal sets is to classify all possible types of singularities – that is to say, minimal cones.

1.1. Local behavior and classification of singularities. The plan for the list of d-dimensional minimal cones in $\mathbb{R}^n$ is very far from clear. Even for $d = 2$, we know very little, except for the case in $\mathbb{R}^3$, where J. Taylor ([**26**]) gave a complete classification in 1976, and the list was in fact already known a century ago in other circumstances (see [**12**] and [**11**]). They are, modulo isomorphism: a plane, a $\mathbb{Y}$ set (the union of three half planes that meet along a straight line where they make angles of 120°), and a $\mathbb{T}$ set (the cone over the 1-skeleton of a regular tetrahedron centered at the origin). See Figure 1.

A $\mathbb{Y}$ set A $\mathbb{T}$ set

FIGURE 1

Based on the above, a natural way to find new types of singularities is by taking unions and products of known minimal cones.

Concerning unions, the minimality of the union of two orthogonal minimal sets of dimension d can be obtained easily from a well known geometric lemma (cf. for example Lemma 5.2 of [**22**]). Thus one suspects that if the angle between two minimal sets is not far from orthogonal, the union of them might also be minimal.

In the case of planes, the author proved in [**14**] and [**17**] that the almost orthogonal union of several d-dimensional planes is Almgren and topologically minimal. When the number of planes is two, this is part of Morgan's conjecture in [**23**] on the angle condition under which a union of two planes is minimal.

As for minimal cones other than unions of planes, since they are all with non isolated singularities (after the structure Theorem 2.25), the situation is much more complicated, as briefly stated in the introduction of [**19**]. Up to now we are able to treat a big part of 2-dimensional cases: in [**19**] we prove that the almost orthogonal union of several 2-dimensional minimal cones (in any ambient dimension) is minimal, provided that all these minimal cones satisfy the following properties: the translational and sliding stabilities and the local uniqueness property. (The theorem is stated separately in the Almgren case and topological case in [**19**].) Moreover, this union still satisfies the translational and sliding stabilities, and the local uniqueness property. This enables us to continue obtaining infinitely many new families of minimal cones by taking a finite number of iterations of almost orthogonal unions.

Here, the uniqueness property of a minimal cone is that in any ball B containing the origin, it is the only minimal set with the given Dirichlet value on ∂B. See Section 2 for the precise definitions. The translational and sliding stabilities of a minimal cone will not be discussed in this paper; see [**19, 18, 20**] for the precise definitions.

The above result of [**19**] makes good sense, because due to the following group of papers (of which the present paper is a part), almost all known 2-dimensional minimal cones satisfy the above mentioned properties (i.e., the translational and sliding stabilities, and the local uniqueness property):

- In the present paper we prove the uniqueness property in $\mathbb{R}^3$: all 2-dimensional minimal cones in $\mathbb{R}^3$ are topological and Almgren unique (Theorems 5.1, 5.2, and 5.6).
- In [**18**] we treat the stability properties: all 2-dimensional minimal cones in $\mathbb{R}^n$ (for any $n \geq 3$) are translational stable, and all 2-dimensional minimal cones in $\mathbb{R}^3$ satisfy the sliding stability.
- For 2-dimensional minimal cones in $\mathbb{R}^n$ for $n \geq 3$, by Theorem 10.1 and Remark 10.5 of [**19**], the almost orthogonal unions of several planes in $\mathbb{R}^n$ are also topological sliding and Almgren sliding stable.
- Besides unions of planes, the only known 2-dimensional minimal cone not contained in $\mathbb{R}^3$ is the set $Y \times Y$, the product of two 1-dimensional $\mathbb{Y}$ sets. The proof of its sliding stability and uniqueness

property are much more involved, so we will treat it in a separate paper [**20**].

After the results of the above papers, together with [**19**], we are able to conclude that if we take a finite number of known 2-dimensional minimal cones, their almost orthogonal union is minimal.

As a small remark, compared to the unions, the case of product is much more mysterious. It is not known in general whether the product of two non trivial minimal cones stays minimal. We even do not know whether the product of a minimal cone with a line stays minimal. Moreover, if we consider the product of two concrete minimal cones (other than planes) one by one, up to now the only known result is the minimality of the product of two 1-dimensional $\mathbb{Y}$ sets (cf. [**16**]). Among all singular minimal cones, 1-dimensional $\mathbb{Y}$ sets are of simplest structure, but still, the proof of the minimality of their product is surprisingly hard.

1.2. About uniqueness of solutions. As mentioned before, we are going to discuss the uniqueness property for 2-dimensional minimal cones in $\mathbb{R}^3$. Roughly speaking, the local uniqueness property for a minimal set is that in a small ball B, it is the unique minimal set with the given Dirichlet value on ∂B. (For cones, we can forget about the word "local".)

Another natural question about Plateau's problem is the uniqueness of solutions.

It is well known that solutions for Plateau's problem are in general not unique, even in codimension 1. The simplest example is the following. Given the union of two parallel circles in $\mathbb{R}^3$, it can be the boundary of at least three types of minimal sets: the union of two disks bounded by the two circles respectively, the part of catenoid, and the third type – a "catenoid" with a central disk. See Figure 2. They admit different topologies and they all exist in soap film experiments.

A catenoid A catenoid with a central disk

FIGURE 2

On the other hand, we know that around a regular point x of a minimal set, the solution is locally unique, because the soap film is locally a minimal graph on a convex part of the tangent plane at x and the uniqueness comes from calibrations for minimal graphs.

The advantage of considering local uniqueness is that we do not have to worry about topology. One can then ask whether this local uniqueness also holds for singular points. Since blow-up limits at singular points are minimal cones, a first step is to study whether each minimal cone is the unique solution, at least under a given topology.

Due to the lack of information on the structure for minimal cones of dimension greater than or equal to 3, we are still far from a general conclusion. However, from the very little information we have, we can still give a positive answer for almost all known 2-dimensional minimal cones. See the account in the last subsection.

1.3. Upper semi-continuity and the organization of the paper. Besides the main results about uniqueness, an indispensable intermediate step in the discussion for the uniqueness property is the upper semi-continuity property for minimal sets with reasonable boundary regularity (Theorem 4.13). It consists of saying that in many cases, when its boundary is not too wild, a minimal set minimizes also the measure in the class of limits of deformations, which is much larger than the class of deformations. This property is helpful in various circumstances. For example, when we have to carry on arguments using Hausdorff limits and some properties do not pass to the limit, the upper semi-continuity can serve as a link. As an example, it plays a very important role throughout [**19**].

The organization of the rest of the article is the following.

In Section 2 we introduce basic definitions and preliminaries for minimal sets, and properties concerning 2-dimensional minimal cones.

The definitions of uniqueness and some related useful properties are given in Section 3.

In Section 4 we prove the upper semi-continuity property for minimal sets with relatively regular boundaries (Theorems 4.1, 4.11, and 4.13). These theorems guarantee in particular that the definition of uniqueness makes good sense for minimal cones and many other minimal sets. It is also useful in many other circumstances; see [**19**] for example.

We prove topological and Almgren uniqueness for each 2-dimensional minimal cone in $\mathbb{R}^3$ in Section 5.

Acknowledgements. This work is supported by China's Recruitement Program of Global Experts, School of Mathematics and Systems Science, Beihang University, and National Natural Science Foundation of China (Grant No. 11871090).

2. Definitions and preliminaries

2.1. Some useful notation. $[a, b]$ is the line segment with endpoints a and b.

$\overrightarrow{ab}$ is the vector $b - a$.

R_{ab} denotes the half line starting from the point a and passing through b.

$B(x, r)$ is the open ball with radius r and centered on x.

$\bar{B}(x, r)$ is the closed ball with radius r and center x.

For any (affine) subspace Q of $\mathbb{R}^n$ and $x \in Q$, $r > 0$, $B_Q(x, r)$ stands for $B(x, r) \cap Q$, the open ball in Q.

For any subset E of $\mathbb{R}^n$, E° denotes the interior of E, $\bar{E}$ denotes the closure of E, and $E^C = \mathbb{R}^n \backslash E$. And for any $m \leq n$ and any m-dimensional dyadic cube Q in $\mathbb{R}^n$, Q° denotes its m-dimensional interior.

For any subset E of $\mathbb{R}^n$, χ_E denotes the characteristic function of E.

For any subset E of $\mathbb{R}^n$ and any $r > 0$, we call $B(E, r) := \{x \in \mathbb{R}^n : \operatorname{dist}(x, E) < r\}$ the r neighborhood of E.

$\mathcal{H}^d$ is the Hausdorff measure of dimension d.

$d_{\mathsf{H}}(E, F) = \max\{\sup\{d(y, F) : y \in E\}, \sup\{d(y, E) : y \in F\}\}$ is the Hausdorff distance between two sets E and F.

For any subset $K \subset \mathbb{R}^n$, the local Hausdorff distance in K d_K between two sets E, F is defined as $d_K(E, F) = \max\{\sup\{d(y, F) : y \in E \cap K\}, \sup\{d(y, E) : y \in F \cap K\}\}$.

For any open subset $U \subset \mathbb{R}^n$, let $\{E_n\}_n$, F be closed sets in U, we say that F is the Hausdorff limit of $\{E_n\}_n$ if for any compact subset $K \subset U$, $\lim_n d_K(E_n, F) = 0$.

$d_{x,r}$: the relative distance with respect to the ball $B(x, r)$ is defined by

$$d_{x,r}(E, F) = \frac{1}{r} \max\{\sup\{d(y, F) : y \in E \cap B(x, r)\}, \\ \sup\{d(y, E) : y \in F \cap B(x, r)\}\}.$$

For any polyhedral complex $\mathcal{S}$ in $\mathbb{R}^n$, let $|\mathcal{S}|$ denote the support of $\mathcal{S}$, that is, $|\mathcal{S}| = \bigcup_{\sigma \in \mathcal{S}} \sigma$. And for any $0 \leq m \leq n$, let $\mathcal{S}_m$ denote the set of all m-faces in $\mathcal{S}$. Then $|\mathcal{S}_m|$ is the m-skeleton of $\mathcal{S}$.

Definition 2.1 (Hausdorff limit in an open set)**.** Let U be an open subset in $\mathbb{R}^n$. Let $\{E_k\}$, E be relatively closed subsets of U. We say that E is the Hausdorff limit of E_k in U if for all compact sets $K \subset U$, $d_K(E_k, E) \to 0$. We also say that E_k converges to E under the Hausdorff limit, and denote this by

$$E_k \xrightharpoonup{U} E.$$

Definition 2.2 (Approximate tangent plane, cf. [**21**, Definition 15.17])**.** Let $A \subset \mathbb{R}^n$, $a \in \mathbb{R}^n$, and V an m-dimensional linear subspace of $\mathbb{R}^n$. We say that V is an m-dimensional approximate tangent plane for A at a if $\theta^{*m}(A,a) > 0$, and for all $0 < s < 1$,

$$\lim_{r\to 0} r^{-m}\mathcal{H}^m(A \cap B(a,r)\backslash X(a,V,s)) = 0.$$

Here $\theta^{*m} = \limsup_{r\to 0} r^{-m}\mathcal{H}^m(A\cap B(a,r))$ is the m-upper density of A at a, and $X(a,V,s) = \{y \in \mathbb{R}^n : d(x-a,V) < s|x-a|\}$.

If E is a d-rectifiable set, denote by T_xE the approximate tangent plane (if it exists and is unique) of E at x.

Remark 2.3*.* We say that V is a true tangent plane of A at a if it is tangent to A at a in the classical sense, that is, for any $0 < s < 1$, there exists $r > 0$, so that

$$A \cap B(a,r)\backslash X(a,V,s) = \emptyset.$$

2.2. Basic definitions and notations about minimal sets. In the next definitions, fix integers $0 < d < n$. We first give a general definition for minimal sets. Briefly, a minimal set is a closed set which minimizes the Hausdorff measure among a certain class of competitors. Different choices of classes of competitors give different kinds of minimal sets.

Definition 2.4 (Minimal sets)**.** Let $0 < d < n$ be integers. Let $U \subset \mathbb{R}^n$ be an open set. A relative closed set $E \subset U$ is said to be minimal of dimension d in U with respect to the competitor class $\mathscr{F}$ (which contains E) if

$$\mathcal{H}^d(E \cap B) < \infty \text{ for every compact ball } B \subset U \tag{2.1}$$

and

$$\mathcal{H}^d(E\backslash F) \le \mathcal{H}^d(F\backslash E) \tag{2.2}$$

for any competitor $F \in \mathscr{F}$.

Definition 2.5 (Almgren competitor (Al competitor for short))**.** Let E be relatively closed in an open subset U of $\mathbb{R}^n$. An Almgren competitor for E is a relatively closed set $F \subset U$ that can be written as $F = \varphi_1(E)$, where $\varphi_t \colon U \to U$, $t \in [0,1]$, is a family of continuous mappings such that

$$\varphi_0(x) = x \text{ for } x \in U; \tag{2.3}$$

$$\text{the mapping } (t,x) \to \varphi_t(x) \text{ of } [0,1]\times U \text{ to } U \text{ is continuous;} \tag{2.4}$$

$$\varphi_1 \text{ is Lipschitz,} \tag{2.5}$$

and if we set $W_t = \{x \in U;\ \varphi_t(x) \neq x\}$ and $\widehat{W} = \bigcup_{t\in[0.1]}[W_t \cup \varphi_t(W_t)]$, then

$$\widehat{W} \text{ is relatively compact in } U. \tag{2.6}$$

Such a φ_1 is called a deformation in U and F is also called a deformation of E in U.

For future convenience, we also have the following more general definition:

Definition 2.6. Let $U \subset \mathbb{R}^n$ be an open set and let $E \subset \mathbb{R}^n$ be a closed set (not necessarily contained in U). We say that E is minimal in U if $E \cap U$ is minimal in U. A closed set $F \subset \mathbb{R}^n$ is called a deformation of E in U if $F = (E\backslash U) \cup \varphi_1(E \cap U)$, where φ_1 is a deformation in U.

Now let $E \subset \mathbb{R}^n$ be closed and denote by $\mathcal{F}(E, U)$ the class of all deformations of E in U as in Definition 2.6. We need to use Hausdorff limits of sequences in $\mathcal{F}(E, U)$. However, if we regard elements of $\mathcal{F}(E, U)$ as sets in $\mathbb{R}^n$ and take the Hausdorff limit, the limit may have positive measure on $\partial U\backslash E$. In other words, sets in $\mathcal{F}(E, U)$ may converge to the boundary. We do not like this. Hence we let $\overline{\mathcal{F}}(E, U)$ be the class of Hausdorff limits in $\mathbb{R}^n$ of sequences in $\mathcal{F}(E, U)$ that essentially do not converge to the boundary. That is, we set

$$\begin{aligned}\overline{\mathcal{F}}(E, U) = \{F \text{ closed} : \exists\{E_k\}_k \subset \mathcal{F}(E, U) \\ \text{such that } E_k \overset{\mathbb{R}^n}{\rightharpoonup} F \text{ and } \mathcal{H}^d(F \cap \partial U\backslash E) = 0\}.\end{aligned} \tag{2.7}$$

It is easy to see that both classes $\mathcal{F}(E, U)$ and $\overline{\mathcal{F}}(E, U)$ are stable with respect to Lipschitz deformations in U.

Definition 2.7 (Almgren minimal sets)**.** Let $0 < d < n$ be integers, and let U be an open set of $\mathbb{R}^n$. An Almgren minimal set E in U is a minimal set defined in Definition 2.4 while taking the competitor class $\mathscr{F}$ to be the class of all Almgren competitors for E.

For our future arguments, we also have the following definition:

Definition 2.8. Let $0 < d < n$ be integers, let U be an open set of $\mathbb{R}^n$. A closed set $E \subset \mathbb{R}^n$ is said to be Almgren minimal in U if $E \cap U$ is Almgren minimal in U.

Next, let us define another type of competitors and minimizers.

Let $k \leq n$. Two subsets A and B of $\mathbb{R}^n$ are said to be k-essentially disjoint if $\mathcal{H}^k(A \cap B) = 0$.

Let U be an open subset of $\mathbb{R}^n$. Since it is a smooth n-manifold, it admits smooth triangulations (cf. [**27**, Chapter IV, §14B, Theorem 12]). As a result, the singular homology and the simplicial homology on U are isomorphic.

For any smooth triangulation $\mathcal{K}$ of U and any k-simplicial G-chain Γ of $\mathcal{K}$, we call Γ a k-simplicial G-chain in U for short.

For any Euclidean k-sphere $S \subset U$, a k-simplicial G-chain Γ is said to be induced by S if $\Gamma = \sum_{i=1}^m \sigma_m$, where σ_m are k-simplices in a triangulation $\mathcal{K}$ of U and S is the k-essentially disjoint union of σ_i, $1 \leq i \leq m$.

Now for any Euclidean k-sphere $S \subset U$, the element represented by S in the simplicial homology group $H_k^\Delta(U; G)$ is the element represented by any k-simplicial G-chain Γ induced by S.

Note that this definition is independent of the choice of smooth triangulation $\mathcal{K}$, since the singular homology on U and the simplicial homology on $\mathcal{K}$ are isomorphic and the singular homology on U is independent of the smooth triangulation.

Definition 2.9 (Topological competitors)**.** Let G be an abelian group. Let E be a relatively closed set in an open set U of $\mathbb{R}^n$. We say that a relatively closed set F is a G-topological competitor of dimension d ($d < n$) of E in U if there exists an open convex set B such that $\bar{B} \subset U$ and

(i) $F \backslash B = E \backslash B$.

(ii) For all Euclidean $(n-d-1)$-sphere $S \subset U \backslash (B \cup E)$, if S represents a nonzero element in the simplicial homology group $H_{n-d-1}^\Delta(U \backslash E; G)$, then it is also nonzero in $H_{n-d-1}^\Delta(U \backslash F; G)$.

We also say that F is a G-topological competitor of dimension d of E with respect to B.

When $G = \mathbb{Z}$, we usually omit $\mathbb{Z}$, and say directly that F is topological competitor of dimension d.

Remark 2.10. Since the singular homology and the simplicial homology are isomorphic both on $U \backslash E$ and on $U \backslash F$, in the above Definition 2.9, it is equivalent to replace condition (ii) by

(ii') For each Euclidean $(n-d-1)$-sphere $S \subset U \backslash (B \cup E)$, if S represents a nonzero element in the singular homology group $H_{n-d-1}(U \backslash E; G)$, then it is also nonzero in $H_{n-d-1}(U \backslash F; G)$.

Definition 2.4 gives the definition of G-topological minimizers of dimension d in an open set U when we take the competitor class to be the class of G-topological competitors of dimension d of E.

The simplest example of a G-topological minimal set is a d-dimensional plane in $\mathbb{R}^n$.

Proposition 2.11 (cf. [**15**, Proposition 3.7 and Corollary 3.17])**.**

(i) *Let $E \subset \mathbb{R}^n$ be closed. Then for any $d < n$ and any open convex set B, B' such that $\bar{B}' \subset B^\circ$, every Almgren competitor of E in B' is a G-topological competitor of E with respect to B of dimension d.*

(ii) *All G-topological minimal sets are Almgren minimal in $\mathbb{R}^n$.*

Remark 2.12. (1) One can see directly from the definition that we have the following transitivity: given a relatively closed set E in an open set $U \subset \mathbb{R}^n$, a deformation of a deformation of E in U is a deformation of E in U, and for any bounded convex open set B so that $\bar{B} \subset U$, a G-topological competitor with respect to B of a G-topological competitor of E with respect to B is a G-topological competitor of E with respect to B of the same dimension.

(2) The class of G-topological competitors of dimension d for a set E is closed under taking supersets. More precisely, given a relatively closed set E in an open set $U \subset \mathbb{R}^n$, if F is a G-topological competitor of E of dimension d in U with respect to B, and $F \subset F'$ where F' is relatively closed, then for any bounded convex open set B' so that $B \subset \bar{B}' \subset U$ and such that $F' \backslash B' = E \backslash B'$, F is a G-topological competitor of E of dimension d in U with respect to B'. In fact, take any $(n-d-1)$-sphere $S \subset U \backslash (B' \cup E)$, it is contained in $S \subset U \backslash (B' \cup F)$, and since F is a G-topological competitor of E of dimension d with respect to B, if S represents a nonzero element in $H_{n-d-1}(U \backslash E, G)$, then it represents a nonzero element in $H_{n-d-1}(U \backslash F, G)$, and thus it represents a nonzero element in $H_{n-d-1}(U \backslash F', G)$ because $F \subset F'$.

(3) The notion of (Almgren or G-topological) minimal sets does not depend much on the ambient dimension. One can easily check that $E \subset U$ is d-dimensional Almgren minimal in $U \subset \mathbb{R}^n$ if and only if E is Almgren minimal in $U \times \mathbb{R}^m \subset \mathbb{R}^{m+n}$ for any integer m. The case of G-topological minimality is proved in [**15**, Proposition 3.18].

Proposition 2.13 (Topological competitors pass to the limit)**.** *Let E be a closed set in an open set U of $\mathbb{R}^n$ and let B' be a open convex set such that $\bar{B}' \subset U$. If $\{F_n\}$ is a sequence of d-dimensional G-topological competitors of E with respect to B', and F_n converge to F in Hausdorff distance, then for any open convex set B such that $\bar{B}' \subset B \subset \bar{B} \subset U$, F is a G-topological competitor of dimension d of E with respect to B.*

Proof: Let us verify the two conditions in Definition 2.9.

Since F_j converge to F and $F_j \backslash B' = E \backslash B'$, we have $F \backslash \bar{B}' = E \backslash \bar{B}'$. Since $\bar{B}' \subset B$, we know that (i) holds.

Now take any $(n-d-1)$-sphere $S \subset U\backslash(B \cup E)$ that represents a nonzero element in $H^{\Delta}_{n-d-1}(U\backslash E; G)$. Since $B' \subset B$, we know that $S \subset U\backslash(B' \cup E)$. We know that each F_j is a G-topological competitor of dimension d for E with respect to B, hence S also represents a nonzero element in $H^{\Delta}_{n-d-1}(U\backslash F_j; G)$.

For (ii), suppose it does not hold. That is, S represents a zero element in $H^{\Delta}_{n-d-1}(U\backslash F; G)$. As a result, there exists a simplicial $n-d$-G-chain σ in $U\backslash F$, and a simplicial $n-d-1$-G-chain $\tilde{S}$ induced by S such that $\partial\sigma = \tilde{S}$. Then the support $|\sigma|$ of σ is compact in $U\backslash F$. Since $U\backslash F$ is open, there exists $\epsilon > 0$ such that the ϵ-neighborhood $B(|\sigma|, \epsilon) \subset U\backslash F$. As a result, since $F_j \to F$, we know that for j large enough, $F_j \cap |\sigma| = \emptyset$. Hence σ is also a simplicial G-chain in $U\backslash F_j$ for j large. Then $\partial\sigma = \tilde{S}$ implies that S represents a zero element in $H_{n-d-1}(U\backslash F_j; G)$ for j large. This contradicts the fact that S represents a nonzero element in $H_{n-d-1}(U\backslash F_j; G)$ for all j.

Hence (ii) holds. □

Definition 2.14 (Reduced set)**.** Let $U \subset \mathbb{R}^n$ be an open set. For every closed subset E of U, denote by

$$E^* = \{x \in E : \mathcal{H}^d(E \cap B(x, r)) > 0 \text{ for all } r > 0\} \tag{2.8}$$

the closed support (in U) of the restriction of $\mathcal{H}^d$ to E. We say that E is reduced if $E = E^*$.

It is easy to see that

$$\mathcal{H}^d(E\backslash E^*) = 0. \tag{2.9}$$

In fact, we can cover $E\backslash E^*$ by countably many balls B_j such that $\mathcal{H}^d(E \cap B_j) = 0$.

Remark 2.15. It is not hard to see that if E is Almgren minimal (resp. G-topologically minimal), then E^* is also Almgren minimal (resp. G-topologically minimal). As a result, it is enough to study reduced minimal sets. An advantage of reduced minimal sets is that they are locally Ahlfors regular (cf. Proposition 4.1 in [**8**]). Hence any approximate tangent plane of them is a true tangent plane (as in Remark 2.3). Since minimal sets are rectifiable (cf. [**8**, Theorem 2.11] for example), reduced minimal sets admit true tangent d-planes almost everywhere.

If we regard two sets to be equivalent if they are equal modulo $\mathcal{H}^d$-null sets, then a reduced set is always considered to be a good (in the sense of regularity) representative of its equivalence class.

In the rest of the article we only consider reduced sets.

Remark 2.16. (1) One can see directly from the definition that we have the following transitivity: given a relatively closed set E in an open set $U \subset \mathbb{R}^n$, a deformation of a deformation of E in U is a deformation of E in U, and for any bounded convex open set B so that $\bar{B} \subset U$, a G-topological competitor with respect to B of a G-topological competitor of E with respect to B is a G-topological competitor of E with respect to B of the same dimension.

(2) The class of G-topological competitors for a set E is closed under taking supersets. More precisely, given a relatively closed set E in an open set $U \subset \mathbb{R}^n$, if F is a G-topological competitor of E of dimension d in U with respect to B, and $F \subset F'$ where F' is relatively closed, then for any bounded convex open set B' so that $B \subset \bar{B}' \subset U$ and such that $F'\backslash B' = E\backslash B'$, F is a G-topological competitor of E of dimension d in U with respect to B'. In fact, take any $(n-d-1)$-sphere $S \subset U\backslash(B' \cup E)$. Then it is contained in $S \subset U\backslash(B' \cup F)$, and since F is a G-topological competitor of E of dimension d with respect to B, if S represents a nonzero element in $H_{n-d-1}(U\backslash E, G)$, then it represents a nonzero element in $H_{n-d-1}(U\backslash F, G)$, and thus it represents a nonzero element in $H_{n-d-1}(U\backslash F', G)$ because $F \subset F'$.

(3) The notion of Almgren or G-topological minimal sets does not depend much on the ambient dimension. One can easily check that $E \subset U$ is d-dimensional Almgren minimal in $U \subset \mathbb{R}^n$ if and only if E is Almgren minimal in $U \times \mathbb{R}^m \subset \mathbb{R}^{m+n}$ for any integer m. The case of G-topological minimality is proved in [**15**, Proposition 3.18].

2.3. Regularity results for minimal sets. We now begin to give regularity results for minimal sets. They are in fact regularity results for Almgren minimal sets, but they also hold for all G-topological minimizers, after Proposition 2.11. By Remark 2.15, from now on all minimal sets are supposed to be reduced.

Definition 2.17 (Blow-up limit)**.** Let $U \subset \mathbb{R}^n$ be an open set, let E be a relatively closed set in U, and let $x \in E$. Denote by $E(r, x) = r^{-1}(E-x)$. A set C is said to be a blow-up limit of E at x if there exists a sequence of numbers r_n, with $\lim_{n\to\infty} r_n = 0$, such that the sequence of sets $E(r_n, x)$ converges to C for the local Hausdorff distance in any compact set of $\mathbb{R}^n$.

Remark 2.18. (1) A set E might have more than one blow-up limit at a point x. However, it is not known yet whether this can happen to minimal sets.

When a set E admits a unique blow-up limit at a point $x \in E$, denote this blow-up limit by $C_x E$.

(2) Let $Q \subset \mathbb{R}^n$ be any subspace and denote by π_Q the orthogonal projection from $\mathbb{R}^n$ to Q. Then it is easy to see that if $E \subset \mathbb{R}^n$, $x \in E$, and C is any blow-up limit of E at x, then $\pi_Q(C)$ is contained in a blow-up limit of $\pi_Q(E)$ at $\pi_Q(x)$.

Proposition 2.19 (c.f. [**6**, Proposition 7.31])**.** *Let E be a reduced Almgren minimal set in an open set U of $\mathbb{R}^n$ and let $x \in E$. Then every blow-up limit of E at x is a reduced Almgren minimal cone F centered at the origin, and $\mathcal{H}^d(F \cap B(0,1)) = \theta(x) := \lim_{r\to 0} r^{-d}\mathcal{H}^d(E \cap B(x,r))$.*

An Almgren minimal cone is just a cone which is also Almgren minimal. We will call them minimal cones throughout this paper, since we will not talk about any other type of minimal cones.

Remark 2.20. (1) The existence of the density $\theta(x)$ is due to the monotonicity of the density function $\theta(x,r) := r^{-d}\mathcal{H}^d(E \cap B(x,r))$ at any given point x for minimal sets. See for example [**6**, Proposition 5.16].

(2) After the above proposition, the set $\Theta(n,d)$ of all possible densities for points in a d-dimension minimal set in $\mathbb{R}^n$ coincides with the set of all possible densities for d-dimensional minimal cones in $\mathbb{R}^n$. When $d = 2$, this is a very small set. For example, we know that π is the density for a plane, $\frac{3}{2}\pi$ is the density for a $\mathbb{Y}$ set, and for any n and any other type of 2-dimensional minimal cone in $\mathbb{R}^n$, its density should be no less than some $d_T = d_T(n) > \frac{3}{2}\pi$, by [**6**, Lemma 14.12].

(3) Obviously, a cone in $\mathbb{R}^n$ is minimal if and only if it is minimal in the unit ball, if and only if it is minimal in any open subset containing the origin.

(4) For future convenience, we also set the following notation: let $U \subset \mathbb{R}^n$ be a open convex set containing the origin. A set $C \subset U$ is called a cone in U if it is the intersection of a cone with U.

We now state some regularity results on 2-dimensional Almgren minimal sets.

Definition 2.21 (Bi-Hölder ball for closed sets)**.** Let E be a closed set of Hausdorff dimension 2 in $\mathbb{R}^n$. We say that $B(0,1)$ is a bi-Hölder ball for E with constant $\tau \in (0,1)$ if we can find a 2-dimensional minimal cone Z in $\mathbb{R}^n$ centered at 0, and $f\colon B(0,2) \to \mathbb{R}^n$ with the following properties:

(i) $f(0) = 0$ and $|f(x) - x| \le \tau$ for $x \in B(0,2)$;

(ii) $(1-\tau)|x-y|^{1+\tau} \le |f(x)-f(y)| \le (1+\tau)|x-y|^{1-\tau}$ for $x, y \in B(0,2)$;

(iii) $B(0, 2-\tau) \subset f(B(0,2))$;

(iv) $E \cap B(0, 2-\tau) \subset f(Z \cap B(0,2)) \subset E$.

We also say that $B(0,1)$ is of type Z for E.

We say that $B(x,r)$ is a bi-Hölder ball for E of type Z (with the same parameters) when $B(0,1)$ is a bi-Hölder ball of type Z for $r^{-1}(E-x)$.

Theorem 2.22 (Bi-Hölder regularity for 2-dimensional Almgren minimal sets, c.f. [**6**, Theorem 16.1])**.** *Let U be an open set in $\mathbb{R}^n$ and E a reduced Almgren minimal set in U. Then for each $x_0 \in E$ and every choice of $\tau \in (0,1)$, there is an $r_0 > 0$ and a minimal cone Z such that $B(x_0, r_0)$ is a bi-Hölder ball of type Z for E with constant τ. Moreover, Z is a blow-up limit of E at x.*

Definition 2.23 (Point of type Z)**.** (i) In the above theorem, we say that x_0 is a point of type Z (or Z point for short) of the minimal set E. The set of all points of type Z in E is denoted by E_Z.

(ii) In particular, we denote by E_P the set of regular points of E and E_Y the set of $\mathbb{Y}$ points of E. Any 2-dimensional minimal cone other than planes and $\mathbb{Y}$ sets are called $\mathbb{T}$ type cone, and any point which admits a $\mathbb{T}$ type cone as a blow-up is called a $\mathbb{T}$ type point. Set $E_T = E \backslash (E_Y \cup E_P)$ the set of all $\mathbb{T}$ type points of E. Set $E_S := E \backslash E_P$ the set of all singular points in E.

Remark 2.24. Again, since we might have more than one blow-up limit for a minimal set E at a point $x_0 \in E$, the point x_0 might be of more than one type (but all the blow-up limits at a point are bi-Hölder equivalent). However, if one of the blow-up limits of E at x_0 admits the "full length" property (see Remark 2.26), then in fact E admits a unique blow-up limit at the point x_0. Moreover, we have the following $C^{1,\alpha}$-regularity around the point x_0.

Theorem 2.25 ($C^{1,\alpha}$-regularity for 2-dimensional minimal sets, c.f. [**7**, Theorem 1.15])**.** *Let E be a 2-dimensional reduced minimal set in the open set $U \subset \mathbb{R}^n$. Let $x \in E$ be given. Suppose in addition that some blow-up limit of E at x is a full length minimal cone (see Remark 2.26). Then there is a unique blow-up limit X of E at x, and $x + X$ is tangent to E at x. In addition, there is a radius $r_0 > 0$ such that, for $0 < r < r_0$, there is a $C^{1,\alpha}$ diffeomorphism (for some $\alpha > 0$) $\Phi \colon B(0,2r) \to \Phi(B(0,2r))$ such that $\Phi(0) = x$ and $|\Phi(y) - x - y| \leq 10^{-2} r$ for $y \in B(0,2r)$ and $E \cap B(x,r) = \Phi(X) \cap B(x,r)$.*

We can also ask that $D\Phi(0) = \mathrm{Id}$. We call $B(x,r)$ a C^1 ball for E of type X.

Remark 2.26 (Full length, union of two full length cones $X_1 \cup X_2$). We are not going to give the precise definition of the full length property. Instead, we just give some information here which is enough for the proofs in this paper.

(1) The three types of 2-dimensional minimal cones in $\mathbb{R}^3$, i.e. the planes, the $\mathbb{Y}$ sets, and the $\mathbb{T}$ sets, all verify the full length property (cf. [**7**, Lemmas 14.4, 14.6, and 14.27]). Hence all 2-dimensional minimal sets E in an open set $U \subset \mathbb{R}^3$ admits the local $C^{1,\alpha}$-regularity at every point $x \in E$. But this was known from [**26**].

(2) Let $n > 3$. Note that the planes, the $\mathbb{Y}$ sets, and the $\mathbb{T}$ sets are also minimal cones in $\mathbb{R}^n$. Denote by $\mathfrak{C}$ the set of all planes, $\mathbb{Y}$ sets, and $\mathbb{T}$ sets in $\mathbb{R}^n$. Let $X = \bigcup_{1\le i\le n} X_i \in \mathbb{R}^n$ be a minimal cone, where $X_i \in \mathfrak{C}$, $1 \le i \le n$, and for any $i \neq j$, $X_i \cap X_j = \{0\}$. Then X also verifies the full length property (cf. [**7**, Remark 14.40]).

Theorem 2.27 (Structure of 2-dimensional minimal cones in $\mathbb{R}^n$, cf. [**6**, Proposition 14.1]). *Let K be a reduced 2-dimensional minimal cone in $\mathbb{R}^n$ and let $X = K \cap \partial B(0, 1)$. Then X is a finite union of great circles and arcs of great circles C_j, $j \in J$. The C_j can only meet at their endpoints, and each endpoint is a common endpoint of exactly three C_j, which meet with $120°$ angles. In addition, the length of each C_j is at least η_0, where $\eta_0 > 0$ depends only on the ambient dimension n.*

An immediate corollary of the above theorem is the following:

Corollary 2.28. (i) *If C is a minimal cone of dimension 2, then for the set of regular points C_P of C, each of its connected components is a planar sector (the cone centered at 0 over an arc of great circle centered at 0).*

(ii) *Let E be a 2-dimensional minimal set in $U \subset \mathbb{R}^n$. Then $\bar{E}_Y = E_S$.*

(iii) *The set $E_S \backslash E_Y$ is composed of isolated points.*

As a consequence of the C^1-regularity for regular points and $\mathbb{Y}$ points, and Corollary 2.28, we have

Corollary 2.29. *Let E be an 2-dimensional Almgren minimal set in an open set $U \subset \mathbb{R}^n$. Then*

(i) *The set E_P is open in E.*

(ii) *The set E_Y is a countable union of C^1 curves. The endpoints of these curves are either in $E_T := E_S \backslash E_Y$, or lie in ∂U.*

We also have a similar quantified version of the $C^{1,\alpha}$-regularity (cf. [**6**, Corollary 12.25]). In particular, we can use the distance between a minimal set and a $\mathbb{P}$ or a $\mathbb{Y}$ cone to control the constants of the

$C^{1,\alpha}$ parametrization. As a direct corollary, we have the following neighborhood deformation retract property for regular and $\mathbb{Y}$ points:

Corollary 2.30. *There exists $\epsilon_2 = \epsilon_2(n) > 0$ such that the following holds: let E be an 2-dimensional Almgren minimal set in an open set $U \subset \mathbb{R}^n$. Then*

(i) *For any $x \in E_P$ and any codimension 1 submanifold $M \subset U$ which contains x, such that M is transversal to the tangent plane T_xE+x, if $r > 0$ satisfies that $d_{x,r}(E, x+T_xE) < \epsilon_2$, then $\mathcal{H}^1(B(x,r) \cap M \cap E) < \infty$, and $B(x,r) \cap M \cap E$ is a Lipschitz deformation retract of $B(x,r) \cap M$.*

(ii) *For any $x \in E_Y$ and any codimension 1 submanifold $M \subset U$ which contains x, such that M is transversal to the tangent cone C_xE+x and its spine, if $r > 0$ satisfies that $d_{x,r}(E, x + C_xE) < \epsilon_2$, then $\mathcal{H}^1(B(x,r) \cap M \cap E) < \infty$, and $B(x,r) \cap M \cap E$ is a Lipschitz deformation retract of $B(x,r) \cap M$.*

As for the regularity for minimal sets of higher dimensions, we know much less. But for points which admit a tangent plane (i.e. some blow-up limit on the point is a plane), we still have the C^1-regularity.

Theorem 2.31 (cf. [**14**, Proposition 6.4])**.** *For $2 \leq d < n < \infty$, there exists $\epsilon_1 = \epsilon_1(n,d) > 0$ such that if E is a d-dimensional reduced minimal set in an open set $U \subset \mathbb{R}^n$, with $B(0,2) \subset U$ and $0 \in E$. Then if E is ϵ_1 near a d-plane P in $B(0,1)$, then E coincides with the graph of a C^1 map $f\colon P \to P^\perp$ in $B\big(0,\frac{3}{4}\big)$. Moreover, $||\nabla f||_\infty < 1$.*

Remark 2.32. (1) This proposition is a direct corollary of Allard's famous regularity theorem for stationary varifolds. See [**1**].

(2) After this proposition, a blow-up limit of a reduced minimal set E at a point $x \in E$ is a plane if and only if the plane is the unique approximate tangent plane of E at x.

After Remark 2.32, for any reduced minimal set E of dimension d, and for any $x \in E$ at which an approximate tangent d-plane exists (which is true for a.e. $x \in E$), T_xE also denotes the tangent plane of E at x and the blow-up limit of E at x.

3. Uniqueness: definitions and properties

Definition 3.1. Let $U \subset \mathbb{R}^n$ be a bounded open set. Let $C \subset \mathbb{R}^n$ be a reduced set so that $C \cap U$ is d-dimensional Almgren minimal in U. We say that

(i) C is Almgren unique in U if $\mathcal{H}^d(C\cap\bar U)=\inf_{F\in\overline{\mathcal{F}}(C,U)}\mathcal{H}^d(F\cap\bar U)$ and

(3.1) $$\forall \text{ reduced set } E\in\overline{\mathcal{F}}(C,U),\ \mathcal{H}^d(E\cap\bar U)=\inf_{F\in\overline{\mathcal{F}}(C,U)}\mathcal{H}^d(F\cap\bar U) \Rightarrow E=C \text{ (or, equivalently, } E\cap\bar U=C\cap\bar U).$$

(ii) C is G-topologically unique in U if $C\cap U$ is d-dimensional G-topological minimal in U, and

(3.2) $$\text{For any reduced } d\text{-dimensional } G\text{-topological competitor } E \text{ of } C\cap U \text{ in U, } \mathcal{H}^d(E)=\mathcal{H}^d(C\cap U) \text{ implies } E=C\cap U.$$

(iii) We say that a d-dimensional Almgren minimal set C in $\mathbb{R}^n$ is Almgren (resp. G-topologically) unique if it is Almgren (resp. G-topologically) unique in every bounded open set $U\subset\mathbb{R}^n$.

When $G=\mathbb{Z}$, we usually omit $\mathbb{Z}$, and say directly topologically unique.

For minimal cones, we immediately have:

Proposition 3.2 (Unique minimal cones). *Let K be a d-dimensional Almgren minimal cone in $\mathbb{R}^n$. Then it is Almgren (resp. G-topologically) unique if and only if it is Almgren (resp. G-topologically) unique in some bounded open convex set U that contains the origin.*

Proof: By definition, the only if part is trivial. So let us prove the converse.

Suppose that K is a d-dimensional Almgren minimal cone in $\mathbb{R}^n$ and is Almgren (resp. G-topologically) unique in a bounded convex open set U that contains the origin. Then since K is a cone centered at the origin, K is Almgren (resp. G-topologically) unique in rU for all $r>0$. Now, for any other bounded open set U', there exists r such that $U'\subset rU$, hence K is Almgren (resp. G-topologically) unique in U'. □

Let us make some important remarks:

Remark 3.3. (1) Note that for an arbitrary d-dimensional reduced set $C\subset\mathbb{R}^n$ which is Almgren minimal in U, by definition, C only minimizes the measure in the class $\mathcal{F}(C,U)$. Hence we do not necessarily have that

(3.3) $$\mathcal{H}^d(C\cap\bar U)=\inf_{F\in\overline{\mathcal{F}}(C,U)}\mathcal{H}^d(F\cap\bar U).$$

On the other hand, by Theorem 4.13, this holds if U is a convex open set and $C\cap\partial U$ is relatively regular.

(2) Unlike the definition of Almgren uniqueness, in the definition for topological uniqueness we do not consider limits of G-topological competitors.
(3) As a corollary of the above term (1) and Proposition 3.2, we know that if K is a d-dimensional minimal cone in $\mathbb{R}^n$, then (3.3) holds automatically.
(4) The condition $\mathcal{H}^d(E \cap \bar{U}) = \inf_{F \in \overline{\mathcal{F}}(C,U)} \mathcal{H}^d(F \cap \bar{U})$ in (3.1) already implies that E is itself Almgren minimal in U, because the class $\overline{\mathcal{F}}(C,U)$ is stable under deformations in U and hence $\mathcal{F}(E,U) \subset \overline{\mathcal{F}}(C,U)$ (cf. Remark 2.16). Also notice that $\mathcal{H}^d(E \cap \bar{U}) = \inf_{F \in \overline{\mathcal{F}}(C,U)} \mathcal{H}^d(F \cap \bar{U})$ is equivalent to the condition $\mathcal{H}^d(E \cap \bar{U}) \leq \inf_{F \in \overline{\mathcal{F}}(C,U)} \mathcal{H}^d(F \cap \bar{U})$ since $E \in \overline{\mathcal{F}}(C,U)$.
(5) Similarly, when U is a convex open set, since the condition $\mathcal{H}^d(E \cap U) = \mathcal{H}^d(C \cap U)$ in (3.2) implies that E minimizes measure among all d-dimensional G-topological competitors of C in U, and all d-dimensional G-topological competitors of E in U are d-dimensional G-topological competitors of C in U (cf. Remark 2.16), we have that E is G-topological minimal of dimension d in U.
(6) If C is an Almgren unique minimal set in U, $V \subset U$ is an open set, then C is also Almgren unique minimal in V.

Proposition 3.4 (Independence of ambient dimension). *Let $K \subset \mathbb{R}^m$ be a d-dimensional Almgren minimal cone in $\mathbb{R}^m$. If K is Almgren (resp. G-topologically) unique, then for all $n \geq m$, K is also Almgren (resp. G-topologically) unique in $\mathbb{R}^n$ while regarded as a subset of $\mathbb{R}^n$ in the natural sense.*

Proof: Fix any $n \geq m$. Write $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ and suppose, without loss of generality, that K is contained in $\mathbb{R}^m \times \{0\}$.

Suppose that K is Almgren unique in $\mathbb{R}^m$. We want to prove that K is Almgren unique in $\mathbb{R}^n$. Let B_n denote the unit ball in $\mathbb{R}^n$. Then by Proposition 3.2, it is enough to prove that $K \cap B_n$ is Almgren unique in B_n. So let $F \in \overline{\mathcal{F}}(K, B_n)$ be reduced, such that

$$\mathcal{H}^d(F \cap \bar{B}_n) = \inf_{E \in \overline{\mathcal{F}}(K,B_n)} \mathcal{H}^d(E \cap \bar{B}_n). \tag{3.4}$$

By Remark 3.3 (5), condition (3.4) implies that F is Almgren minimal in B_n. As a result, by the convex hull property of minimal sets, a reduced minimal set must be contained in the convex hull of its boundary, hence we know that F must be included in the convex hull of $F \cap \partial B_n = K \cap \partial B_n = K \cap \partial B_m \subset \bar{B}_m$.

As a result, $F \in \overline{\mathcal{F}}(K, B_m)$. Since $\overline{\mathcal{F}}(K, B_m) \subset \overline{\mathcal{F}}(K, B_n)$, hence

$$\inf_{E \in \overline{\mathcal{F}}(K,B_n)} \mathcal{H}^d(E \cap \bar{B}_n) \leq \inf_{E \in \overline{\mathcal{F}}(K,B_m)} \mathcal{H}^d(E \cap \bar{B}_m). \tag{3.5}$$

Combined with (3.4), we obtain

$$\mathcal{H}^d(F \cap \bar{B}_m) = \mathcal{H}^d(F \cap \bar{B}_n) \leq \inf_{E \in \overline{\mathcal{F}}(K,B_m)} \mathcal{H}^d(E \cap \bar{B}_m). \tag{3.6}$$

By (3.6), and the Almgren uniqueness of K in $\mathbb{R}^m$, we know that F must be $K \cap B_m = K \cap B_n$.

The proof for the case of G-topological uniqueness is similar and we leave it to the reader. □

The next proposition shows that for relatively regular d-dimensional minimal cones, G-topological uniqueness implies Almgren uniqueness:

Proposition 3.5. *Let $K \subset \mathbb{R}^n$ be a G-topologically unique minimal cone of dimension d. Then it is also Almgren unique of dimension d.*

Proof: Let K be a G-topologically unique minimal cone of dimension d in $\mathbb{R}^n$. By Proposition 3.2, it is enough to prove that K is Almgren unique of dimension d in the unit ball $B = B(0,1)$.

Let $F \in \overline{\mathcal{F}}(K, B)$ be reduced such that

$$\mathcal{H}^d(F \cap \bar{B}) = \inf_{E \in \overline{\mathcal{F}}(K,B)} \mathcal{H}^d(E \cap \bar{B}). \tag{3.7}$$

Note that by Propositions 2.11 and 2.13, we know that F is a G-topological competitor of dimension d for K in $\mathbb{R}^n$ with respect to $2B$. Since K is topologically minimal of dimension d,

$$\mathcal{H}^d(F \cap 2B) \geq \mathcal{H}^d(K \cap 2B). \tag{3.8}$$

Note that $F \backslash \bar{B} = K \backslash \bar{B}$, hence

$$\mathcal{H}^d(F \cap \bar{B}) \geq \mathcal{H}^d(K \cap \bar{B}). \tag{3.9}$$

But $K \in \overline{\mathcal{F}}(K, B)$. Combined with (3.7), we get

$$\mathcal{H}^d(F \cap \bar{B}) \geq \mathcal{H}^d(K \cap \bar{B}) \geq \inf_{E \in \overline{\mathcal{F}}(K,B)} \mathcal{H}^d(E \cap \bar{B}) = \mathcal{H}^d(F \cap \bar{B}), \tag{3.10}$$

hence

$$\mathcal{H}^d(F \cap \bar{B}) = \mathcal{H}^d(K \cap \bar{B}). \tag{3.11}$$

Again because $F \backslash \bar{B} = K \backslash \bar{B}$, we get that

$$\mathcal{H}^d(F \cap 2B) = \mathcal{H}^d(K \cap 2B). \tag{3.12}$$

Now since K is a G-topologically unique minimal cone of dimension d, it is topologically unique of dimension d in $2B$. Since F is a G-topological competitor of dimension d for K in $\mathbb{R}^n$ with respect to $2B$, (3.12) implies that $F = K$. □

4. Upper semi-continuity

In this section we prove the upper semi-continuity property for minimal sets with reasonable boundary regularity. This consists of saying that in many cases, when its boundary is not too wild, a minimal set minimizes also the measure in the class of limits of deformations. This serves as an indispensable part in the definition of uniqueness, as we have already seen in the last section (Remark 3.3). This property also plays a very important role in [**19**].

For each $k \in \mathbb{N}$, let Δ_k denote the family of (closed) dyadic cubes of side-length 2^{-k}. For $j \le n$, let $\Delta_{k,j}$ denote the set of all j-dimensional faces of elements in Δ_k. For each cube Q, denote by $\Delta_j(Q)$ the set of all j-faces of Q. Set $|\Delta_{k,j}| = \bigcup_{\sigma \in \Delta_{k,j}} \sigma$ the j-skeleton of Δ_k.

Theorem 4.1. *Let $0 < d < n$, let $U \subset \mathbb{R}^n$ be a bounded convex open set, and E be a closed set with finite d-Hausdorff measure such that $E \subset \bar{U}$. Let C denote the convex hull of E. Suppose that*

$$C \cap \partial U = E \cap \partial U \tag{4.1}$$

and

$$E \cap \partial U \subset |\Delta_{k_0, d-1}| \text{ for some } k_0 \in \mathbb{N}, \tag{4.2}$$

where Q_0 denotes the unit cube $[0,1]^n$. Then

(i) $\inf_{F \in \overline{\mathcal{F}}(E,U)} \mathcal{H}^d(F) = \inf_{F \in \mathcal{F}(E,U)} \mathcal{H}^d(F)$.

(ii) *If E is a d-dimensional minimal set in U, then*

$$\mathcal{H}^d(E) = \inf_{F \in \overline{\mathcal{F}}(E,U)} \mathcal{H}^d(F). \tag{4.3}$$

Proof of Theorem 4.1: (i) Since $\mathcal{F}(E,U) \subset \overline{\mathcal{F}}(E,U)$, we have automatically $\inf_{F \in \overline{\mathcal{F}}(E,U)} \mathcal{H}^d(F) \le \inf_{F \in \mathcal{F}(E,U)} \mathcal{H}^d(F)$. So let us prove the converse.

Set $\partial E = E \cap \partial U$.

Now we need the following theorem.

Theorem 4.2 (Existence of minimal sets; c.f. [**10**, Théorème. 6.1.7])**.** *Let $U \subset \mathbb{R}^n$ be an open set, $0 < d < n$, and let $\mathfrak{F}$ be a class of nonempty sets relatively closed in U and satisfying* (2.1)*, which is stable by deformations in U. Suppose that*

$$\inf_{F \in \mathfrak{F}} \mathcal{H}^d(F) < \infty. \tag{4.4}$$

Then there exists $M > 0$ (depending only on d and n), a sequence (F_k) of elements of $\mathfrak{F}$, and a set E of dimension d relatively closed in U that verifies (2.1) *such that:*

(i) *For all compact subsets K of U,*

$$\lim_{k\to\infty} d_{\mathsf{H}}(F_k\cap K, E\cap K)=0. \tag{4.5}$$

(ii) *For all open sets V such that $\bar{V}$ is relatively compact in U, there exists $k_0\in\mathbb{N}$ such that for all $k>k_0$,*

$$F_k \text{ is } (M,+\infty)\text{-quasiminimal in } V. \tag{4.6}$$

*(See [**8**] for a precise definition.)*

(iii) $H^d(E)\leq \inf_{F\in\mathfrak{F}} H^d(F)$.

(iv) *E is minimal in U.*

Remark 4.3. Note that in general, the local Hausdorff distance $d_K(E,F)$ between two sets E and F are not the same as $d_{\mathsf{H}}(E\cap K, F\cap K)$, but it is easy to see that for any compact set K, and any two sets E and F, $d_K(E,F)\leq d_{\mathsf{H}}(E\cap K,F\cap K)$. In particular, (i) implies that

$$F_k \xrightharpoonup{U} E. \tag{4.7}$$

We get back to the proof of Theorem 4.1.

Since U is bounded, there exists $R>0$ so that $\bar{U}\subset B(0,R)$. Set $V=B(0,R)\backslash\partial E$. Then V is an open set that contains U and $\bar{U}\backslash\partial E\subset V$.

Proposition 4.4. *Let n, d, E, U, and C be as in the statement of Theorem 4.1, so that* (4.1) *holds. Let V be as defined above. Then there exists $\{G_k\}_{k\in\mathbb{N}}\in\mathcal{F}(E,U)$ and $G_0\in\overline{\mathcal{F}}(E,U)$ such that the following holds:*

(i) *G_0 is minimal in V, $G_0\subset C$, and $G_0\cap\partial U=C\cap\partial U=\partial E$;*

(ii) $\lim_{k\to\infty} d_{\mathsf{H}}(G_k,G_0)=0$;

(iii) $\mathcal{H}^d(G_0)\leq \inf_{F\in\overline{\mathcal{F}}(E,U)}\mathcal{H}^d(F)$.

Proof: Let $\mathfrak{F}=\{F\in\overline{\mathcal{F}}(E,V): F \text{ satisfy } (2.1)\}$. Then $\mathfrak{F}$ is stable by deformations in V.

We apply Theorem 4.2 to the class $\mathfrak{F}$ and get a sequence $F_k\in\mathfrak{F}$ and a set F_0 such that F_0 is minimal in V, with

$$F_k \xrightharpoonup{V} F_0 \tag{4.8}$$

and

$$\mathcal{H}^d(F_0)\leq \inf_{F\in\mathfrak{F}}\mathcal{H}^d(F). \tag{4.9}$$

Since $\mathcal{H}^d(\partial E)=0$ and $V=B(0,R)\backslash\partial E$, we may suppose $\partial E\subset F_0$. Otherwise, we just replace F_0 by $F_0\cup\partial E$ and still satisfies all the above properties.

Note that each F_k belongs to $\overline{\mathcal{F}}(E,V)$, hence for each k there exists a sequence $\{F_k^j\}_j \subset \mathcal{F}(E,V)$ so that $F_k^j \xrightarrow{\mathbb{R}^n} F_k$. Note that $\bar{B}(0,2R)$ is a compact subset in $\mathbb{R}^n$. Hence we have

$$d_{\bar{B}(0,2R)}(F_k^j, F_k) \to 0,\ j \to \infty. \tag{4.10}$$

But the F_k and F_k^j are subsets of $\bar{B}(0,R)$, hence $d_{\mathsf{H}}(F_k^j, F_k) \to 0$, $j \to \infty$. Thus, modulo extracting a subsequence, we may suppose that $d_{\mathsf{H}}(F_k^j, F_k) < \frac{1}{j}$, $\forall j,k$. As a result, since $F_k \xrightarrow{V} F_0$, we know that $F_k^k \xrightarrow{V} F_0$ as well.

Now let π_C denote the nearest point projection from $\mathbb{R}^n$ to C. Then π_C is 1-Lipschitz (cf. [**4**, Proposition 5.3]). Set $G_0 = \pi_C(F_0)$. Then $G_0 \subset C$ and

$$\mathcal{H}^d(G_0) = \mathcal{H}^d(\pi_C(F_0)) \le \mathcal{H}^d(F_0). \tag{4.11}$$

We would like to construct the sequence G_k from F_k^k, so that $G_k \xrightarrow{U} G_0$.

Since F_k^k is a deformation of E in V, by definition of deformations in V, there exists a deformation φ^k in V such that $F_k^k = \varphi^k(E)$.

For each k, let $\delta_k \in \big(0, \frac{1}{k}\operatorname{diam} C\big)$ be such that $\varphi^k = \mathrm{id}$ on $B(\partial E, \delta_k)$. This is possible because $\partial E \subset \partial V$ and φ^k is a deformation in V.

Let D_k denote the convex hull of $C\backslash B(\partial E, \delta_k)$. Then we know that $C \subset \bar{B}(C\backslash B(\partial E, \delta_k), \delta_k)$, and hence $D_k \subset C \subset \bar{B}(D_k, \delta_k)$ for all $k \in \mathbb{N}$.

We also have that D_k is a compact subset of U. In fact, since $E \subset \bar{U}$ and $E \cap \partial U = \partial E$, we have $d(E\backslash B(\partial E, \delta_k), \partial U) > 0$. Since U is convex, the map $d(\cdot, \partial U)\colon \bar{U} \to \mathbb{R}$ is convex. Hence $d(E\backslash B(\partial E, \delta_k), \partial U) > 0$ implies that $d(D_k, \partial U) > 0$.

Let π_k be the nearest point projection to the convex set D_k. Then π_k is 1-Lipschitz (cf. [**4**, Proposition 5.3]).

Let us prove that

$$\sup_{x \in V} |\pi_k(x) - \pi_C(x)| \le 4R\delta_k. \tag{4.12}$$

Take any $x \in V$ and let $y = \pi_C(x)$. Then since C is convex, we know that

$$\langle z-y, x-y \rangle \le 0, \quad \forall z \in C. \tag{4.13}$$

Now let $z = \pi_k(x)$. Then $z \in D_k \subset C$. Since $D_k \subset C \subset \bar{B}(D_k, \delta_k)$, we have $d(x,C) \le d(x,D_k) \le d(x,C) + \delta_k$, that is,

$$|y-x| \le |z-x| \le |y-x| + \delta_k. \tag{4.14}$$

By the cosine formula and (4.13), we know that

$$|z-x|^2 = |x-y|^2 + |z-y|^2 - 2\langle z-y, x-y\rangle |x-y||z-y| \\ \geq |x-y|^2 + |z-y|^2, \tag{4.15}$$

hence

$$|z-y|^2 \leq |z-x|^2 - |x-y|^2 \\ = (|z-x| + |x-y|)(|z-x| - |x-y|) \leq 4R\delta_k, \tag{4.16}$$

because $x, y, z \in V \subset B(0, R)$.

Now we define $\psi_k \colon E \to (E \cap B(\partial E, \delta_k)) \cup D_k$:

$$\psi_k(x) = \begin{cases} x, & x \in E \cap \bar{B}(\partial E, \delta_k), \\ \pi_k \circ \varphi^k(x), & x \in E \backslash B(\partial E, \delta_k). \end{cases} \tag{4.17}$$

By definition, ψ_k is Lipschitz both on $E \cap \bar{B}(\partial E, \delta_k)$ and $E \backslash B(\partial E, \delta_k)$. On their intersection $E \cap \partial B(\partial E, \delta_k)$, by definition we know that $\varphi^k(x) = x$, and since $E \cap \partial B(\partial E, \delta_k) \subset C \cap \partial B(\partial E, \delta_k) \subset D_k$, we know that

$$\pi_k \circ \varphi^k(x) = \pi_k(x) = x, \tag{4.18}$$

hence ψ_k is well defined and Lipschitz.

Set $\epsilon_k = d(D_k, \partial U)$. Let $C_k = \bar{B}\big(D_k, \frac{1}{2}\epsilon_k\big)$. Then C_k is a compact convex subset of U, and we set $\psi_k(x) = x$ for $x \in U \backslash C_k$, and then extend ψ_k to a Lipschitz map $U \to U$. Then $W_k := \{x \in U : \psi_k(x) \neq x\}$ is compact in U, and hence $\psi_k(W_k) \cup W_k$ is compact. Therefore, ψ_k is a deformation in U.

We claim that

$$d_{\mathsf{H}}(\pi_k(F_k^k), \psi_k(E)) < 2\delta_k. \tag{4.19}$$

Let us first prove

$$\psi_k(E) \subset B(\pi_k(F_k^k), 2\delta_k). \tag{4.20}$$

Take any $y \in \psi_k(E)$. Then there exists $x \in E$ so that $y = \psi_k(x)$. By definition, if $x \in E \backslash B(\partial E, \delta_k)$, then $y = \pi_k \circ \varphi^k(x) \in \pi_k(F_k^k)$, because $\varphi^k(E) = F_k^k$; if $x \in E \cap B(\partial E, \delta_k)$, then $\psi(x) = x$, and hence $\psi(x) \subset B(\partial E, \delta_k)$. But note that $\partial E \subset F_k^k$ and $\partial E \subset \bar{B}(D_k, \delta_k)$, hence $\partial E \subset \bar{B}(\pi_k(\partial E), \delta_k) \subset \bar{B}(\pi_k(F_k^k), \delta_k)$. As a result, $\psi(x) \in B(\pi_k(F_k^k), 2\delta_k)$. Altogether we have (4.20).

Next, we prove that

$$\pi_k(F_k^k) \subset B(\psi_k(E), 2\delta_k). \tag{4.21}$$

Take any $y \in \pi_k(F_k^k)$. Then there exists $x \in E$ so that $y = \pi_k \circ \varphi^k(x)$. By definition, if $x \in E \backslash B(\partial E, \delta_k)$, then $y = \pi_k \circ \varphi^k(x) = \psi_k(x)$, and hence $y \in \psi_k(E)$; if $x \in E \cap B(\partial E, \delta_k)$, then $y = \pi_k \circ \varphi^k(x) = \pi_k(x)$.

Since $d(\pi_k(x), \partial E) \le d(\pi_k(x), x) + d(x, \partial E) \le \delta_k + \delta_k = 2\delta_k$, we know that $y \in B(\partial E, 2\delta_k) \subset B(\psi_k(E), 2\delta_k)$. Altogether we have (4.21). And (4.20) and (4.21) yield Claim (4.19).

By (4.12), we know that

$$d_{\mathsf{H}}(\pi_k(F_k^k), \pi_C(F_k^k)) < 4R\delta_k. \tag{4.22}$$

Combined with (4.19), we get

$$d_{\mathsf{H}}(\psi_k(E), \pi_C(F_k^k)) < (4R+2)\delta_k, \tag{4.23}$$

and hence

$$\lim_{k\to\infty} d_{\mathsf{H}}(\psi_k(E), \pi_C(F_k^k)) = 0. \tag{4.24}$$

Set $G_k = \psi_k(E) \in \mathcal{F}(E,U)$. Then, since $F_k^k \xrightarrow{V} F_0$, we know that $\pi_C(F_k^k) \xrightarrow{V} \pi_C(F_0) = G_0$. By (4.24) we have $\psi_k(E) \xrightarrow{V} G_0$, that is, $G_k \xrightarrow{V} G_0$.

Let us now prove (ii). Fix any $\epsilon > 0$. Let $K = \bar{U} \backslash B(\partial E, \epsilon)$. Then K is a compact subset of V. So there exists $k_0 > 0$ so that for each $k > k_0$, $d_K(G_k, G_0) < \epsilon$.

Now for any $x \in G_k \backslash B(\partial E, \epsilon)$, since $d_K(G_k, G_0) < \epsilon$ and $G_k \subset K$, we get $x \in B(G_0, \epsilon)$; for $x \in G_k \cap B(\partial E, \epsilon)$, since $\partial E \subset G_0$, we have again $x \in B(G_0, \epsilon)$. Hence $G_k \subset B(G_0, \epsilon)$. By symmetry we have also $G_0 \subset B(G_k, \epsilon)$. Hence $d_{\mathsf{H}}(G_k, G_0) < \epsilon$ for any $k > k_0$.

The above holds for any $\epsilon > 0$, hence we have (ii).

Note that $G_0 \subset C$, and hence $G_0 \cap \partial U \backslash \partial E = \emptyset$. By (4.11), we know that (2.1) holds for G_0. Since $G_k \in \mathcal{F}(E,U)$, by (ii) and the definition of $\overline{\mathcal{F}}(E,U)$, we have $G_0 \in \overline{\mathcal{F}}(E,U)$.

Let us now prove (i) and (iii). We already know that $G_0 \subset C$, so let us prove that G_0 is minimal in U. In fact, (4.9) yields

$$\mathcal{H}^d(F_0) \le \inf_{F \in \overline{\mathcal{F}}(E,V)} \mathcal{H}^d(F), \tag{4.25}$$

because for any set $E \in \overline{\mathcal{F}}(E,V) \backslash \mathfrak{F}$, E must have infinite $\mathcal{H}^d$ measure, and hence $\mathcal{H}^d(E) > \mathcal{H}^d(F_0)$.

As a result, by (4.11), we know that $\mathcal{H}^d(G_0) \le \inf_{F \in \overline{\mathcal{F}}(E,V)} \mathcal{H}^d(F) \le \inf_{F \in \overline{\mathcal{F}}(E,U)} \mathcal{H}^d(F)$, which yields (iii). But we know that $\overline{\mathcal{F}}(E,V)$ is stable under deformations in V and $G_0 \in \overline{\mathcal{F}}(E,U) \subset \overline{\mathcal{F}}(E,V)$, hence G_0 is minimal in V. Finally, we know that $\partial E \subset F_0$ and $\partial E \subset C$, hence $\partial E = \pi_C(\partial E) \subset \pi_C(F_0) = G_0$. On the other hand, since $G_0 \subset C$, we have $G_0 \cap \partial U \subset C \cap \partial U = \partial E$. Hence $G_0 \cap \partial U = \partial E = C \cap \partial U$. Thus we get all claims in (i). □

Our idea of proving Theorem 4.1 is that, when G_k is sufficiently close to G_0, by the local regularity of the minimal set G_0, we can deform a big part of G_k to G_0. Note that the local regularity is for a reduced minimal set, so we will need that the closed support G_k^* is sufficiently close to G_0^* as well. Hence we need the following.

Definition 4.5. Let $0 < d < n$. Let $F \subset \mathbb{R}^n$ be closed, let $W \subset \mathbb{R}^n$ be an open set. Set

$$\mathcal{F}_d^*(F,W) = \{H \text{ closed: } \exists M \in \mathcal{F}(F,W) \text{ and } N \subset W \text{ with } \mathcal{H}^d(N) = 0 \text{ such that } M = H \cup N\}. \tag{4.26}$$

It is easy to see from the definition that $\mathcal{F}_d^*(F,W)$ is stable under Lipschitz deformations in W and

$$\inf_{K\in\mathcal{F}_d^*(F,W)} \mathcal{H}^d(K) = \inf_{K\in\mathcal{F}_d(F,W)} \mathcal{H}^d(K). \tag{4.27}$$

Proposition 4.6. *Let n, d, E, U, and C be as in the statement of Theorem 4.1, so that* (4.1) *holds. Let V be as above. Then there exists $\{E_k\}_{k\in\mathbb{N}} \in \mathcal{F}_d^*(E,U)$ and a closed set $E_0 \subset \bar{U}$ such that the following holds:*

(i) *$E_0 \cap \partial U = \partial E$ and $E_0 \subset C$;*

(ii) *$E_0 \cap U = E_0 \cap V$ is a reduced minimal set in V and $\mathcal{H}^d(E_0) \le \inf_{F\in\overline{\mathcal{F}}(E,U)} \mathcal{H}^d(F)$;*

(iii) *$E_k \subset B(E_0, 2^{-k})$.*

Proof: Let G_0 and G_k be as obtained in Proposition 4.4. We set $E_0 = G_0^* \cup \partial G_0$. Then $G_0^* \subset E_0 \subset G_0 \subset C$, and (i) and (ii) hold directly for E_0.

The idea of constructing E_k is roughly the following: we fix a small neighborhood of E_0, say W, and look at the part $G_k \backslash W$. Note that $G_0 \backslash W$ is compact and the measure of $G_0 \backslash W$ is zero, hence we can use a Federer–Fleming projection ψ to project $G_0 \backslash W$ to a union of $(d-1)$-faces of dyadic cubes and a part in W. Then $\psi(G_k) \backslash W$ will be very close to this union of $(d-1)$-faces, and hence we can use a deformation retract to map this part to this union of $(d-1)$-faces. And after this deformation the closed support of the image of G_k will be contained in $\psi(G_k) \backslash W$, which is contained in $B(E_0, 2^{-k})$. Then we set E_k to be the closed support of $\psi(G_k)$.

So let us do it more precisely. Fix any k.

Let $A_1 = G_0 \cap B(E_0, 2^{-k-5})$ and $A_2 = G_0 \backslash B(E_0, 2^{-k-5})$. Then G_0 is the disjoint union of A_1 and A_2, and $\mathcal{H}^d(A_2) = 0$.

Let $l \in \mathbb{N}$ be such that $2^l > \max\{10, \sqrt{n}\}$, and $d(A_2, \partial U) > 2^{-l}$.

Set

$$\mathcal{S} = \{Q \in \Delta_{k+4l} : Q \subset U,\ Q \cap \bar{B}(E_0, 2^{-k-2l}) = \emptyset\}. \tag{4.28}$$

Then the support $|\mathcal{S}|$ of $\mathcal{S}$ satisfies that

$$|\mathcal{S}| \cap B(E_0, 2^{-k-2l}) = \emptyset \text{ and } d(|\mathcal{S}|, \partial U) > 0. \tag{4.29}$$

Also by definition, since $d(A_2, \partial U) > 2^{-l}$ and $d(A_2, E_0) \geq 2^{-k-5} > 2^{-k-l}$, we have

$$A_2 \subset |\mathcal{S}|^\circ \text{ and } d(A_2, \partial|\mathcal{S}|) > 2^{-k-2l}, \tag{4.30}$$

and hence

$$B(A_2, 2^{-k-2l}) \subset |\mathcal{S}|^\circ. \tag{4.31}$$

By definition of A_2, we know that

$$\mathcal{H}^d(A_2) = 0. \tag{4.32}$$

Since $|\mathcal{S}| \subset U$, hence we can find a Federer–Fleming projection $\psi \colon U \to U$ so that the following holds:

$$\psi(x) = x \text{ for } x \in U \backslash |\mathcal{S}|^\circ; \tag{4.33}$$

$$\psi(x) = x \text{ for } x \in \mathcal{S}_{d-1}, \tag{4.34}$$

where $\mathcal{S}_{d-1}$ denotes the union of $(d-1)$-faces of $\mathcal{S}$;

$$\psi(A_2) \subset \mathcal{S}_{d-1} \cup \partial|\mathcal{S}|; \tag{4.35}$$

$$\psi(Q) \subset Q \text{ for every } Q \in \mathcal{S}. \tag{4.36}$$

Note that the set $\{x \in U : \psi(x) \neq x\} \subset |\mathcal{S}|$ and $|\mathcal{S}|$ is compact, hence ψ is a deformation in U. Moreover, (4.36) implies that

$$|\psi(x) - x| \leq \sqrt{n} 2^{-k-4l} < 2^{-k-3l}. \tag{4.37}$$

By (4.30) and (4.37), we get

$$d(\psi(A_2), |\mathcal{S}|^C) > 2^{-k-3l}, \tag{4.38}$$

and hence by (4.35),

$$\psi(A_2) \subset \mathcal{S}_{d-1}. \tag{4.39}$$

Let $L \geq 1$ denote the Lipschitz constant of ψ.

Now modulo taking a subsequence, we suppose that

$$G_k \subset B(G_0, L^{-1} 2^{-k-5l}).$$

Then

$$G_k \subset B(A_1, L^{-1} 2^{-k-5l}) \cup B(A_2, L^{-1} 2^{-k-5l}). \tag{4.40}$$

Since ψ is L-Lipschitz, we have

$$\psi(G_k)\subset B(\psi(G_0),2^{-k-5l})\subset B(\psi(A_1),2^{-k-5l})\cup B(\psi(A_2),2^{-k-5l}). \tag{4.41}$$

By (4.39), we have

$$\psi(G_k)\cap B(\psi(A_2),2^{-k-5l})\subset B(\mathcal{S}_{d-1},2^{-k-5l}) \tag{4.42}$$

and

$$d(\psi(G_k)\cap B(\psi(A_2),2^{-k-5l}),|\mathcal{S}|^C)>2^{-k-4l}. \tag{4.43}$$

Let $\mathcal{T}=\{Q\in\mathcal{S}:Q\cap B(\psi(A_1),2^{-k-5l})=\emptyset\}$. Then $|\mathcal{T}|\cap B(\psi(A_1), 2^{-k-5l})=\emptyset$, and by (4.41) and (4.42) we have

$$\psi(G_k)\cap|\mathcal{T}|\subset\psi(G_k)\cap B(\psi(A_2),2^{-k-5l})\subset B(\mathcal{S}_{d-1},2^{-k-5l}). \tag{4.44}$$

Now we will define a map from $|\mathcal{T}|$ to $|\mathcal{T}|$ that deforms $\psi(G_k)\cap|\mathcal{T}|$ to $\mathcal{S}_{d-1}\cup\partial|\mathcal{T}|$. The idea will be the same as the Federer–Fleming projection. So we need the following lemma:

Lemma 4.7. *Let $d-1<m$. Let Q be an m-dimensional cube of side-length $l(Q)$. For $1\le k\le m$, let Q_k denote the union of its k-faces. Then there exists a Lipschitz map $\varphi_Q\colon Q\to Q$ such that*

$$\varphi_Q\left(B\left(Q_{d-1},\frac{1}{10}l(Q)\right)\cap Q\right)\subset\left(B\left(Q_{d-1},\frac{1}{10}l(Q)\right)\cap Q_{m-1}\right), \tag{4.45}$$

$$\varphi_Q\left(B\left(\partial Q,\frac{1}{10}l(Q)\right)\cap Q\right)\subset\partial Q, \tag{4.46}$$

$$\varphi_Q|_{\partial Q}=\mathrm{id}\,. \tag{4.47}$$

Proof: For $x\in\mathbb{R}^m$, we write its coordinates as $x=(x_1,\dots,x_m)$.

Fix any $d-1<m$. Let us first look at the cube $Q=[-1,1]^m\subset\mathbb{R}^m$. It is a cube of dimension m, with side-length 2.

Let o be the origin and let $f\colon Q\backslash\{o\}\to\partial Q$ be the radial projection, that is, for any $x\in Q\backslash\{o\}$, let $\delta_x=\min_{1\le i\le m}d(x_i,\{-1,1\})$, and then

$$f(x)=\frac{x}{1-\delta_x}. \tag{4.48}$$

Now suppose that $x\in B\big(Q_{d-1},\frac15\big)\cap Q$. Then there exists a $(d-1)$-face σ of Q so that $x\in B\big(\sigma,\frac15\big)$. Without loss of generality, suppose $\sigma=[-1,1]^{d-1}\times\{1,\dots,1\}=\{y\in Q:y_d=y_{d+1}=\cdots=y_m=1\}$. Then for any point $y\in Q$,

$$d(y,\sigma)=\sqrt{\sum_{i=d}^m(1-y_i)^2}. \tag{4.49}$$

Thus, for the point x, we have

$$d(f(x),\sigma) = d\left(\frac{x}{1-\delta_x},\sigma\right) = \sqrt{\sum_{i=d}^{m}\left(1-\frac{x_i}{1-\delta_x}\right)^2}. \tag{4.50}$$

Since $1-\delta_x \le 1$, we have

$$d(f(x),\sigma) \le \sqrt{\sum_{i=d}^{m}(1-x_i)^2} \le d(x,\sigma) < \frac{1}{5}. \tag{4.51}$$

As a result, we know that

$$f\left(B\left(Q_{d-1},\frac{1}{5}\right)\cap Q\right) \subset B\left(Q_{d-1},\frac{1}{5}\right)\cap Q_{m-1}. \tag{4.52}$$

Now we set $\varphi_Q\colon Q\to Q$:

$$\varphi_Q(x) = \begin{cases} f(x), & d(x,\partial Q)\le \frac{1}{5},\\ (1-t)f(x)+tx, & d(x,\partial Q) = \frac{1}{5}+\frac{t}{5},\ t\in[0,1],\\ x, & d(x,\partial Q)\ge \frac{2}{5}. \end{cases} \tag{4.53}$$

Then φ_Q satisfy (4.45), (4.46), and (4.47).

Now for a general cube Q', let $\psi\colon Q\to Q'$ be an isometry, then it is enough to set $\varphi_{Q'} = \psi\circ\varphi_Q\circ\psi^{-1}$. □

Let us now construct the aforementioned Lipschitz map from $|\mathcal{T}|$ to $|\mathcal{T}|$ that deforms $\psi(G_k)\cap|\mathcal{T}|$ to $\mathcal{S}_{d-1}\cup\partial|\mathcal{T}|$. We will recurrently define a sequence of maps $\varphi_m\colon |\mathcal{T}_m|\cup\partial|\mathcal{T}|\to|\mathcal{T}_{m-1}|\cup\partial|\mathcal{T}|$ for $d\le m\le n$, so that

$$\varphi_m(Q)\subset Q,\quad \forall Q\in\mathcal{T}_m, \tag{4.54}$$

$$\varphi_m(x) = x,\quad \forall x\in\partial|\mathcal{T}|\cup|\mathcal{T}_{d-1}|, \tag{4.55}$$

and

$$\varphi_m(x)\in Q_{d-1}\cup\partial|\mathcal{T}|,\quad \forall x\in Q\cap B\left(\mathcal{T}_{d-1},\frac{1}{10}2^{-k-4l}\right),\quad \forall Q\in\mathcal{T}_m. \tag{4.56}$$

Let us first define φ_d. Take any $x\in|\mathcal{T}_d|\cup\partial|\mathcal{T}|$. Set $\varphi_d(x)=x$ if $x\in\partial|\mathcal{T}|$. Otherwise, there exists $Q\in\mathcal{T}_d$ so that $Q^\circ\cap|\mathcal{T}_d| = \emptyset$, and set $x=\varphi_Q(x)$, where φ_Q is the one obtained in Lemma 4.7. Then by (4.47), φ_d is well defined.

Now suppose that φ_{m-1} is already defined and satisfies (4.54)–(4.56) replacing m by $m-1$. Let us define φ_m.

Take any $x\in|\mathcal{T}_m|\cup\partial|\mathcal{T}|$. Set $\varphi_m(x)=x$ if $x\in\partial|\mathcal{T}|$. For any $Q\in\mathcal{T}_m$ so that $Q^\circ\cap\partial|\mathcal{T}| = \emptyset$, we first define $f_Q(x) = \varphi_{m-1}\circ\varphi_Q(x)$ $\forall x\in\left[B\left(\partial Q,\frac{1}{10}l(Q)\right)\cap Q\right]\cup\partial|\mathcal{T}|$, where φ_Q is the one obtained in Lemma 4.7.

Note that φ_{m-1} and φ_Q are both the identity on $\partial|\mathcal{T}|\cap Q$, hence f_Q is well defined. Also by definition of φ_Q, for each Q it is easy to see that $\varphi_Q=\varphi_{Q'}$ for any $Q,Q'\in\mathcal{T}_m$ so that $Q\cap Q'\neq\emptyset$. Hence $f_Q=f_{Q'}$.

Then we extend f_Q to a map from $Q\to Q$. And set $\varphi_m(x)=f_Q(x)$ for $x\in Q$ and for all $Q\in\mathcal{T}_m$.

Let us verify that φ_m is well defined and satisfies (4.54)–(4.56).

Take any $Q_1,Q_2\in\mathcal{T}_m$ and let $x\in Q_1\cap Q_2$. Then $x\in\partial Q_1\cap\partial Q_2$. By definition of φ_Q, we know that $\varphi_{Q_1}(x)=\varphi_{Q_2}(x)=x$, and hence $f_{Q_1}(x)=\varphi_{m-1}\circ\varphi_{Q_1}(x)=\varphi_{m-1}\circ\varphi_{Q_1}(x)=f_{Q_2}(x)$. Hence φ_m is well defined.

Since $f_Q(Q)\subset Q$, we have (4.54).

To check (4.55), we know that $f_Q|_{\partial|\mathcal{T}|}=\mathrm{id}$ by definition. For $x\in|\mathcal{T}_{d-1}|$, let $Q\in\mathcal{T}$ be such that $x\in Q_{d-1}$. Then by definition, $\varphi_m(x)=\varphi_{m-1}\circ\varphi_Q(x)$. But $\varphi_Q(x)=x$ for $x\in\partial Q$, and $\varphi_{m-1}(x)=x$ for $x\in|\mathcal{T}_{d-1}|$ by hypothesis of induction, hence $\varphi_m(x)=x$. Thus we get (4.55).

Finally, to verify (4.56), take any $Q\in\mathcal{T}_m$ and any $x\in Q\cap B\big(\mathcal{T}_{d-1},\frac{1}{10}2^{-k-4l}\big)$. By definition, we know that $\varphi_m(x)=f_Q(x)=\varphi_{m-1}\circ\varphi_Q(x)$. By Lemma 4.7, $x\in Q\cap B\big(\mathcal{T}_{d-1},\frac{1}{10}2^{-k-4l}\big)$ implies that $\varphi_Q(x)\in\partial Q\cap B\big(\mathcal{T}_{d-1},\frac{1}{10}2^{-k-4l}\big)$. Hence by hypothesis of induction for $m-1$ in (4.56), we have that $\varphi_{m-1}(\varphi_Q(x))\in Q_{d-1}\cup\partial|\mathcal{T}|$.

By induction, we get that φ_m satisfies (4.54)–(4.56), $d\leq m\leq n$.

Now we set $\varphi\colon U\to U$:

$$\varphi(x)=\begin{cases}\varphi_n(x), & x\in|\mathcal{T}|,\\ x, & x\in U\backslash|\mathcal{T}|.\end{cases}\tag{4.57}$$

Note that $\varphi_n(x)=x$ for $x\in\partial|\mathcal{T}|$. Hence φ is well defined, and is a Lipschitz deformation in U.

Now set $H_k=\varphi\circ\psi(G_k)$. Then $H_k\in\mathcal{F}(E,U)$. Set $E_k=H_k^*$. Then $E_k\in\mathcal{F}_d^*(E,U)$. Let us now verify (iii) of Proposition 4.6.

By (4.44), $\psi(G_k)\cap|\mathcal{T}|\subset B(\mathcal{S}_{d-1},2^{-k-5l})\cap|\mathcal{T}|\subset B(\mathcal{T}_{d-1},2^{-k-5l})$ and $\varphi|_{|\mathcal{T}|}=\varphi_n|_{|\mathcal{T}|}$, hence by (4.56), we obtain

$$\varphi\circ\psi(G_k\cap|\mathcal{T}|)\subset|\mathcal{T}_{d-1}|\cup\partial|\mathcal{T}|.\tag{4.58}$$

As a result, we have

$$\mathcal{H}^d(\varphi\circ\psi(G_k\cap|\mathcal{T}|)\cap|\mathcal{T}|^\circ)=\mathcal{H}^d(\varphi\circ\psi(G_k\cap|\mathcal{T}|)\backslash\partial|\mathcal{T}|)=0.\tag{4.59}$$

Since

$$H_k=\varphi\circ\psi(G_k)=[\varphi\circ\psi(G_k\cap|\mathcal{T}|)]\cup[\varphi\circ(\psi(G_k)\backslash|\mathcal{T}|^\circ)],\tag{4.60}$$

by (4.58) and (4.59), we get

$$E_k=H_k^*\subset[\varphi\circ\psi(G_k\cap|\mathcal{T}|)\cap\partial|\mathcal{T}|]\cup[\varphi\circ(\psi(G_k)\backslash|\mathcal{T}|^\circ)].\tag{4.61}$$

Let us look at $\varphi\circ\psi(G_k\cap|\mathcal{T}|)\cap\partial|\mathcal{T}|$. Take any $y\in\varphi\circ\psi(G_k\cap|\mathcal{T}|)\cap\partial|\mathcal{T}|$, then there exists $x\in G_k$ so that $y=\varphi\circ\psi(x)$. But by definition of φ and ψ, we have

$$\begin{aligned}||y-x|| &\le ||\varphi\circ\psi(x)-\psi(x)||+||\psi(x)-x|| \\ &\le \sqrt{n}2^{-k-4l}+\sqrt{n}2^{-k-4l}<2^{-k-2l}.\end{aligned} \tag{4.62}$$

Since $y\in\partial|\mathcal{T}|$, we get $x\in B(\partial|\mathcal{T}|,2^{-k-2l})$. Now by definition of $\mathcal{T}$ and $\mathcal{S}$, we know that for any $z\in\partial|\mathcal{T}|$, $d(z,\partial U\cup\bar{B}(E_0,2^{-k-2l})\cup B(\psi(A_1),2^{-k-5l}))<2^{-k-2l}$. Hence

$$\begin{aligned}&d(x,\partial U\cup\bar{B}(E_0,2^{-k-2l})\cup B(\psi(A_1),2^{-k-5l})) \\ &\qquad\qquad<2^{-k-2l}+2^{-k-2l}<2^{-k-l}.\end{aligned} \tag{4.63}$$

So there are three cases:

If $d(x,\partial U)<2^{-k-l}$, by definition of l, we have $x\notin B(A_2,2^{-k-5})$, and by (4.40) we see that $x\in B(A_1,2^{-k-5l})$, and thus by (4.62) and the definition of A_1,

$$\begin{aligned}y\in B(A_1,2^{-k-5l}+2^{-k-2l})&\subset B(E_0,2^{-k-5l}+2^{-k-l}+2^{-k-2l}) \\ &\subset B(E_0,2^{-k}).\end{aligned} \tag{4.64}$$

If $d(x,\bar{B}(E_0,2^{-k-2l}))<2^{-k-l}$, then by (4.62)

$$\begin{aligned}y\in B(\bar{B}(E_0,&2^{-k-2l}),2^{-k-l}+2^{-k-2l}) \\ &\subset B(E_0,2^{-k-2l}+2^{-k-l}+2^{-k-2l}) \\ &\subset B(E_0,2^{-k}).\end{aligned} \tag{4.65}$$

If $d(x,B(\psi(A_1),2^{-k-5l}))<2^{-k-l}$, by definition of ψ and A_1, we know that $\psi(A_1)\subset B(A_1,2^{-k-4l})\subset B(E_0,2^{-k-l}+2^{-k-4l})$, and hence

$$\begin{aligned}x\in B(B(\psi(A_1),&2^{-k-5l}),2^{-k-l}) \\ &\subset B(\psi(A_1),2^{-k-5l}+2^{-k-l}) \\ &\subset B(E_0,2^{-k-l}+2^{-k-4l}+2^{-k-5l}+2^{-k-l}).\end{aligned} \tag{4.66}$$

Thus by (4.62), we have again

$$\begin{aligned}y\in B(x,2^{-k-2l})&\subset B(E_0,2^{-k-l}+2^{-k-4l}+2^{-k-5l}+2^{-k-l}+2^{-k-2l}) \\ &\subset B(E_0,2^{-k}).\end{aligned} \tag{4.67}$$

Altogether, we have

$$\varphi\circ\psi(G_k\cap|\mathcal{T}|)\cap\partial|\mathcal{T}|\subset B(E_0,2^{-k}). \tag{4.68}$$

Now for the set $\varphi(\psi(G_k)\backslash|\mathcal{T}|^\circ)$, take any $y\in\varphi(\psi(G_k)\backslash|\mathcal{T}|^\circ)$. Again there exists $x\in G_k$ so that $y=\varphi\circ\psi(x)$, and (4.62) holds. Note that

$\psi(x) \notin |\mathcal{T}|^\circ$, by definition of φ, $\psi(x) = \varphi(\psi(x)) = y$. Thus $y \notin |\mathcal{T}|^\circ$. By definition of $\mathcal{T}$, we know that

$$d(y, \partial U \cup \bar{B}(E_0, 2^{-k-2l}) \cup B(\psi(A_1), 2^{-k-5l})) < 2^{-k-2l}, \tag{4.69}$$

and hence x satisfies again (4.63), and the exact argument as above gives that

$$y \in B(E_0, 2^{-k}). \tag{4.70}$$

Hence

$$\varphi(\psi(G_k) \backslash |\mathcal{T}|^\circ) \subset B(E_0, 2^{-k}). \tag{4.71}$$

Combined with (4.61) and (4.68), we have

$$E_k \subset B(E_0, 2^{-k}). \tag{4.72}$$

This completes the proof of Proposition 4.6. □

Now let us fix the set E_0 and the sequence $\{E_k\}_k$ as obtained in Proposition 4.6. We want to prove that, when E_k is sufficiently close to E_0, we can deform E_k into the union of E_0 and a set of very small measure, so that the measure after the deformation can be arbitrarily close to $\mathcal{H}^d(E_0) = \inf_{F \in \mathcal{F}(E,U)} \mathcal{H}^d(F)$, which yields (i) of Theorem 4.1.

The construction of such a deformation is similar to the construction in [**5**]. By minimality of E_0, around each regular point x of E_0 there is a neighborhood retract to E_0 in some ball centered at x with a uniform Lipschitz constant. We use a finite number of such balls to cover a big part of E_0 and the measure of E_0 which is not covered is very small. When E_k is close enough to E_0, a big part of E_k is contained in the union of these balls, so we can deform E_k onto E_0 in each of these balls, and then extend this deformation to the whole space with the same Lipschitz constant. Outside these balls, since each E_k is very close to E_0, we expect that measures of E_k are comparable to the measure of E_0, and so the measures of the image of E_k outside the above balls are still small.

But in our case there is no reason why the measures of E_k should be uniformly comparable to that of E_0 at small scales. This issue results in more works. In other words, we have to first deform $\{E_k\}$ into a new sequence $\{E_k'\}$ whose local measures can be controlled by that of E_0 and they are still very close to E_0 for k large.

Now let us give more details:

Set

$$Q'_k := \{Q \in \Delta_k : Q \cap E_0 \neq \emptyset\} \tag{4.73}$$

and

$$Q_k = \{Q \in \Delta_k : \exists Q' \in Q'_k \text{ such that } Q \cap Q' \neq \emptyset\}, \tag{4.74}$$

that is, Q'_k is the family of elements in Δ_k that are neighbors of E_0, and we get Q_k by adding another layer of cubes in Δ_k to Q_k. Let $|Q_k| = \bigcup_{Q\in Q_k} Q$ be the union of elements in Q_k, and for each $j \le n$, let $Q_{k,j}$ be the set of all j-faces of elements in Q_k, and let $\mathcal{S}_{k,j} = \bigcup_{\sigma\in Q_{k,j}} \sigma$ denote the j-skeleton of Q_k.

Set $\partial E_0 = E_0 \cap \partial U$, which is equal to ∂E by Proposition 4.6, and set

$$R_k := \{Q \in \Delta_k : \exists Q' \in \Delta_k \text{ such that } Q' \cap \partial E_0 \neq \emptyset \text{ and } Q \cap Q' \neq \emptyset\}. \tag{4.75}$$

Let $|R_k| = \bigcup_{Q\in R_k} Q$, and for each $j \le n$, let $R_{k,j}$ be the set of all j-faces of elements in R_k, and let $\mathcal{T}_{k,j} = \bigcup_{\sigma\in R_{k,j}} \sigma$ denote the j-skeleton of R_k.

It is easy to see that

$$Q'_k \subset Q_k \text{ and } R_k \subset Q_k, \tag{4.76}$$

and hence

$$|R_k| \subset |Q_k|,\ R_{k,j} \subset Q_{k,j},\ \text{and } \mathcal{T}_{k,j} \subset \mathcal{S}_{k,j} \text{ for all } j \le n. \tag{4.77}$$

Let us first give some properties for the sets $\mathcal{S}_{k,d}$ and $\mathcal{T}_{k,d}$, where d is the dimension of E_0.

Proposition 4.8. *Let E, U, and C be as in the statement of Theorem 4.1, so that* (4.1) *holds, and so that $E \cap \partial U$ is of finite $(d-1)$-Hausdorff measure. Let E_0 and $\{E_k\}_k$ be as obtained in Proposition 4.6. Let Q'_k, Q_k, $Q_{k,j}$, $\mathcal{S}_{k,j}$, $R_k, \mathcal{T}_{k,j}$ be as defined above. Then*

(i) $\lim_{k\to\infty} \mathcal{H}^d(\mathcal{T}_{k,d}) \to 0$.

(ii) *There exists $M > 0$ which depends only on n and d such that for each $k > k_0$, and each $Q \in Q_k$ and $Q^\circ \cap |R_{k-2}| = \emptyset$, we have*

$$\mathcal{H}^d(\mathcal{S}_{k,d} \cap Q) < M\mathcal{H}^d(E_0 \cap V(Q)), \tag{4.78}$$

where $V(Q)$ denotes the union of cubes that touch some cube that touches Q, that is:

$$V(Q) := \cup\{Q' \in \Delta_k : \textit{there exists } Q'' \in \Delta_k \textit{ such that } Q'' \cap Q \neq \emptyset \textit{ and } Q'' \cap Q' \neq \emptyset\}. \tag{4.79}$$

Proof: (i) Since $\partial E_0 = \partial E$ is of finite $(d-1)$-Hausdorff measure, we apply [**9**, Theorem 3.2.39] and get

$$\mathcal{M}^{d-1}(\partial E_0) = \mathcal{H}^{d-1}(\partial E_0) < \infty, \tag{4.80}$$

where $\mathcal{M}^{d-1}$ stands for the $(d-1)$-dimensional Minkowski content.

By definition of Minkowski content, we know that

$$\lim_{r\to 0+}\frac{\mathcal{H}^n(B(\partial E_0,r))}{r^{n-d+1}}<\infty, \tag{4.81}$$

and hence, when k is large, we have

$$\mathcal{H}^n(B(\partial E_0,2^{-k}))<C_0 2^{-k(n-d+1)}. \tag{4.82}$$

We know that $|R_k|\subset B(\partial E_0,2^{-k+3})$. Hence for k large,

$$\begin{aligned}\mathcal{H}^n(|R_k|)&\le \mathcal{H}^n(B(\partial E_0,2^{-k+3}))\\ &<C_0 2^{(-k+3)(n-d+1)}=C_1 2^{-k(n-d+1)}.\end{aligned} \tag{4.83}$$

On the other hand,

$$\mathcal{H}^d(\mathcal{T}_{k,d})=\sum_{\sigma\in R_{k,d}}\mathcal{H}^d(\sigma)\le \sum_{Q\in R_k}\sum_{\sigma\in\Delta_d(Q)}\mathcal{H}^d(\sigma). \tag{4.84}$$

Now for each $Q\in R_k$, we know that $\sum_{\sigma\in\Delta_d(Q)}\mathcal{H}^d(\sigma)=\alpha_{n,d}2^{-kd}$, where $\alpha_{n,d}$ is the d-Hausdorff measure of the d-skeleton of a unit cube, which is a constant that depends only on n and d. As a result, by (4.84),

$$\mathcal{H}^d(\mathcal{T}_{k,d})\le\sum_{Q\in R_k}\alpha_{n,d}2^{-kd}=\alpha_{n,d}2^{-kd}\sharp R_k, \tag{4.85}$$

where $\sharp R_k$ is the number of cubes in R_k.

Meanwhile, since the $\mathcal{H}^n$ measure of each cube in R_k is 2^{-kn} we have, for k large,

$$\sharp R_k=\frac{\mathcal{H}^n(|R_k|)}{2^{-kn}}\le\frac{C_1 2^{-k(n-d+1)}}{2^{-kn}}=C_1 2^{kd-k}, \tag{4.86}$$

where the second inequality is by (4.83). Combined with (4.85), we get

$$\mathcal{H}^d(\mathcal{T}_{k,d})\le\alpha_{n,d}2^{-kd}\times C_1 2^{kd-k}=C_1\alpha_{n,d}2^{-k}\to 0,\ \text{ as } k\to\infty, \tag{4.87}$$

which yields (i).

(ii) Fix any $Q\in Q_k$. By definition, there exists $Q'\in\Delta_k$ such that $Q'\cap E_0\neq\emptyset$ and $Q\cap Q'\neq\emptyset$. Take $y\in Q'\cap E_0$. Then by definition of $V(Q)$, $B(y,2^{-k})\subset V(Q)$. On the other hand, since $Q^\circ\cap|R_{k-2}|=\emptyset$, we know that $d(Q',\partial E_0)>2^{-k+2}$. In particular, $d(y,\partial E_0)>2\times 2^{-k}$, which means $B(y,2\times 2^{-k})\subset V$ (V is as defined before Proposition 4.4). Since E_0 is a reduced minimal set in V, by Ahlfors regularity for reduced minimal sets (cf. [**8**, Proposition 4.1]),

$$C_2^{-1}2^{-kd}\le\mathcal{H}^d(E_0\cap B(y,2^{-k}))\le C_2 2^{-kd}, \tag{4.88}$$

where C_2 is a constant that depends only on n and d. As a result, we have

$$\begin{aligned}\mathcal{H}^d(\mathcal{S}_{k,d}\cap Q) = \alpha(n,d)2^{-kd} &\leq C_2\alpha(n,d)\mathcal{H}^d(E_0\cap B(y,2^{-k})) \\ &\leq C_2\alpha(n,d)\mathcal{H}^d(E_0\cap V(Q)).\end{aligned} \tag{4.89}$$ □

Next, let us construct the new sequence.

Proposition 4.9. *Let E, U, and C be as in the statement of Theorem 4.1, so that* (4.1) *and* (4.2) *hold. Let E_0 and $\{E_k\}_k$ be obtained as in Proposition 4.6. Let $\mathcal{S}_{k,j}$ be as defined above. Then for each $\epsilon > 0$, there exist a sequence of deformations f_k in $\mathbb{R}^n$ and $u_k > 0$ such that*

$$\mathcal{H}^d(E_k\cap B(\partial E_0, u_k)) < \epsilon, \tag{4.90}$$

$$f_k = \mathrm{id} \ \ \text{in } B(\partial E_0, u_k), \tag{4.91}$$

$$f_k(E_k) \subset B(E_0, \sqrt{n}2^{-k+1}), \tag{4.92}$$

and for k large,

$$\mathcal{H}^d(f_k(E_k)\backslash \mathcal{S}_{k,d}) < \epsilon. \tag{4.93}$$

Proof: Fix any $k > k_0$. Let Q_k', Q_k, $Q_{k,j}$, R_k, $\mathcal{T}_{k,j}$ be as defined above.

Since $E_k \subset B(E_0, 2^{-k})$, $E_k \subset |Q_k|$. And we know that E_k is contained in a deformation of E, hence E_k has finite d-Hausdorff measure. As a result, by a standard Federer–Fleming argument (cf. Section 4.2 of [**9**], or Section 3 of [**8**]), there exists a Lipschitz map $\varphi_k\colon |Q_k| \to |Q_k|$ (the Lipschitz constant L_k depends on k and $L_k \geq 1$) such that

$$\varphi_k(Q) \subset Q, \quad \forall Q \in Q_k, \tag{4.94}$$

$$\varphi_k(E_k) \subset \mathcal{S}_{k,d}, \tag{4.95}$$

$$||\varphi_k(x) - x|| \leq \sqrt{n}2^{-k}, \tag{4.96}$$

and

$$\varphi_k(x) = x, \quad \forall x \in \mathcal{S}_{k,d}. \tag{4.97}$$

In particular, we have $\varphi_k(E_k) \subset \mathcal{S}_{k,d}$.

We will modify φ_k to f_k so that f_k satisfies (4.91).

Fix $\epsilon > 0$. Let $\mu = \mathcal{H}^d\lfloor_{E_k}$, then μ is a finite measure. In particular, we have

$$\begin{aligned}\lim_{r\to 0}\mu(B(\partial E_0, r)) = \mu(\partial E_0) &= \mathcal{H}^d(E_k\cap \partial E_0) \\ &\leq \mathcal{H}^d(\partial E_0) = \mathcal{H}^d(\partial E) = 0,\end{aligned} \tag{4.98}$$

because $\partial E_0 = \partial E$.

Take $u_k > 0$ such that $\mu(B(\partial E_0, 2u_k)) < (3L_k+2)^{-d}\epsilon$, that is, $\mathcal{H}^d(E_k\cap B(\partial E_0, 2u_k)) < (3L_k+2)^{-d}\epsilon$. Also, take $R > 1$ so that $E_0 \subset B(0, R)$.

For any $x \in B(0,R)$, set

$$t_x = \begin{cases} 0, & x \in B(\partial E_0, u_k), \\ \frac{d(x,\partial E_0)}{u_k} - 1, & x \in B(\partial E_0, 2u_k)\backslash B(\partial E_0, u_k), \\ 1, & x \in B(0,R)\backslash B(\partial E_0, 2u_k), \end{cases} \tag{4.99}$$

and set $f_k(x) = (1-t_x)x + t_x\varphi_k(x)$.

Then $f_k\colon B(0,R) \to \mathbb{R}^n$ is $2L_k+1$-Lipschitz. In fact, for any x, y, suppose that $d(x,\partial E_0) \geq d(y,\partial E_0)$. Then we get

(4.100)

$$\begin{aligned}
&||f_k(x) - f_k(y)|| \\
&\quad = ||[(1-t_x)x + t_x\varphi_k(x)] - [(1-t_y)y + t_y\varphi_k(y)]|| \\
&\quad = ||(1-t_x)(x-y) + t_x(\varphi_k(x) - \varphi_k(y)) + (t_x - t_y)(\varphi_k(y) - y)|| \\
&\quad \leq ||(1-t_x)(x-y)|| + ||t_x(\varphi_k(x) - \varphi_k(y))|| + ||(t_x - t_y)(\varphi_k(y) - y)|| \\
&\quad \leq (1-t_x)||x-y|| + (t_x)L_k||x-y|| + ||(t_x - t_y)(\varphi_k(y) - y)|| \\
&\quad \leq L_k||x-y|| + ||(t_x - t_y)(\varphi_k(y) - y)||.
\end{aligned}$$

To estimate the second term, when $d(y,\partial E_0) \geq 2u_k$, we know that $t_x = t_y = 1$, and this term vanishes. So suppose that $d(y,\partial E_0) < 2u_k$. Let $z \in \partial E_0$ be such that $d(y,\partial E_0) = d(z,y)$. Then we have

$$\varphi_k(y) - y = \varphi_k(y) - \varphi_k(z) + \varphi_k(z) - y. \tag{4.101}$$

Since $\partial E_0 \subset \mathcal{T}_{k,d}$, we know that φ_k is identity on ∂E_0, and hence $\varphi_k(z) = z$. Therefore

$$\begin{aligned} ||\varphi_k(y) - y|| = ||\varphi_k(y) - \varphi_k(z) + (z-y)|| &\leq (1+L_k)||z-y|| \\ = (1+L_k)d(y,\partial E_0) &\leq 2(1+L_k)u_k. \end{aligned} \tag{4.102}$$

On the other hand, since $d(x,\partial E_0) \geq d(y,\partial E_0)$, we have $t_x \geq t_y$, and hence

$$\begin{aligned} 0 \leq t_x - t_y &\leq \left(\frac{d(x,\partial E_0)}{u_k} - 1\right) - \left(\frac{d(y,\partial E_0)}{u_k} - 1\right) \\ &= \frac{1}{u_k}[d(x,\partial E_0) - d(y,\partial E_0)], \end{aligned} \tag{4.103}$$

hence

$$||t_x - t_y|| \leq \frac{1}{u_k}||d(x,\partial E_0) - d(y,\partial E_0)|| \leq \frac{1}{u_k}||x-y||. \tag{4.104}$$

Combine (4.102) and (4.104) to obtain

$$\begin{aligned}||(t_x - t_y)(\varphi_k(y) - y)|| &\le \frac{1}{u_k}||x - y|| \times 2(1 + L_k)u_k \\ &\le 2(1 + L_k)||x - y||.\end{aligned} \tag{4.105}$$

Together with (4.100), we get

$$||f_k(x) - f_k(y)|| \le (3L_k + 2)||x - y||. \tag{4.106}$$

We extend f_k to a $(3L_k + 2)$-Lipschitz map in $\mathbb{R}^n$, so that $f_k = \mathrm{id}$ outside a compact set. Then f_k is a deformation in $\mathbb{R}^n$.

By definition, for $x \in E_k$ we know that

$$||f(x)-x|| = ||(1-t_x)x+t_x\varphi_k(x)-x|| \le ||\varphi_k(x)-x|| \le \sqrt{n}2^{-k}, \tag{4.107}$$

where the last inequality is by (4.96). Hence

$$f_k(E_k) \subset B(E_k, \sqrt{n}2^{-k}) \subset B(E_0, \sqrt{n}2^{-k+1}),$$

which yields (4.92).

Moreover, by definition of f_k, $f_k(E_k \backslash B(\partial E_0, 2u_k)) \subset \mathcal{S}_{k,d}$, and hence

$$\begin{aligned}\mathcal{H}^d(f_k(E_k)\backslash \mathcal{S}_{k,d}) &\le \mathcal{H}^d(f_k(E_k \cap B(\partial E_0, 2u_k))) \\ &\le (3L_k + 2)^d \mathcal{H}^d(E_k \cap B(\partial E_0, 2u_k)) < \epsilon,\end{aligned} \tag{4.108}$$

which gives (4.93). □

Now for k large, we will deform a big part of E_k to E_0:

Proposition 4.10. *Let E, U, and C be as in the statement of Theorem 4.1, so that* (4.1) *and* (4.2) *hold. Let E_0 and $\{E_k\}_k$ be obtained as in Proposition 4.6. Then for k large, for each $\epsilon > 0$, there exist $s_k > 0$ and a deformation h_k in U such that $h_k = \mathrm{id}$ in $B(\partial E_0, s_k)$ and*

$$\mathcal{H}^d(h_k(E_k)) < \mathcal{H}^d(E_0) + \epsilon. \tag{4.109}$$

Proof: Since E_0 is a reduced and minimal in $V = B(0, R)\backslash \partial E_0$, the set of regular points E_{0P} of E_0 is of full measure: $\mathcal{H}^d(E_0 \backslash E_{0P}) = 0$. By the C^1-regularity (Theorem 2.25) for regular points, for each $x \in E_{0P}$, there exists $r_x > 0$ with $B(x, 2r_x) \subset U$ such that for all $r < r_x$, there is a Lipschitz deformation retraction $\varphi_{x,r}$ from $\bar{B}(x, r) \to E_0 \cap \bar{B}(x, r)$, with Lipschitz constant no more than 2, and such that $|\varphi_{x,r}(y) - y| \le 2\,\mathrm{dist}(y, E_0)$. Note that $\mathcal{H}^d(E_0 \backslash E_{0P}) = 0$.

We apply the Vitali covering theorem (cf. for example [**21**, Theorem 2.8]) to the family $\mathcal{B} := \{\bar{B}(x, r) : x \in E_{0P}, r < r_x\}$, the measure $\mu = \mathcal{H}^d|_{E_{0P}}$, and get that for any fixed $\epsilon > 0$, there exist a finite set of points $\{x_j\}_{1 \le j \le m} \subset E_{0P}$ and $r_j \in (0, r_{x_j})$ such that the balls $\bar{B}(x_j, r_j)$

are disjoint, $B(x_j,2r_j)\cap\partial E_0=\emptyset$, and $\mathcal{H}^d\big(E_{0P}\backslash\bigcup_{j=1}^m\bar{B}(x_j,r_j)\big)<\frac{\epsilon}{3M\times 2^{d+1}\times 7^n}$. Take $t_j<r_j$ so that $\mathcal{H}^d\big(E_{0P}\backslash\bigcup_{j=1}^m B(x_j,t_j)\big)<\frac{\epsilon}{4M\times 2^d\times 7^n}$.

Let $r=\min_j r_j$ and $t=\min_j(r_j-t_j)$. Set $W=\big\{x\colon d\big(x,\big(\bigcup_{j=1}^m B(x_j,r_j)\big)\cup E_0\big)>t\big\}\cup B(\partial E_0,r)$. Then $d\big(W,\big(\bigcup_{j=1}^m B(x_j,r_j)\big)\big)\geq t$. Define a Lipschitz map $g\colon\big(\bigcup_{j=1}^m B(x_j,r_j)\big)\cup W\to\mathbb{R}^n$ by $g(x)=\varphi_{x_j,r_j}(x)$ when $x\in B(x_j,r_j)$ and $g(x)=x$ for $x\in W$. Then g is 2-Lipschitz in each $B(x_j,r_j)$ and in W, and for any $x\in\bigcup_{j=1}^m B(x_j,r_j)$, we have

$$|g(x)-x|\leq 2\operatorname{dist}(x,E_0). \tag{4.110}$$

For each k such that $\sqrt{n}2^{-k+2}<t/2$, we claim that the restriction of g to $\big[f_k(E_k)\cap\big(\bigcup_{j=1}^m B(x_j,r_j)\big)\big]\cup W$ is 2-Lipschitz, where f_k and u_k are obtained as in Proposition 4.9 with respect to $\frac{\epsilon}{4\times 2^d}$. So take $x,y\in\big[f_k(E_k)\cap\big(\bigcup_{j=1}^m B(x_j,r_j)\big)\big]\cup W$. We know that g is 2-Lipschitz in each $B(x_j,r_j)$ and in W, hence the rest of the argument deals with the case when x, y do not belong to the same $B(x_j,r_j)$ or W.

If x, y belong to two different $B(x_j,r_j)\cap f_k(E_k)$, we know that $|x-y|\geq t$. Then by (4.110), we have

$$\begin{aligned}|g(x)-g(y)|&\leq|g(x)-x|+|x-y|+|g(y)-y|\\&\leq 2d(x,E_0)+2d(y,E_0)+|x-y|.\end{aligned} \tag{4.111}$$

But k is such that $\sqrt{n}2^{-k+2}<t/2$, and $x,y\in f_k(E_k)\subset B(E_0,\sqrt{n}2^{-k+1})\subset B\big(E_0,\frac{t}{4}\big)$, hence

$$|g(x)-g(y)|\leq t+|x-y|\leq 2|x-y|, \tag{4.112}$$

because $|x-y|\geq t$.

If $x\in W$ and $y\in B(x_j,r_j)\cap f_k(E_k)$ for some j, then we have similarly

$$\begin{aligned}|g(x)-g(y)|&=|x-g(y)|\leq|x-y|+|y-g(y)|\\&\leq|x-y|+2d(y,E_0)\leq|x-y|+t/2\leq 2|x-y|,\end{aligned} \tag{4.113}$$

because $d(y,E_0)<\frac{t}{4}$ and $|x-y|\geq t$.

Hence g is 2-Lipschitz on $\big[f_k(E_k)\cap\big(\bigcup_{j=1}^m B(x_j,r_j)\big)\big]\cup W$. So we can extend it to a 2-Lipschitz map g_k from $\mathbb{R}^n$ to $\mathbb{R}^n$.

We would like to control the measure of $\mathcal{H}^d\big(f_k(E_k)\backslash\big(\bigcup_{j=1}^m B(x_j,r_j)\big)\big)$. Since the major part of $f_k(E_k)$ is included in $\mathcal{S}_{k,d}$, let us first estimate $\mathcal{H}^d\big(\mathcal{S}_{k,d}\backslash\big(\bigcup_{j=1}^m B(x_j,r_j)\big)\big)$.

Take any $Q\in Q_k$ and $Q^\circ\cap|R_{k-2}|=\emptyset$. Then by Proposition 4.8 (ii), we know that for $k>k_0$,

$$\mathcal{H}^d(\mathcal{S}_{k,d}\cap Q)\leq M\mathcal{H}^d(E_0\cap V(Q)). \tag{4.114}$$

Now if k is such that $3 \times \sqrt{n}2^{-k} < t$, for each Q such that $Q \backslash \big(\bigcup_{j=1}^m B(x_j, r_j)\big) \neq \emptyset$, we know that $d\big(Q, \big(\bigcup_{j=1}^m B(x_j, t_j)\big)\big) > t - \sqrt{n}2^{-k}$, and hence $d\big(V(Q), \big(\bigcup_{j=1}^m B(x_j, t_j)\big)\big) > t - 3 \times \sqrt{n}2^{-k} > 0$, that is $V(Q) \cap \big(\bigcup_{j=1}^m B(x_j, t_j)\big) = \emptyset$. Hence we have

$$
\begin{aligned}
&\mathcal{H}^d\left(\mathcal{S}_{k,d} \backslash \left(|R_{k-2}| \cup \left(\bigcup_{j=1}^m B(x_j, r_j)\right)\right)\right) \\
&\quad \leq \sum \Big\{\mathcal{H}^d(\mathcal{S}_{k,d} \cap Q) : Q \in Q_k,\, Q^\circ \cap |R_{k-2}| = \emptyset, \\
&\qquad\qquad \text{and } Q \backslash \Big(\bigcup_{j=1}^m B(x_j, r_j)\Big) \neq \emptyset\Big\} \\
&\quad \leq \sum \Big\{M\mathcal{H}^d(E_0 \cap V(Q)) : Q \in Q_k \text{ and } V(Q) \cap \Big(\bigcup_{j=1}^m B(x_j, t_j)\Big) = \emptyset\Big\} \\
&\quad = M \int_{E_0} \sum \Big\{\chi_{V(Q)} : Q \in Q_k \text{ and } V(Q) \cap \Big(\bigcup_{j=1}^m B(x_j, t_j)\Big) = \emptyset\Big\}\, d\mathcal{H}^d \\
&\quad \leq M \int_{E_0 \backslash (\bigcup_{j=1}^m B(x_j, t_j))} \Big[\sum_{Q \in Q_k} \chi_{V(Q)}\Big]\, d\mathcal{H}^d.
\end{aligned} \tag{4.115}
$$

Note that $\sum_{Q \in Q_k} \chi_{V(Q)} \leq \sum_{Q \in \Delta_k} \chi_{V(Q)} \leq 7^n$, hence

$$
\begin{aligned}
&\mathcal{H}^d\Big(\mathcal{S}_{k,d} \backslash \Big(|R_{k-2}| \cup \Big(\bigcup_{j=1}^m B(x_j, r_j)\Big)\Big)\Big) \\
&\quad \leq M \int_{E_0 \backslash (\bigcup_{j=1}^m B(x_j, t_j))} \sum_{Q \in Q_k} \chi_{V(Q)} \\
&\quad \leq 7^n M \int_{E_0 \backslash (\bigcup_{j=1}^m B(x_j, t_j))} d\mathcal{H}^d = 7^n M \mathcal{H}^d\Big(E_0 \backslash \Big(\bigcup_{j=1}^m B(x_j, t_j)\Big)\Big) \\
&\quad < 7^n M \times \frac{\epsilon}{4M \times 2^d \times 7^n} = \frac{\epsilon}{4 \times 2^d}.
\end{aligned} \tag{4.116}
$$

Next let us estimate $\mathcal{H}^d(\mathcal{S}_{k,d} \cap |R_{k-2}|)$. For each $Q \in \Delta_{k-2}$, we know that

$$
\mathcal{H}^d(\mathcal{S}_{k,d} \cap Q) = 4^{n-d} \mathcal{H}^d(S_{k-2,d} \cap Q), \tag{4.117}
$$

hence

$$\begin{aligned}\mathcal{H}^d(\mathcal{S}_{k,d}\cap|R_{k-2}|)&\leq \sum_{Q\in R_{k-2}}\mathcal{H}^d(\mathcal{S}_{k,d}\cap Q)\\ &= 4^{n-d}\sum_{Q\in R_{k-2}}\mathcal{H}^d(\mathcal{S}_{k-2,d}\cap Q)\leq C_3 4^{n-d}\mathcal{H}^d(\mathcal{T}_{k-2,d}),\end{aligned}\tag{4.118}$$

where $C_3=C_3(n,d)$ is the number of cubes $Q\in\Delta_k$ that share a same d-face. This is a constant that only depends on n and d.

By Proposition 4.8 (i), we know that for k large, $\mathcal{H}^d(\mathcal{S}_{k,d}\cap|R_{k-2}|)<\frac{\epsilon}{4\times 2^d}$.

Recall that u_k is such that

$$\mathcal{H}^d(f_k(E_k)\backslash\mathcal{S}_{k,d})<\frac{\epsilon}{4\times 2^d},\tag{4.119}$$

hence for k large, we have

$$\begin{aligned}
&\mathcal{H}^d(g_k(f_k(E_k)))\leq \mathcal{H}^d(g_k(\mathcal{S}_{k,d}))+\mathcal{H}^d(g_k(f_k(E_k)\backslash\mathcal{S}_{k,d}))\\
&\quad\leq \mathcal{H}^d\Big(g_k\Big(\mathcal{S}_{k,d}\cap\Big(\bigcup_{j=1}^m B(x_j,r_j)\Big)\Big)\Big)+\mathcal{H}^d(g_k(\mathcal{S}_{k,d}\cap|R_{k-2}|))\\
&\qquad+\mathcal{H}^d\Big(g_k\Big(\mathcal{S}_{k,d}\backslash\Big(|R_{k-2}|\cup\Big(\bigcup_{j=1}^m B(x_j,r_j)\Big)\Big)\Big)\Big)+\mathcal{H}^d(g_k(f_k(E_k)\backslash\mathcal{S}_{k,d}))\\
&\quad\leq \mathcal{H}^d(E_0)+2^d\Big[\mathcal{H}^d(\mathcal{S}_{k,d}\cap|R_{k-2}|)\\
&\qquad\qquad+\mathcal{H}^d\Big(\mathcal{S}_{k,d}\backslash\Big(|R_{k-2}|\cup\Big(\bigcup_{j=1}^m B(x_j,r_j)\Big)\Big)\Big)+\mathcal{H}^d(f_k(E_k)\backslash\mathcal{S}_{k,d})\Big]\\
&\quad\leq \mathcal{H}^d(E_0)+2^d\Big(\frac{\epsilon}{4\times 2^d}+\frac{\epsilon}{4\times 2^d}+\frac{\epsilon}{4\times 2^d}\Big)=\mathcal{H}^d(E_0)+\frac{3}{4}\epsilon.
\end{aligned}\tag{4.120}$$

Note that $g_k\circ f_k$ is the identity map in the neighborhood $B(\partial E_0,s_k)$ of ∂E_0, with $s_k=\min\{u_k,r\}$. But $g_k\circ f_k$ might even not be a deformation in $\mathbb{R}^n\backslash\partial E_0$, because the image of $g_k\circ f_k$ might touch ∂E_0.

We still have to modify this sequence $g_k\circ f_k(E_k)$ to a sequence of deformations of E_k in U.

For this purpose, let D_k denote the convex hull of $E_k\backslash B(\partial E_0,s_k)$. Then D_k is a compact subset of U. In fact, since $E_k\subset\bar{U}$ and $E_k\cap\partial U=\partial E_0$, we have $d(E_k\backslash B(\partial E_0,s_k),\partial U)>0$. Since U is convex, the map $d(\cdot,\partial U)\colon\bar{U}\to\mathbb{R}$ is convex. Hence $d(E_k\backslash B(\partial E_0,s_k),\partial U)>0$ implies that $d(D_k,\partial U)>0$.

Let π_k be the nearest point projection to the convex set D_k. Then π_k is 1-Lipschitz (cf. [**4**, Proposition 5.3]). We define $h_k \colon E_k \to (E_k \cap B(\partial E_0, s_k)) \cup D_k$ by

$$h_k(x) = \begin{cases} x, & x \in E_k \cap \bar{B}(\partial E_0, s_k), \\ \pi_k \circ g_k \circ f_k(x), & x \in E_k \backslash B(\partial E_0, s_k). \end{cases} \tag{4.121}$$

Note that by definition, h_k is Lipschitz both on $E_k \cap \bar{B}(\partial E_0, s_k)$ and $E_k \backslash B(\partial E_0, s_k)$. On their intersection $E_k \cap \partial B(\partial E_0, s_k)$, by definition we know that $g_k \circ f_k(x) = x$, and since $E_k \cap \partial B(\partial E_0, s_k) \subset D_k$, we know that

$$\pi_k \circ g_k \circ f_k(x) = \pi_k(x) = x, \tag{4.122}$$

hence h_k is well defined and Lipschitz.

Set $\delta_k = d(D_k, \partial U)$. Let $C_k = \bar{B}\big(D_k, \frac{1}{2}\delta_k\big)$. Then C_k is a compact convex subset of U.

Then we set $h_k(x) = x$ for $x \in U \backslash C_k$ and then extend h_k to a Lipschitz map $U \to U$. Then $W_k := \{x \in U : h_k(x) \neq x\}$ is compact in U, and hence $h_k(W_k) \cup W_k$ is compact. Therefore, h_k is a deformation in U.

Moreover, we know that

$$\begin{aligned} \mathcal{H}^d(h_k(E_k)) &\leq \mathcal{H}^d(h_k(E_k \backslash B(\partial E_0, s_k))) + \mathcal{H}^d(h_k(E_k \cap B(\partial E_0, s_k))) \\ &= \mathcal{H}^d(\pi_k \circ g \circ f_k(E_k \backslash B(\partial E_0, s_k))) + \mathcal{H}^d(E_k \cap B(\partial E_0, s_k)) \\ &\leq \mathcal{H}^d(g_k \circ f_k(E_k \backslash B(\partial E_0, s_k))) + \mathcal{H}^d(E_k \cap B(\partial E_0, u_k)) \\ &\leq \mathcal{H}^d(g_k \circ f_k(E_k)) + \frac{\epsilon}{4 \times 2^d} \\ &\leq \mathcal{H}^d(E_0) + \frac{3}{4}\epsilon + \frac{\epsilon}{4 \times 2^d} < \mathcal{H}^d(E_0) + \epsilon. \end{aligned} \tag{4.123}$$

□

Now after Proposition 4.10, for any $\epsilon > 0$, take k large and h_k as obtained in Proposition 4.10 such that (4.109) holds. Then since $E_k \in \mathcal{F}_d^*(E, U)$, so does $h_k(E_k)$. Since ϵ is arbitrary, we have

$$\inf_{F \in \mathcal{F}_d^*(E,U)} \mathcal{H}^d(F) \leq \mathcal{H}^d(E_0) \leq \inf_{F \in \overline{\mathcal{F}}(E,U)} \mathcal{H}^d(F). \tag{4.124}$$

Then by (4.27), we have

$$\inf_{F \in \mathcal{F}(E,U)} \mathcal{H}^d(F) \leq \inf_{F \in \overline{\mathcal{F}}(E,U)} \mathcal{H}^d(F). \tag{4.125}$$

On the other hand, since $\mathcal{F}(E, U) \subset \overline{\mathcal{F}}(E, U)$, we have that (i) of Theorem 4.1 follows directly. And (ii) is a direct corollary of (i).

Theorem 4.11. *Let $U \subset \mathbb{R}^n$ be a bounded convex open set and E be a reduced closed set with finite d-Hausdorff measure such that $E \subset \bar{U}$. Let C denote the convex hull of E. Suppose that* (4.1) *holds and*

$$\text{there exists a bi-Lipschitz map } \psi\colon \mathbb{R}^n \to \mathbb{R}^n \text{ such that } \psi^{-1}(E \cap \partial U) \subset |\Delta_{k,d-1}| \text{ for some } k \in \mathbb{N}, \tag{4.126}$$

where Q_0 denotes the unit cube $[0,1]^n$. Then

(i) $\inf_{F \in \overline{\mathcal{F}}(E,U)} \mathcal{H}^d(F) = \inf_{F \in \mathcal{F}(E,U)} \mathcal{H}^d(F)$.

(ii) *If E is a d-dimensional minimal set in U, then*

$$\mathcal{H}^d(E) = \inf_{F \in \overline{\mathcal{F}}(E,U)} \mathcal{H}^d(F). \tag{4.127}$$

Remark 4.12. (1) We will see in Theorem 4.13 that condition (4.1) is not needed.

(2) Condition (4.126) can be relaxed, with essentially the same proof, but with more technical details. Here we only give proof under this hypotheses, which is enough for purpose of use.

Proof of Theorem 4.11: Note that in the proof of Theorem 4.1, we only used condition (4.1) before Proposition 4.8. Hence we can obtain the sequence $\{E_k\} \subset \mathcal{F}_d^*(E,U)$ and $E_0 \subset \bar{U}$ such that (i)–(iii) in Proposition 4.6 hold.

Set

$$Q_k' := \{Q \in \Delta_k : Q \cap \psi^{-1}(E_0) \neq \emptyset\} \tag{4.128}$$

and

$$Q_k = \{Q \in \Delta_k : \exists Q' \in Q_k' \text{ such that } Q \cap Q' \neq \emptyset\}. \tag{4.129}$$

Let $|Q_k| = \bigcup_{Q \in Q_k} Q$, and for each $j \leq n$, let $Q_{k,j}$ be the set of all j-faces of elements in Q_k, and let $\mathcal{S}_{k,j} = \bigcup_{\sigma \in Q_{k,j}} \sigma$ denote the j-skeleton of Q_k.

Set

$$R_k := \{Q \in \Delta_k : Q \cap \psi^{-1}(E_0 \cap U) \neq \emptyset\}. \tag{4.130}$$

Let $|R_k| = \bigcup_{Q \in R_k} Q$, and for each $j \leq n$, let $R_{k,j}$ be the set of all j-faces of elements in R_k, and let $\mathcal{T}_{k,j} = \bigcup_{\sigma \in R_{k,j}} \sigma$ denote the j-skeleton of R_k.

Then by the same argument as in Proposition 4.8 (which only used the hypothesis that $E \cap \partial U$ is of finite $(d-1)$-Hausdorff measure, and this is also guaranteed by (4.126)), we get that

$$\lim_{k \to \infty} \mathcal{H}^d(\mathcal{T}_{k,d}) = 0 \tag{4.131}$$

and there exists $M > 0$ which depends only on n and d such that, for each $k > k_0$, and each $Q \in Q_k$ such that $Q^\circ \cap |R_{k-2}| = \emptyset$,

$$\mathcal{H}^d(\mathcal{S}_{k,d} \cap Q) < M\mathcal{H}^d(\psi^{-1}(E_0) \cap V(Q)). \tag{4.132}$$

Now the same argument as in Proposition 4.9 gives that, for each $\epsilon > 0$, there exist a sequence of deformations f_k in $\mathbb{R}^n$ and $u_k > 0$ such that

$$\mathcal{H}^d(\psi^{-1}(E_k \cap B(\partial E_0, u_k))) < \epsilon, \tag{4.133}$$

$$f_k = \mathrm{id} \ \text{ in } \psi^{-1}(B(\partial E_0, u_k)), \tag{4.134}$$

$$f_k(\psi^{-1}(E_k)) \subset \psi^{-1}(B(E_0, \sqrt{n}2^{-k+1})), \tag{4.135}$$

and for k large,

$$\mathcal{H}^d(f_k(\psi^{-1}(E_k))\backslash \mathcal{S}_{k,d}) < \epsilon. \tag{4.136}$$

We apply ψ to (4.132)–(4.136) and get that, for each $\epsilon > 0$, and for k large, we have

(4.137) for each $Q \in Q_k$ such that $Q^\circ \cap |R_{k-2}| = \emptyset$,

$$\mathcal{H}^d(\psi(\mathcal{S}_{k,d} \cap Q)) < M'\mathcal{H}^d(E_0 \cap \psi(V(Q))),$$

where M only depends on n, d, and the Lipschitz constant L of ψ; and there exist deformations $f_k' = \psi \circ f_k$ in $\mathbb{R}^n$ and $u_k > 0$ such that

$$\mathcal{H}^d(E_k \cap B(\partial E_0, u_k)) < \epsilon, \tag{4.138}$$

$$f_k' = \mathrm{id} \ \text{ in } B(\partial E_0, u_k), \tag{4.139}$$

$$f_k'(E_k) \subset B(E_0, L\sqrt{n}2^{-k+1}), \tag{4.140}$$

and

$$\mathcal{H}^d(f_k'(E_k)\backslash \psi(\mathcal{S}_{k,d})) < \epsilon. \tag{4.141}$$

Then by exactly the same operation as in Proposition 4.10, we deform the major part of $f_k'(E_k)$ to E_0 by a map g_k, so that $\mathcal{H}^d(g_k \circ f_k'(E_k)) \leq \mathcal{H}^d(E_0) + C\epsilon$, where C only depends on the Lipschitz constant L of ψ. Then we similarly define the deformation h_k in U as in (4.121) replacing f_k by f_k'. Then we get the same conclusion as in Proposition 4.10. And thus Theorem 4.11 is proved. □

Finally we will get rid of condition (4.1).

Theorem 4.13 (Upper semi-continuity). *Let $U \subset \mathbb{R}^n$ be a bounded convex open set and E be a closed set in $\bar{U}$ with finite d-Hausdorff measure. Let C denote the convex hull of E. Suppose that* (4.126) *holds. Then*

(i) $\inf_{F\in\overline{\mathcal{F}}(E,U)} \mathcal{H}^d(F) = \inf_{F\in\mathcal{F}(E,U)} \mathcal{H}^d(F)$.

(ii) *If E is a d-dimensional minimal set in U. Then*

$$\mathcal{H}^d(E) = \inf_{F\in\overline{\mathcal{F}}(E,U)} \mathcal{H}^d(F). \tag{4.142}$$

Proof: In case that U is strictly convex, then (4.1) holds directly, hence (4.126) is enough for getting conclusions (i) and (ii) of Theorem 4.11.

Now if U is not strictly convex, modulo translation, we can suppose that $0 \in U$.

Set $p\colon \mathbb{R}^n \to [0, \infty)$ be the Minkowski functional of U: $p(x) = \inf\big\{r > 0 : \frac{r}{x} \in U\big\}$. Then p is convex; $p(\lambda x) = \lambda p(x), \forall \lambda > 0, \forall x \in \mathbb{R}^n$; $U = \{x \in \mathbb{R}^n : p(x) < 1\}$; and there exists $M_1 > 0$ so that $p(x) \leq M_1|x|$, $\forall x \in \mathbb{R}^n$ (cf. [**4**, Lemma 1.2]). Since U is bounded, there exists $M_2 \in (0, M_1)$ so that $p(x) \geq M_2|x|$, $\forall x \in \mathbb{R}^n$.

Let us prove that p is M_1-Lipschitz. That is

$$|p(x) - p(y)| < M_1|x - y|, \quad \forall x, y \in \mathbb{R}^n. \tag{4.143}$$

For any $x, y \in \mathbb{R}^n$, if $p(x) = p(y)$, then (4.143) holds directly. Otherwise, suppose without loss of generality that $p(x) > p(y)$. Set $z = x - y$. Then since p is convex and homogenous, we know that

$$\frac{|p(x) - p(y)|}{|x - y|} = \frac{p(x) - p(y)}{|x - y|} = \frac{p(y + z) - p(y)}{|z|} \leq \frac{p(z)}{|z|} \leq M_1 \tag{4.144}$$

which again gives (4.143).

Now for any $\epsilon > 0$, set $p_\epsilon(x) = p(x) + \epsilon|x|$. Let $U_\epsilon = \{x \in \mathbb{R}^n : p_\epsilon(x) < 1\}$. Note that $\bar{U}_\epsilon$ is strictly convex. Indeed, for any $x, y \in \partial U_\epsilon$, we have $p_\epsilon(x) = p_\epsilon(y) = 1$, and for any $\alpha \in (0, 1)$, we have

$$p_\epsilon(\alpha x + (1 - \alpha)y) = p(\alpha x + (1 - \alpha)y) + \epsilon|\alpha x + (1 - \alpha)y|. \tag{4.145}$$

Since p is convex, $p(\alpha x + (1 - \alpha)y) \leq \alpha p(x) + (1 - \alpha)p(y)$; and since $|\cdot|$ is strictly convex, we have $|\alpha x + (1 - \alpha)y| < \alpha|x| + (1 - \alpha)|y|$. Hence

$$\begin{aligned} p_\epsilon(\alpha x + (1 - \alpha)y) &< \alpha p(x) + (1 - \alpha)p(y) + \epsilon(\alpha|x| + (1 - \alpha)|y|) \\ &= \alpha p_\epsilon(x) + (1 - \alpha)p_\epsilon(y). \end{aligned} \tag{4.146}$$

As a result, U_ϵ is strictly convex, $0 \subset U_\epsilon \subset U$, and p_ϵ is the Minkowski functional of U_ϵ.

Let $f_\epsilon\colon \bar{U} \to \bar{U}_\epsilon$: $f_\epsilon(x) = x\frac{p(x)}{p_\epsilon(x)}$. Then we know that

$$p_\epsilon(f(x)) = p_\epsilon\Big(x\frac{p(x)}{p_\epsilon(x)}\Big) = \frac{p(x)}{p_\epsilon(x)}p_\epsilon(x) = p(x). \tag{4.147}$$

Hence f_ϵ is a bijection.

Let us estimate the bi-Lipschitz constant of f_ϵ. Take any $x, y \in \bar{U}$, we have

$$\begin{aligned}|f_\epsilon(x) - f_\epsilon(y)| &= \left| x\frac{p(x)}{p_\epsilon(x)} - y\frac{p(y)}{p_\epsilon(y)} \right| \\ &= \left| \frac{p(x)}{p_\epsilon(x)}(x-y) + y\left(\frac{p(x)}{p_\epsilon(x)} - \frac{p(y)}{p_\epsilon(y)}\right) \right|.\end{aligned} \tag{4.148}$$

Note that

$$\begin{aligned}&\frac{p(x)}{p_\epsilon(x)} \in \left[\frac{1}{1+\epsilon M_1}, \frac{1}{1+\epsilon M_2}\right] \\ &\quad \Rightarrow \frac{1}{1+\epsilon M_1}|x-y| \leq \frac{p(x)}{p_\epsilon(x)}|x-y| \leq \frac{1}{1+\epsilon M_2}|x-y|,\end{aligned} \tag{4.149}$$

and

$$\begin{aligned}|y|\left|\frac{p(x)}{p_\epsilon(x)} - \frac{p(y)}{p_\epsilon(y)}\right| &= |y|\left|\frac{p(x)}{p(x)+\epsilon|x|} - \frac{p(y)}{p(y)+\epsilon|y|}\right| \\ &= |y|\left|\frac{p(x)(p(y)+\epsilon|y|) - p(y)(p(x)+\epsilon|x|)}{(p(x)+\epsilon|x|)(p(y)+\epsilon|y|)}\right| \\ &= \epsilon|y|\frac{|p(x)|y| - p(y)|x||}{(p(x)+\epsilon|x|)(p(y)+\epsilon|y|)} \\ &\leq \epsilon\Bigg[\frac{|y|p(x)||y|-|x||}{(p(x)+\epsilon|x|)(p(y)+\epsilon|y|)} \\ &\qquad + \frac{|y||x||p(x)-p(y)|}{(p(x)+\epsilon|x|)(p(y)+\epsilon|y|)}\Bigg] \\ &= \epsilon\Bigg[\frac{|y|}{p(y)+\epsilon|y|}\frac{p(x)}{p(x)+\epsilon|x|}||y|-|x|| \\ &\qquad + \frac{|y|}{p(y)+\epsilon|y|}\frac{|x|}{p(x)+\epsilon|x|}|p(x)-p(y)|\Bigg] \\ &\leq \epsilon\left[\frac{|y|}{p(y)}|y-x| + \frac{|y|}{p(y)}\frac{|x|}{p(x)}|p(x)-p(y)|\right] \\ &\leq \epsilon\left[\frac{1}{M_2}|y-x| + \left(\frac{1}{M_2}\right)^2 M_1|x-y|\right] \\ &= \epsilon\left[\frac{M_1+M_2}{M_2^2}\right]|x-y|.\end{aligned} \tag{4.150}$$

(4.148), (4.149), and (4.150) give

$$\begin{aligned}(4.151)\quad \Big[\frac{1}{1+\epsilon M_1}-\epsilon\Big[\frac{M_1+M_2}{M_2^2}\Big]\Big]|x-y| &\le |f_\epsilon(x)-f_\epsilon(y)|\\ &\le \Big[\frac{1}{1+\epsilon M_2}+\epsilon\Big[\frac{M_1+M_2}{M_2^2}\Big]\Big]|x-y|.\end{aligned}$$

Hence we have that f_ϵ is L_ϵ-bi-Lipschitz, with $L_\epsilon \to 1$ as $\epsilon \to 0$. As a result we have

$$(4.152)\quad \mathcal{F}(f_\epsilon(E),U_\epsilon)=\{f_\epsilon(F): F\in\mathcal{F}(E,U)\} \text{ and } \overline{\mathcal{F}}(f_\epsilon(E),U_\epsilon)=\{f_\epsilon(F): F\in\overline{\mathcal{F}}(E,U)\}.$$

Since U_ϵ is strictly convex, we can apply Theorem 4.11 to the open set U_ϵ and the set $f_\epsilon(E)$, and get

$$(4.153)\quad \inf_{F\in\mathcal{F}(f_\epsilon(E),U_\epsilon)}\mathcal{H}^d(F)=\inf_{F\in\overline{\mathcal{F}}(f_\epsilon(E),U_\epsilon)}\mathcal{H}^d(F),$$

hence by (4.152),

$$(4.154)\quad \inf_{F\in\mathcal{F}(E,U)}\mathcal{H}^d(f_\epsilon(F))=\inf_{F\in\overline{\mathcal{F}}(E,U)}\mathcal{H}^d(f_\epsilon(F)).$$

Note that for each $F\subset U$, since $\lim_{\epsilon\to 0}L(\epsilon)=1$, we have

$$(4.155)\quad \mathcal{H}^d(F)=\lim_{\epsilon\to 0}\mathcal{H}^d(f_\epsilon(F)),$$

hence (4.154) gives conclusion (i) of Theorem 4.13. Then conclusion (ii) follows directly. □

5. Uniqueness properties for 2-dimensional minimal cones in $\mathbb{R}^3$

In this section we prove the topological and Almgren uniqueness for all 2-dimensional minimal cones in $\mathbb{R}^3$. Hence in the following text, Almgren and G-topological uniqueness refer to Almgren and G-topological uniqueness of dimension 2.

5.1. Planes.

Theorem 5.1. *A 2-dimensional linear plane P is Almgren and G-topologically unique in $\mathbb{R}^n$ for all $n\ge 3$ and all abelian group G.*

Proof: Let $P\subset\mathbb{R}^n$ be a 2-dimensional plane containing the origin. By Propositions 3.2 and 3.4, to prove that P is Almgren and G-topologically unique, it is enough to prove that P is G-topologically unique in the unit ball B.

Suppose that E is a reduced G-topological competitor of dimension 2 for P in B, so that

$$\mathcal{H}^2(E \cap B) = \mathcal{H}^2(P \cap B). \tag{5.1}$$

By Remark 3.3 (5), we know that E is G-topological and hence Almgren minimal in B. By the convex hull property for Almgren minimal sets, $E \cap B$ is contained in the convex hull of $E \cap \partial B = P \cap \partial B$, which is $P \cap B$. Hence $E \cap B \subset P \cap B$. Then since both P and E are reduced sets, (5.1) gives that $E = P$. Hence P is G-topologically unique, and hence Almgren unique. □

5.2. The $\mathbb{Y}$ sets.

Theorem 5.2. *Any 2-dimensional $\mathbb{Y}$ set is Almgren and G-topologically unique in $\mathbb{R}^n$ for all $n \geq 3$ and all abelian groups G.*

Proof: By Propositions 3.4 and 3.5, it is enough to prove that $\mathbb{Y}$ sets are G-topologically unique in $\mathbb{R}^3$.

So let Y be a 2-dimensional $\mathbb{Y}$ set in $\mathbb{R}^3$. Modulo changing the coordinate system, we can suppose that the spine of Y is the vertical line $Z = \{(x, y, z) \in \mathbb{R}^3 : x = y = 0\}$, and that the intersection of Y with the horizontal plane $Q := \{z = 0\}$ is the union Y_1 of the three half lines R_{0a_i}, $1 \leq i \leq 3$, where $a_1 = (1, 0)$, $a_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and $a_3 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ under the coordinate in Q.

We regard Q and Z as subspaces of $\mathbb{R}^3$, and write $\mathbb{R}^3 = Q \times Z$.

Then $Y = Y_1 \times Z$.

By Proposition 3.2, it is enough to prove that Y is G-topologically unique in the cylinder $D := B_Q(0, 1) \times (-1, 1)$.

For $t \in (-1, 1)$, let $a_i^t = (a_i, t) \in Q \times (-1, 1)$.

Let $f \colon \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = z$. For any set $F \subset \mathbb{R}^3$, and each $t \in \mathbb{R}$, set $F_t = f^{-1}\{t\} \cap F$ the slice of F at level t.

Let $\widehat{a_i^t a_j^t}$ denote the open minor arc of circle of $\partial B_Q(0, 1) \times \{t\} = \partial D_t$ between a_i^t and a_j^t, $1 \leq i \neq j \leq 3$. Then these arcs belong to $\mathbb{R}^3 \backslash D$. Since $\widehat{a_i^t a_j^t}$, $1 \leq i < j \leq 3$, lie in three different connected components of $\mathbb{R}^3 \backslash Y$, for any 2-dimensional G-topological competitor F of Y with respect to D, they also lie in three different connected components of $\mathbb{R}^3 \backslash F$. In particular, they belong to three different connected components of $\bar{D}_t \backslash F_t$.

Lemma 5.3. *If F is a G-topological competitor for Y of dimension 2 with respect to D, then for each $t \in (-1, 1)$, $F_t \cap D_t$ must connect the three points in $Y_t \cap \partial D_t = \{a_i^t, 1 \leq i \leq 3\}$, i.e. the three points a_i^t, $1 \leq i \leq 3$ lie in the same connected component of $(F_t \cap D_t) \cup \{a_i^t, 1 \leq i \leq 3\}$.*

Proof: Take any $t \in [-1,1]$.

Suppose that the three points a_i^t, $1 \leq i \leq 3$, do not belong to the same connected component of $(F_t \cap D_t) \cup \{a_i^t,\ 1 \leq i \leq 3\}$. Suppose for example that the connected component of $(F_t \cap D_t) \cup \{a_i^t,\ 1 \leq i \leq 3\}$ contains a_1^t but does not contain a_2^t and a_3^t. Then there exist two relatively closed subsets C_1 and C_2 of $(F_t \cap D_t) \cup \{a_i^t,\ 1 \leq i \leq 3\}$, so that $C_1 \cap C_2 = \emptyset$, $C_1 \cup C_2 = (F_t \cap D_t) \cup \{a_i^t,\ 1 \leq i \leq 3\}$, and $a_1^t \in C_1$, $a_2^t, a_3^t \in C_2$ (cf. [**24**, §37, Ex. 4]). Since $(F_t \cap D_t) \cup \{a_i^t,\ 1 \leq i \leq 3\}$ is compact, so are C_1 and C_2. Hence there exists a curve $\gamma\colon [0,1] \to \bar{D}_t$ with $\gamma(0), \gamma(1) \in \partial D_t$, which separates C_1 and C_2. That is, $\operatorname{im}\gamma \subset \bar{D}_t \backslash ((F_t \cap D_t) \cup \{a_i^t,\ 1 \leq i \leq 3\})$, and the sets C_1 and $\{a_2^t, a_3^t\}$ belong to different connected components of $\bar{D}_t \backslash \operatorname{im}\gamma$.

As a consequence, there exist $t_2, t_3 \in [0,1]$ such that $\gamma(t_j)$ belong to the open minor arc of circle $\widehat{a_1^t a_j^t}$ of ∂D_t between a_1^t, a_j^t, $j = 2,3$. As a result, $b_j := \gamma(t_j)$ belong to different connected components of $\mathbb{R}^3_t \backslash Y_t$, and hence they belong to different connected components of $\mathbb{R}^3 \backslash Y$, since $Y = Y_t \times \mathbb{R}$.

Since Y is a cone, $b_j \notin Y$, we have the segment $[b_j, 2b_j] \subset \mathbb{R}^3 \backslash Y$. Note that $(b_j, 2b_j] \subset \mathbb{R}^3 \backslash \bar{D}$ and $Y \backslash \bar{D} = F \backslash \bar{D}$, hence $(b_j, 2b_j] \subset \mathbb{R}^3 \backslash F$. Since $b_j \in \bar{D}_t \backslash F_t$, we know that $b_j \in \mathbb{R}^3 \backslash F$ as well, hence $[b_j, 2b_j] \subset \mathbb{R}^3 \backslash F$.

Let β denote the curve $[2b_2, b_2] \cup \gamma([t_2, t_3]) \cup [b_3, 2b_3]$. Then $\beta \subset \mathbb{R}^3 \backslash F$, and it connects $2b_2$ and $2b_3$. Hence the two points $2b_2$ and $2b_3$ belong to the same connected component of $\mathbb{R}^3 \backslash F$.

On the other hand, we know that b_j, $j = 2,3$, belong to different connected components of $\mathbb{R}^3 \backslash Y$. Since $[b_j, 2b_j] \subset \mathbb{R}^3 \backslash Y$, $j = 2,3$, we have that $2b_j$, $j = 2,3$, belong to different connected components of $\mathbb{R}^3 \backslash Y$. This contradicts the fact that F is a G-topological competitor for Y of codimension 1 (which, by Remark 3.2 of [**15**], corresponds to Mumford–Shah competitors, as defined in [**6**, Section 19]). □

Proposition 5.4. *Let $E \subset \bar{B}_Q(0,1)$ be a closed set with finite $\mathcal{H}^1$ measure such that $E \cap \partial B_Q(0,1) = \{a_1, a_2, a_3\}$, and a_i, $1 \leq i \leq 3$, belong to the same connected component of E. Then*

$$\mathcal{H}^1(E) \geq \mathcal{H}^1(Y_1 \cap \bar{B}_Q(0,1)), \tag{5.2}$$

and equality holds if and only if $E = Y_1 \cap \bar{B}_Q(0,1)$ modulo a $\mathcal{H}^1$-null set.

Proof: Let B denote $\bar{B}_Q(0,1)$ for short. Let E be as in the statement. Let F_0 be the connected component of E that contains $\{a_1, a_2, a_3\}$. Then $\mathcal{H}^1(F_0) \leq \mathcal{H}^1(E)$.

If $\mathcal{H}^1(E)=\infty$, then it is automatically true. Otherwise, it is enough to prove that, for any $\epsilon>0$,

$$\mathcal{H}^1(F_0)\geq \mathcal{H}^1(Y_1\cap \bar{B}_Q(0,1))=\sum_{i=1}^{3}\mathcal{H}^1([o,a_i])-\epsilon. \tag{5.3}$$

Note that, for this purpose, it is enough to look at $\epsilon\in(0,||a_1-a_2||)$.

So fix $\epsilon\in(0,||a_1-a_2||)$. Let $\mathcal{F}=\{\gamma\subset F_0:\gamma$ is connected and closed, and $\{a_1,a_2\}\subset\gamma\}$, and let $\gamma_{12}\subset\mathcal{F}$ be such that $\mathcal{H}^1(\gamma_{12})<\inf_{\gamma\in\mathcal{F}}\mathcal{H}^1(\gamma)+\frac{\epsilon}{2}$.

Next, we will find a connected set γ_3 such that $a_3\in\gamma_3$, $\gamma_3\cup\gamma_{12}$ is connected, and $\gamma_3\cap\gamma_{12}$ is a single point.

If $a_3\in\gamma_{12}$, we just set $\gamma_3=\{a_3\}$. Otherwise, let $\gamma'=F_0\backslash\gamma_{12}$. Then $\gamma'\cup\gamma_{12}$ is connected and $a_3\in\gamma'$. Let γ_4 be the connected component of γ' that contains a_3. Then we claim that $\gamma_4\cup\gamma_{12}$ is connected. In fact, if $\gamma_4=\gamma'$, then it is clear. Otherwise, suppose that $\gamma_4\cup\gamma_{12}$ is not connected. Then, since both γ_4 and γ_{12} are connected, they are the two connected components of $\gamma_4\cup\gamma_{12}$, and hence there exist two disjoint open sets U_1 and U_2 of $\mathbb{R}^3$ such that $\gamma_4\subset U_1$ and $\gamma_{12}\subset U_2$. Similarly, since γ_4 is a connected component of γ', there exist two disjoint open sets U_3 and U_4 of $\mathbb{R}^3$ such that $\gamma_4\subset U_3$ and $\gamma'\backslash\gamma_4\subset U_4$. Then let $U=U_1\cap U_2$ and $V=U_3\cup U_4$. Then U and V are disjoint, and $\gamma_4\subset U$, $F_0\backslash\gamma_4=\gamma_{12}\cup\gamma'\backslash\gamma_4\subset V$. This contradicts the fact that F_0 is connected.

Hence $\gamma_4\cup\gamma_{12}$ is connected. As a result, $\bar{\gamma}_4\cap\gamma_{12}\neq\emptyset$, because γ_{12} and $\bar{\gamma}_4$ are both closed and their union is connected.

Take $p\in\bar{\gamma}_4\cap\gamma$. Let $\gamma_3=\gamma_4\cup\{p\}$. Then γ_3 is connected, contains a_3, and $\gamma_3\cap\gamma_{12}=\{p\}$. As a result, $\mathcal{H}^1(\gamma_3)\geq\mathcal{H}^1([p,a_3])$.

Let $\gamma=\gamma_{12}\cup\gamma_3$. Then $\gamma\subset F_0$, and thus

$$\mathcal{H}^1(F_0)\geq\mathcal{H}^1(\gamma)=\mathcal{H}^1(\gamma_{12})+\mathcal{H}^1(\gamma_3)\geq\mathcal{H}^1(\gamma_{12})+\mathcal{H}^1([p,a_3]). \tag{5.4}$$

Recall that $\gamma_{12}\subset\mathcal{F}$ is such that $\mathcal{H}^1(\gamma_{12})<\inf_{\gamma\in\mathcal{F}}\mathcal{H}^1(\gamma)+\frac{\epsilon}{2}$, where $\mathcal{F}=\{\gamma\subset F_0:\gamma$ is connected and closed, and $\{a_1,a_2\}\subset\gamma\}$. Since $\epsilon<||a_1-a_2||$, we know that at least one of a_1, a_2 is in $\gamma_{12}\backslash B\big(p,\frac{\epsilon}{2}\big)$. But $p\in\gamma_{12}$, which is connected, hence $\gamma_{12}\cap B\big(p,\frac{\epsilon}{2}\big)$ connects p to the boundary $\partial B\big(p,\frac{\epsilon}{2}\big)$. As a result $\mathcal{H}^1\big(\gamma_{12}\cap B\big(p,\frac{\epsilon}{2}\big)\big)\geq\frac{\epsilon}{2}$, and thus $\mathcal{H}^1\big(\gamma_{12}\backslash B\big(p,\frac{\epsilon}{2}\big)\big)<\inf_{\gamma\in\mathcal{F}}\mathcal{H}^1(\gamma)$. Hence by definition of $\mathcal{F}$, the closed set $\gamma_{12}\backslash B\big(p,\frac{\epsilon}{2}\big)$ does not contain any element in $\mathcal{F}$.

Recall that at least one of a_1, a_2 is in $\gamma_{12}\backslash B\big(p,\frac{\epsilon}{2}\big)$. Without loss of generality we suppose $a_1\in\gamma_{12}\backslash B\big(p,\frac{\epsilon}{2}\big)$. Let γ_1 be the connected component of $\gamma_{12}\backslash B\big(p,\frac{\epsilon}{2}\big)$ that contains a_1. Since γ_{12} is closed, γ_1 is also closed. Since γ_1 is a closed connected subset of $\gamma_{12}\backslash B\big(p,\frac{\epsilon}{2}\big)$ which does not contain any element in $\mathcal{F}$, and $a_1\in\gamma_1$, by definition we get $a_2\notin\gamma_1$. In particular, $\gamma_{12}\backslash\gamma_1\neq\emptyset$.

We claim that $\gamma_1\cap\partial B\big(p,\frac{\epsilon}{2}\big)\neq\emptyset$. Otherwise, let $U_1=B\big(p,\frac{\epsilon}{2}\big)$, $U_2=\bar{B}\big(p,\frac{\epsilon}{2}\big)^C$, then U_i, $i=1,2$, are disjoint open sets, and $\gamma_1\subset U_2$. On the other hand, since γ_1 is a connected component of $\gamma_{12}\backslash B\big(p,\frac{\epsilon}{2}\big)$, there exist disjoint open sets V_1 and V_2 such that $\gamma_1\subset V_1$ and $\big(\gamma_{12}\backslash B\big(p,\frac{\epsilon}{2}\big)\big)\backslash\gamma_1\subset V_2$ ($\big(\gamma_{12}\backslash B\big(p,\frac{\epsilon}{2}\big)\big)\backslash\gamma_1$ might be empty, but it does not matter). Then we have $\gamma_1\subset V_1\cap U_1$ and $\gamma_{12}\backslash\gamma_1\subset\big[\big(\gamma_{12}\backslash B\big(p,\frac{\epsilon}{2}\big)\big)\backslash\gamma_1\big]\cup\big[\gamma_{12}\cap B\big(p,\frac{\epsilon}{2}\big)\big]\subset V_2\cup U_2$. But we already know that $a_2\in\gamma_{12}\backslash\gamma_1\neq\emptyset$, hence the above contradicts the fact that γ_{12} is connected.

Thus $\gamma_1\cap\partial B\big(p,\frac{\epsilon}{2}\big)\neq\emptyset$. Let $p_1\in\gamma_1\cap\partial B\big(p,\frac{\epsilon}{2}\big)$, then $[p,p_1]\cup\gamma_1$ is a connected set that contains p and a_1. As a result, $\mathcal{H}^1([p,p_1]\cup\gamma_1)\geq\mathcal{H}^1([p,a_1])$, and hence

$$\mathcal{H}^1(\gamma_1)\geq\mathcal{H}^1([p,a_1])-\mathcal{H}^1([p,p_1])=\mathcal{H}^1([p,a_1])-\frac{\epsilon}{2}. \tag{5.5}$$

For a_2, we have two cases:

If $a_2\in\gamma_{12}\backslash B\big(p,\frac{\epsilon}{2}\big)$ as well, set γ_2 be the connected component of $\gamma_{12}\backslash B\big(p,\frac{\epsilon}{2}\big)$ that contains a_2. Then $\gamma_1\cap\gamma_2=\emptyset$ and the exact same argument as above gives

$$\mathcal{H}^1(\gamma_2)\geq\mathcal{H}^1([p,a_2])-\frac{\epsilon}{2}. \tag{5.6}$$

If $a_2\in\gamma_{12}\cap B\big(p,\frac{\epsilon}{2}\big)$, then let $\gamma_2=\gamma_{12}\cap B\big(p,\frac{\epsilon}{2}\big)\neq\emptyset$. Then $\gamma_1\cap\gamma_2=\emptyset$, and (5.6) holds automatically, because $\mathcal{H}^1([p,a_2])-\frac{\epsilon}{2}<0$.

Now in both cases, γ_i, $i=1,2$, are disjoint parts of γ_{12}, and (5.5) and (5.6) hold. Hence we have

$$\mathcal{H}^1(\gamma_{12})\geq\mathcal{H}^1(\gamma_1)+\mathcal{H}^1(\gamma_2)\geq\mathcal{H}^1([p,a_1])+\mathcal{H}^1([p,a_2])-\epsilon. \tag{5.7}$$

Combined with (5.4), this yields

$$\mathcal{H}^1(F_0)\geq\sum_{i=1}^{3}\mathcal{H}^1([p,a_i])-\epsilon. \tag{5.8}$$

Obviously, the point p belongs to B. And it is well known that the quantity $\sum_{i=1}^3\mathcal{H}^1([p,a_i])$ attains its minimum if and only if p is the Fermat point of the triangle $\Delta_{a_1a_2a_3}$, which is just the origin 0. In this case,

$$\sum_{i=1}^{3}\mathcal{H}^1([0,a_i])=\mathcal{H}^1(Y_1\cap\bar{B}). \tag{5.9}$$

Together with (5.8) we have

$$\begin{aligned}\mathcal{H}^1(E) \geq \mathcal{H}^1(F_0) &\geq \sum_{i=1}^{3} \mathcal{H}^1([p, a_i]) - \epsilon \\ &\geq \sum_{i=1}^{3} \mathcal{H}^1([0, a_i]) - \epsilon = \mathcal{H}^1(Y_1 \cap \bar{B}) - \epsilon,\end{aligned} \tag{5.10}$$

where the third inequality only holds when $p = 0$. Since this is true for arbitrary $\epsilon \in (0, ||a_1 - a_2||)$, we have that (5.2) holds, and we have equality if and only if $p = 0$. This leads to the conclusion of Proposition 5.4. □

Now let us return to the proof of Theorem 5.2. Let F be a reduced G-topological competitor of dimension 2 of Y with respect to D such that

$$\mathcal{H}^2(F \cap D) = \mathcal{H}^2(Y \cap D), \tag{5.11}$$

we would like to show that $F = Y$.

By Lemma 5.3, we know that F_t connects the three points a_i^t, $1 \leq i \leq 3$. Then Proposition 5.4 tells that

$$\mathcal{H}^1(F_t \cap D_t) \geq \mathcal{H}^1(Y_t \cap D_t). \tag{5.12}$$

We apply the coarea formula (cf. [**9**, 3.2.22]) to the Lipschitz function f and the set $F \cap D$, and get

$$\mathcal{H}^2(F \cap D) \geq \int_{-1}^{1} \mathcal{H}^1(F_t \cap D_t) \geq \int_{-1}^{1} \mathcal{H}^1(Y_t \cap D_t) = \mathcal{H}^2(Y \cap D). \tag{5.13}$$

Then (5.11) tells us that

$$\mathcal{H}^1(F_t \cap D_t) = \mathcal{H}^1(Y_t \cap D_t) \text{ for a.e. } t \in (0, 1), \tag{5.14}$$

and hence

$$F_t \cap D_t = Y_t \cap D_t \text{ for a.e. } t \in (0, 1) \tag{5.15}$$

by Proposition 5.4. Hence we know that $F \cap D = Y \cap D$ modulo $\mathcal{H}^2$-null sets. But F is reduced, hence $F \cap D = Y \cap D$. Hence Y is G-topologically unique in D, and thus Y is G-topologically unique in $\mathbb{R}^3$ (by Proposition 3.2), and therefore also in $\mathbb{R}^n$ (by Proposition 3.4).

By Proposition 3.5, $\mathbb{Y}$ sets are also Almgren unique in $\mathbb{R}^n$. □

Remark 5.5. It is also possible to prove Theorem 5.2 by paired calibration (cf. [**13**] and [**3**]). In fact, we will use this method to prove the uniqueness for $\mathbb{T}$ sets in $\mathbb{R}^3$ in the next subsection, and interested readers can easily find a similar proof for $\mathbb{Y}$ sets. The proof in this section is more elementary in some sense, mainly using elementary geometry.

5.3. The $\mathbb{T}$ sets.

Theorem 5.6. *Any 2-dimensional $\mathbb{T}$ set is Almgren and ($\mathbb{Z}$-)topologically unique in $\mathbb{R}^n$ for all $n \geq 3$.*

Proof: By Propositions 3.4 and 3.5, it is enough to prove that $\mathbb{T}$ sets are topologically unique in $\mathbb{R}^3$.

Let T be a $\mathbb{T}$ set centered at the origin in $\mathbb{R}^3$. That is, T is the cone over the 1-skeleton of a regular tetrahedron C centered at the origin and inscribed in the closed unit ball B.

By Proposition 3.2, to prove that T is topologically unique in $\mathbb{R}^3$, it is enough to prove that T is topologically unique in B. So suppose that E is a reduced topological competitor of dimension 2 for T in B such that

$$\mathcal{H}^2(E \cap B) = \mathcal{H}^2(T \cap B). \tag{5.16}$$

By Remark 3.3 (5), we know that E is minimal, and thus is rectifiable. Hence for almost all $x \in E$, the tangent plane T_xE exists.

As mentioned in the last subsection, our proof will profit from the paired calibration, so let use first give necessary details.

Denote by a_i, $1 \leq i \leq 4$, the four singular points of $T \cap \partial B$. Let Ω_i, $1 \leq i \leq 4$, be the four equivalent connected spherical regions of $\partial B \backslash T$, Ω_i being on the opposite of a_i.

Since E is a topological competitor for T in B, we know that $\partial B \backslash E = \partial B \backslash T = \bigcup_{i=1}^4 \Omega_i$, and the four Ω_i live in different connected components of $B \backslash E$.

For $1 \leq i \leq 4$, let C_i be the connected component of $B \backslash E$ that contains Ω_i. Let $E_i = \partial C_i \backslash \partial B = \partial C_i \backslash \Omega_i$. Then we know that the four C_i, $1 \leq i \leq 4$, are disjoint and $E_i \subset E$. Also note that $E_i \cap \Omega_i \subset E \cap \partial B = T \cap \partial B$ is of $\mathcal{H}^2$ measure zero, hence we have the essentially disjoint unions

$$\partial C_i = E_i \cup \Omega_i, \quad 1 \leq i \leq 4. \tag{5.17}$$

Since C_i are disjoint regions in $\mathbb{R}^3$, we know that for almost all $x \in E$, they belong to at most two of the E_i's. So for $i \neq j$, let $E_{ij} = E_i \cap E_j$. Let E_{i0} denote $E_i \backslash \left(\bigcup_{j \neq i} E_i\right)$, the set of points x that belong only to E_i. Let $F = \bigcup_{1 \leq i \leq 4} E_i \subset E \cap B$, then we have the disjoint union

$$F = \left[\bigcup_{1 \leq i \leq 4} E_{i0}\right] \cup \left[\bigcup_{1 \leq i < j \leq 4} E_{ij}\right]. \tag{5.18}$$

For points $x \in \partial C_i$, let $n_i(x)$ denote the normal vector pointing into the region C_i. Note that since $\partial C_i \subset E \cup \partial B$, it is rectifiable, and hence

$n_i(x)$ is well defined for $\mathcal{H}^2$-a.e. $x \in \partial C_i$. Moreover, for $i \neq j$, we have $n_i(x) = -n_j(x)$ for $\mathcal{H}^2$-a.e. $x \in E_{ij}$.

Now by Stoke's formula, we have, for $1 \le i \le 4$,

$$\begin{aligned} 0 &= \int_{\partial C_i} \langle a_i, n_i(x)\rangle \, d\mathcal{H}^2(x) \\ &= \int_{E_i} \langle a_i, n_i(x)\rangle \, d\mathcal{H}^2(x) + \int_{\Omega_i} \langle a_i, n_i(x)\rangle \, d\mathcal{H}^2(x), \end{aligned} \tag{5.19}$$

and hence

$$\int_{E_i} \langle -a_i, n_i(x)\rangle \, d\mathcal{H}^2(x) = \int_{\Omega_i} \langle a_i, n_i(x)\rangle \, d\mathcal{H}^2(x) = \mathcal{H}^2(\pi_i(\Omega_i)), \tag{5.20}$$

where π_i is the orthogonal projection from $\mathbb{R}^3$ to the plane orthogonal to a_i, $1 \le i \le 4$. We sum over i, and get

$$\sum_{1\le i\le 4} \int_{E_i} \langle -a_i, n_i(x)\rangle \, d\mathcal{H}^2(x) = \sum_{1\le i\le 4} \mathcal{H}^2(\pi_i(\Omega_i)). \tag{5.21}$$

For the left-hand-side, by the disjoint union (5.18), we have

(5.22)

$$\begin{aligned} &\sum_{1\le i\le 4} \int_{E_i} \langle -a_i, n_i(x)\rangle \, d\mathcal{H}^2(x) \\ &\quad = \sum_{1\le i\le 4} \Big[\int_{E_{i0}} \langle -a_i, n_i(x)\rangle \, d\mathcal{H}^2(x) + \Big(\sum_{i\neq j} \int_{E_{ij}} \langle -a_i, n_i(x)\rangle \, d\mathcal{H}^2(x) \Big) \Big] \\ &\quad = \sum_{1\le i\le 4} \int_{E_{i0}} \langle -a_i, n_i(x)\rangle \, d\mathcal{H}^2(x) \\ &\qquad + \sum_{1\le i<j\le 4} \int_{E_{ij}} \big(\langle -a_i, n_i(x)\rangle + \langle -a_j, n_j(x)\rangle \big) \, d\mathcal{H}^2(x) \\ &\quad = \sum_{1\le i\le 4} \int_{E_{i0}} \langle -a_i, n_i(x)\rangle \, d\mathcal{H}^2(x) + \sum_{1\le i<j\le 4} \int_{E_{ij}} \langle n_j(x), a_i - a_j\rangle \, d\mathcal{H}^2(x) \\ &\quad \le \sum_{1\le i\le 4} \int_{E_{i0}} ||a_i|| \, d\mathcal{H}^2(x) + \sum_{1\le i<j\le 4} \int_{E_{ij}} ||a_i - a_j|| \, d\mathcal{H}^2(x) \\ &\quad = \sum_{1\le i\le 4} |a_i| \mathcal{H}^2(E_{i0}) + \sum_{1\le i<j\le 4} ||a_i - a_j|| \mathcal{H}^2(E_{ij}). \end{aligned}$$

Note that $||a_j||=1$, $1\le j\le 4$, and $||a_i-a_j||=\frac{2\sqrt2}{\sqrt3}$, hence

$$\begin{aligned}\sum_{j=1}^4\int_{E_j}\langle -a_i,n_i(x)\rangle\, d\mathcal{H}^2(x) &\le \sum_{1\le i\le 4}\mathcal{H}^2(E_{i0})+\sum_{1\le i<j\le 4}\frac{2\sqrt2}{\sqrt3}\mathcal{H}^2(E_{ij})\\ &\le \frac{2\sqrt2}{\sqrt3}\Big[\sum_{1\le i\le 4}\mathcal{H}^2(E_{i0})+\sum_{1\le i<j\le 4}\mathcal{H}^2(E_{ij})\Big]\\ &= \frac{2\sqrt2}{\sqrt3}\mathcal{H}^2(F)\le \frac{2\sqrt2}{\sqrt3}\mathcal{H}^2(E\cap B),\end{aligned}\tag{5.23}$$

where the second last equality is again because of the disjoint union (5.18).

As a result, we have

$$\mathcal{H}^2(E\cap B)\ge \frac{\sqrt3}{2\sqrt2}\sum_{1\le i\le 4}\mathcal{H}^2(\pi_i(\Omega_i)).\tag{5.24}$$

On the other hand, either by chasing the condition of equality for the inequalities of (5.22) and (5.23) (since T is a topological competitor of dimension 2 for itself), or by a direct calculation, it is easy to see that

$$\mathcal{H}^2(T\cap B)=\frac{\sqrt3}{2\sqrt2}\sum_{1\le i\le 4}\mathcal{H}^2(\pi_i(\Omega_i)).\tag{5.25}$$

By hypothesis (5.16), we know that for the set E, equality in (5.24) holds, and hence all the inequalities in (5.22) and (5.23) are equalities, which implies, in particular, that

$$\begin{gathered}\text{For almost all } x\in E_{ij},\ \text{we have } T_xE_{ij}\perp v_i-v_j.\\ \text{Denote by } P_{ij} \text{ the plane perpendicular to } v_i-v_j.\\ \text{Then for almost all } x\in E_{ij},\ \text{we have } T_xE=T_xE_{ij}=P_{ij};\end{gathered}\tag{5.26}$$

$$\text{For all } j,\ \mathcal{H}^2(E_{j0})=0;\tag{5.27}$$

$$\mathcal{H}^2\Big(E\cap B\backslash\bigcup_{j=1}^4 E_j\Big)=0.\tag{5.28}$$

Now since E is minimal, if $x\in E_P\cap B^\circ$ is a regular point of E, then by Theorem 2.25, there exists $r=r(x)>0$ such that in $B(x,r)$, E is the graph of a C^1 function from T_xE to $T_xE^\perp$. Hence for all $y\in E\cap B(x,r)$, the tangent plane T_yE exists, and the map $f\colon E\cap B(x,r)\to G(3,2)\colon y\mapsto T_yE$ is continuous. But by (5.26), we have only six choices (which are isolated points in $G(3,2)$) for T_yE, hence f is constant, and $T_yE = T_xE$ for all $y \in E \cap B(x,r)$. As a result,

$$E \cap B(x,r) = (T_xE + x) \cap B(x,r) \tag{5.29}$$

is a disk parallel to one of the P_{ij}.

Still by the C^1-regularity Theorem 2.25, the set $E_P \cap B^\circ$ is a C^1 manifold, and is open in E. Thus we deduce that

(5.30) Each connected component of $E_P \cap B^\circ$ is part of a plane that is parallel to one of the P_{ij}.

Let us look at E_Y. First, $E_Y \neq \emptyset$: otherwise, by Corollary 2.28 (ii), $E \cap B^\circ = E_P \cap B^\circ$, and hence is a union of planes. But $E \cap \partial B$ does not coincide with any union of planes.

Take any $x \in E_Y$, then by the C^1-regularity around $\mathbb{Y}$ points (Theorem 2.25 and Remark 2.26), there exists $r = r(x) > 0$ such that in $B(x,r)$, E is the image of a C^1 diffeomorphism φ of a $\mathbb{Y}$ set Y, and Y is tangent to E at x. Denote by L_Y the spine of Y and by R_i, $1 \leq i \leq 3$, the three open half planes of Y. Then $\varphi(R_i)$, $1 \leq i \leq 3$, are connected subsets E_P, hence each of them is a part of a plane parallel to one of the P_{ij}, $1 \leq i < j \leq 4$. As a consequence, $\varphi(L_Y) \cap B(x,r)$ is an open segment passing through x and parallel to one of the spines D_j, $1 \leq j \leq 4$, of T. Here D_j is the intersection of the three P_{ij}, $i \neq j$.

As a result, $E_Y \cap B^\circ$ is a union of open segments $I_1, I_2, \ldots$, each of which is parallel to one of the D_j, $1 \leq j \leq 4$, and every endpoint is either a point on the boundary ∂B, or a point of type $\mathbb{T}$. Moreover,

(5.31) For each $x \in E_Y$ such that $T_xE_Y = D_j$, there exists $r > 0$ such that, in $B(x,r)$, E is a $\mathbb{Y}$ set whose spine is $x + D_j$.

Next, since we are in dimension 3, the only other possible type of singular point is of type $\mathbb{T}$. So we are going to discuss two cases: when there exists a $\mathbb{T}$ point, or there are no $\mathbb{T}$ points.

Case 1: There exists a point $x \in E_T$.

Lemma 5.7. *If there exists a point $x \in E_T$, then $T \cap B^\circ = E$.*

Proof: By the same argument as above, and by Theorem 2.25 and Remark 2.26, the unique blow-up limit C_xE of E at x must be the set T, and there exists $r > 0$ such that in $B(x,r)$, E coincides with $T + x$. As a result, for each segment I_i, at least one of its endpoints is in the unit sphere, because two parallel $\mathbb{T}$ sets cannot be connected by a $\mathbb{Y}$ segment.

Hence all the segments I_i touch the boundary ∂B. That is,

$$L_i\backslash\{\{x\}\cup\partial B)\subset E_Y. \tag{5.32}$$

Denote by L_i, $1\le i\le 4$, the four spines of $T+x$. Then $L_i\cap B^\circ\subset E_Y$, because $L_i\cap B(x,r)$ is part of some $I_j\subset E_Y$, which already has an endpoint x that does not belong to ∂B, hence the other endpoint must lie in ∂B, which yields $I_j=L_i\cap B^\circ$.

Now we take a one parameter family of open balls B_s with radii $r\le s\le 1$, with $B_r=B(x,r)$, $B_1=B^\circ$, such that

(i) $B_s\subsetneqq B_{s'}$ for all $s<s'$;

(ii) $\bigcap_{1>t>s}B_t=\bar{B}_s$ and $\bigcup_{t<s}B_t=B_s$ for all $r\le s\le 1$.

Set $R=\inf\{s>r,\ (T+x)\cap B_s\neq E\}$. We claim that $R=1$.

Suppose this is not true. By definition of B_s, we know that the four spines and the six faces of $T+x$ are never tangent to ∂B_s for any $r<s<1$. Then we know that $\partial B_R\cap(T+x)\subset E_P\cup E_Y$. In fact, if y belongs to one of the L_i, then by (5.32), $y\in L_i\cap\partial B_s\subset E_Y$. Otherwise, suppose y does not lie in the four L_i, $1\le i\le 4$. Then y belongs to $x+P_{ij}$ for some $i\neq j$. As a result, for any $t>0$ small, we know that $E\cap B(y,t)\cap B_R=(x+P_{ij})\cap B(y,t)\cap B_R$. Note that the set $(x+P_{ij})\cap B(y,t)\cap B_R$ is almost a half disk when t is sufficiently small, hence in particular $E\cap B(y,t)$ cannot coincide with a $\mathbb{Y}$ set or a $\mathbb{T}$ set, and thus $y\in E_P$.

If $y\in E_P$, then $y\in x+P_{ij}$ for some $i\neq j$. Then $T_yE=P_{ij}$. By (5.30), and the fact that $R<1$, there exists $r_y>0$ such that $B(y,r_y)\subset B^\circ$ and $E\cap B(y,r_y)=(P_{ij}+y)\cap B(y,r_y)$. In other words,

(5.33) there exists $r_y>0$ such that E coincides with $T+x$ in $B(y,r_y)$.

If y is a $\mathbb{Y}$ point, then it lies in one of the L_i. By the same argument as above, using (5.31), we also have (5.33).

Thus (5.33) holds for all $y\in\partial B_R\cap(T+x)$. Since $\partial B_R\cap(T+x)$ is compact, we get an $r>0$ such that $E\cap B(B_R,r)=(T+x)\cap B(B_R,r)$. By the continuous condition (ii) for the family B_s, there exists $R'\in(R,1)$ such that $B_{R'}\subset B(B_R,r)$. As a consequence, $E\cap B_{R'}=(T+x)\cap B_{R'}$. This contradicts the definition of R.

Hence $R=1$, and by definition of R, we have $(T+x)\cap B^\circ=E\cap B^\circ$. Since $E\cap\partial B=T\cap\partial B$ and E is closed and reduced, x must be the origin. Thus we get the conclusion of Lemma 5.7. □

Case 2: $E_T=\emptyset$. In this case, the same kind of argument as in Lemma 5.7 gives the following:

Lemma 5.8. *Let x be a $\mathbb{Y}$ point in E and $T_xE_Y=D_j$. Denote by Y_j the Y set whose spine is D_j and whose three half planes lie in P_{ij}, $i\neq j$. Then $(Y_j+x)\cap B=E$.*

But this is impossible, because $E \cap \partial B = T \cap \partial B$, which does not contain $(Y_j + x) \cap \partial B$ for any x and j.

Hence we have $E \cap \bar{B} = T \cap \bar{B}$, and thus T is topologically unique in B. We thus get Theorem 5.6. $\square$

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Primera versió rebuda el 5 d'abril de 2019,
darrera versió rebuda el 30 de març de 2020.