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ITERATION OF FUNCTIONS AND CONTRACTIBILITY OF ACYCLIC 2-COMPLEXES

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Abstract: We show that there can be no algorithm to decide whether infinite recursively described acyclic aspherical 2-complexes are contractible. We construct such a complex that is contractible if and only if the Collatz conjecture holds.

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1. Introduction

The existence of an algorithm to determine which finite 2-dimensional simplicial complexes are contractible is a well-known open problem. There are good algorithms to compute the homology of a finite complex, and so the problem quickly reduces to the case of *acyclic* complexes, i.e., those having the same homology as a point. This problem can be stated as a problem about finite presentations of groups: is there an algorithm to decide which finite balanced presentations of perfect groups are trivial? The problem was stated in this form and attributed to Magnus in the first edition of the Kourovka notebook [**5**, 1.12], and appears as problem (FP1) in [**1**]. A presentation is *balanced* if it has the same numbers of generators and relators. The conditions that the group be perfect and that the presentation be balanced are equivalent to the corresponding 2-complex being acyclic.

If a finite presentation presents the trivial group, then a systematic search will eventually find a proof of this fact. Thus there is a partial algorithm that will verify that a finite simplicial complex (of any dimension) is contractible, but that will fail to halt in general.

In general, there is no algorithm to decide whether a finite presentation describes the trivial group. This implies that there can be no algorithm to decide whether a finite 2-dimensional simplicial complex is simply-connected. However, this does not imply that there can be no

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algorithm to decide whether a finite 2-complex is contractible, because all known families of problematic group presentations have many more relators than generators. The corresponding presentation 2-complexes cannot be contractible because they have non-trivial second homology.

We have nothing to say about this well-known problem, but instead we consider an infinite analogue. In contrast to the finite case, there is no algorithm for computing the homology of recursively described infinite complexes. For example, it has been known since the work of Collatz in the 1930's that it is difficult to decide such questions as whether a recursively described graph is connected [**7**]. (We will also justify this assertion in Theorem 11 below.) For this reason we consider only acyclic complexes. Here is our main result.

Theorem 1. *There is no algorithm to decide whether an infinite, recursively described, aspherical, acyclic presentation* 2*-complex is contractible.*

Moreover, there is no partial algorithm to find all the contractible complexes in this class, and there is no partial algorithm to find all the non-contractible complexes.

Usually decidability results in group theory rely on the existence of a non-recursive, recursively enumerable set. Instead we rely on the dynamics of functions on the set $\mathbb{N}_+$ of strictly positive integers. We associate a 2-complex $P(f)$ to each $f\colon \mathbb{N}_+ \to \mathbb{N}_+$ whose contractibility is controlled by the orbits of f. For such a function f, define a group presentation $\mathcal{P}(f)$ in which both the generators and relators are indexed by $\mathbb{N}_+$:

$$\mathcal{P}(f) := \langle a_i,\, i \in \mathbb{N}_+ : a_{f(i)}^{-1} a_i a_{f(i)} = a_i^2 \rangle.$$

Now let $P(f)$ be the presentation 2-complex associated to $\mathcal{P}(f)$, so that the 1- and 2-cells of $P(f)$ are indexed by $\mathbb{N}_+$. Recall also that a *forward orbit* for f is a subset of $\mathbb{N}_+$ of the form $\{i, f(i), f^2(i), \dots\}$ for some $i \in \mathbb{N}_+$.

Lemma 2. *For each f, the* 2*-complex $P(f)$ is both acyclic and aspherical. The following are equivalent:*

- *$P(f)$ is contractible.*
- *$\mathcal{P}(f)$ presents the trivial group.*
- *Every forward orbit of f is eventually periodic of period at most* 3.

There is a striking corollary concerning the Collatz function. Recall that the Collatz or hailstone function $C\colon \mathbb{N}_+ \to \mathbb{N}_+$ is defined by

$$C(n) = \begin{cases} 3n+1 & \text{for } n \text{ odd,} \\ n/2 & \text{for } n \text{ even.} \end{cases}$$

The *Collatz conjecture* states that every forward orbit of C contains 1.

Corollary 3. *$P(C)$ is contractible if and only if the Collatz conjecture holds.*

This corollary follows easily from Lemma 2. The proof of Theorem 1 depends also on a result of Kurtz and Simon concerning the decidability of properties of functions [**6**].

2. Proofs

The presentation 2-complex associated to a group presentation is a CW-complex with one 0-cell, with 1-cells in bijective correspondence with the generators, and 2-cells in bijective correspondence with the relators. We shall assume that each relator is a cyclically reduced word in the generators. The 1-skeleton of the 2-complex is a rose, whose fundamental group is naturally identified with the free group on the set of generators in the given presentation. An element of this group describes a based homotopy class of maps from the circle to the 1-skeleton. The relator corresponding to a 2-cell is used in this way to describe its attaching map. For details see [**3**, p. 50]. A presentation 2-complex is *acyclic* if it has the same homology as a point, and is *aspherical* if its universal covering space is contractible, or equivalently if it is an Eilenberg–Mac Lane space for the group presented. Some authors use the term 'aspherical presentation' for a more general situation that arises when some of the relators are proper powers [**8**]; this will not concern us.

There is an algorithm to pass from presentation 2-complexes as described above to homotopy equivalent simplicial complexes. Each 2-cell corresponding to a relator of length n should be viewed as an n-gon; the second barycentric subdivision of the polygonal cell complex obtained in this way is a simplicial complex. Each petal of the original 1-skeleton rose is triangulated as the boundary of a square, and the 2-cell corresponding to a relator of length n is built from $12n$ triangles.

Conversely, if K is a connected 2-dimensional simplicial complex and T is a maximal tree in K, the quotient space K/T is naturally a CW-complex with one 0-cell, and so it may be viewed as a presentation complex, with generators corresponding to the edges of $K - T$ and relators corresponding to the 2-simplices of K. This process too is algorithmic, given K and T, but when K is infinite there may be no algorithm to find a maximal tree T.

The discussion in the two paragraphs above leads to the following proposition.

Proposition 4. *The existence of an algorithm to decide contractibility for finite presentation* 2*-complexes is equivalent to the existence of an algorithm to decide contractibility for finite* 2*-dimensional simplicial complexes.*

Proof: Follows from the discussion above, since there is an algorithm to find a maximal tree inside a finite connected 2-dimensional simplicial complex. $\square$

The discussion above also leads to a corollary to our main theorem.

Corollary 5. *For the class of recursively defined acyclic aspherical* 2*-dimensional simplicial complexes, there is no partial algorithm to find all contractible complexes in the class, and no partial algorithm to find all non-contractible complexes.*

Proof: Since there is an algorithmic way to pass from a recursively described presentation 2-complex to a recursively defined 2-dimensional simplicial complex, a partial algorithm of the type mentioned in the statement of the corollary would contradict Theorem 1. $\square$

Two infinite families of finite group presentations will play a role in our proofs, the families $\mathcal{B}(n)$ and $\mathcal{H}(n)$ given below. As above, we use the notation $B(n)$ and $H(n)$ for the corresponding presentation 2-complexes.

$$\mathcal{B}(n) := \langle a_1, \ldots, a_{n+1} : a_i^{a_{i+1}} = a_i^2 \rangle,$$

$$\mathcal{H}(n) := \langle a_i,\, i \in \mathbb{Z}/n : a_i^{a_{i+1}} = a_i^2 \rangle.$$

These presentations first appeared as steps in Higman's construction of an infinite finitely generated simple group [**4**]. To establish properties of $B(n)$, $H(n)$, and of the complexes $P(f)$, we will use two well-known propositions.

Proposition 6. *Let X, Y, and Z be Eilenberg–Mac Lane spaces for the groups H, K, and L respectively, and let $i\colon Z \to X$ and $j\colon Z \to Y$ be based maps such that the induced maps on fundamental groups are injective: $i_*\colon L \to H$ and $j_*\colon L \to K$.*

(1) *The double mapping cylinder or homotopy colimit*

$$(X \sqcup Z \times I \sqcup Y)/(z,0) = i(z),\, (z,1) = j(z)$$

is an Eilenberg–Mac Lane space for the free product with amalgamation $H *_L K = \langle H, K : i_*(l) = j_*(l),\, l \in L\rangle$.

(2) *If moreover $X = Y$ so that $K = H$, then the homotopy colimit of the smaller diagram with two arrows and just two spaces, $(X \sqcup Z \times I)/(z,0) = i(z)$, $(z,1) = j(z)$, is an Eilenberg–Mac Lane space for the HNN-extension $H{*_L} = \langle H, t : ti_*(l)t^{-1} = j_*(l),\, l \in L\rangle$.*

Proof: Both assertions are special cases of a theorem concerning arbitrary graphs of groups; see for example [**3**, 1.B.11], although note that the directed graph used in [**3**, 1.B] is the barycentric subdivision (with edges oriented from the larger edge midpoint to the vertex) of the graph usually used to index a graph of groups. □

Proposition 7. *With hypotheses as in part (1) of Proposition 6, if in addition the maps $i\colon Z \to X$ and $j\colon Z \to Y$ are isomorphisms from Z to subcomplexes of X and Y, then the coproduct $X \sqcup Y/i(z) = j(z)$ is also an Eilenberg–Mac Lane space for $H *_L K$.*

Proof: The inclusion of a subcomplex in a CW-complex is a cofibration, and hence the pair $(X, i(Z))$ is homotopy equivalent to the pair (M_i, Z), where M_i denotes the mapping cylinder of $i\colon Z \to X$ and similarly $(X, j(Z))$ is homotopy equivalent to the pair (M_j, Z). The claim follows. □

Corollary 8. *The presentation 2-complex $B(n)$ for $\mathcal{B}(n)$ is aspherical, and each of the generators a_i represents an element of its fundamental group, $\pi_1(B(n))$, of infinite order. The presentation obtained by adding the relation $a_{n+1} = 1$ to $\mathcal{B}(n)$ presents the trivial group. The subgroup of $\pi_1(B(n))$ generated by a_1 and a_{n+1} is free of rank two provided that $n \geq 2$.*

Proof: The group presented by $\mathcal{B}(1)$ is the Baumslag–Solitar group $BS(1,2)$, which is an HNN-extension of the infinite cyclic group $\langle a_1 \rangle$ with stable letter a_2. The asphericity claim for $n = 1$ follows from part (2) of Proposition 6. Given the relation $a_2 = 1$, the relator in $\mathcal{B}(1)$ reduces to $a_1 = a_1^2$, which immediately implies $a_1 = 1$. This completes the proof when $n = 1$.

All of the claims except the final one are proved by induction using Proposition 7, since a complex isomorphic to $B(n+1)$ can be obtained by identifying the circle labelled a_{n+1} in $B(n)$ with the circle labelled a_1 in a second copy of $B(1)$.

For the final claim, note that the cyclic groups $\langle a_1 \rangle$ and $\langle a_{n+1} \rangle$ have trivial intersection inside $\pi_1(B(n))$ for each $n \geq 1$. It follows that $\langle a_1, a_{n+2} \rangle$ is a free group by applying the Normal Form Theorem for free products with amalgamation to the given decomposition of $\pi_1(B(n+1))$ [**8**, IV.2.6]. □

Corollary 9. *The presentation 2-complex $H(n)$ for $\mathcal{H}(n)$ is aspherical for all n, and is contractible if and only if $n \leq 3$. For $n \geq 4$, each a_i generates an infinite cyclic subgroup of $\pi_1(H(n))$.*

Proof: It can be checked readily with a computer algebra package that $\mathcal{H}(n)$ presents the trivial group for $n \leq 3$, or see [**4**, pp. 63–64] for a direct proof. For $n \geq 4$, the complex $H(n)$ can be obtained from a copy of $B(n-2)$ and a copy of $B(2)$ by identifying the 2-petalled rose labelled by a_1 and a_{n-1} inside $B(n-2)$ with the 2-petalled rose labelled by a_3 and a_1 respectively inside $B(2)$. The remaining claims concerning $H(n)$ now follow from Proposition 7 and Corollary 8. □

We are now ready to start the proof of Lemma 2.

Proof: It is immediate that each $P(f)$ is acyclic, and so we concentrate on the homotopy groups. We view the 1- and 2-cells of $P(f)$ as being indexed by $\mathbb{N}_+$, so that the generator a_i corresponds to the loop around the ith edge, and the ith 2-cell is the 2-cell corresponding to the relation $a_{f(i)}^{-1}a_i a_{f(i)} = a_i^2$.

First, consider the case of the function $f(i) := i+1$. In this case, the 2-cells indexed by $1,\ldots,n$ and the 1-cells indexed by $1,\ldots,n+1$ form a subcomplex isomorphic to $B(n)$, and the whole complex is the ascending union of these subcomplexes. It follows that in this case $P(f)$ is aspherical as claimed. To see that $P(f)$ is not contractible, let $r\colon \mathbb{N}_+ \to \mathbb{Z}/4$ be the map defined by $r(i) := [i] := i + 4\mathbb{Z}$. This induces a cellular map $P(f) \to H(4)$ which is surjective on fundamental groups. Since the subgroup $\langle a_{r(i)}\rangle$ is infinite, so is the subgroup $\langle a_i\rangle$ of the group presented by $\mathcal{P}(f)$.

Now we move on to the general case. We fix a function $f\colon \mathbb{N}_+ \to \mathbb{N}_+$, and so we write $\mathcal{P}$ instead of $\mathcal{P}(f)$ and P in place of $P(f)$, and prove the claim by induction on certain subcomplexes of P. The boundary of the 2-cell indexed by $i \in \mathbb{N}_+$ meets only the 1-cells a_i, $a_{f(i)}$, and the 0-cell. Hence if $S \subseteq \mathbb{N}_+$ is any subset such that $f(S) \subseteq S$, the cells indexed by S (together with the 0-cell) form a subcomplex of P. Denote this subcomplex by $P(S)$. For each $i \in \mathbb{N}_+$, write $F(i)$ for the forward orbit of i: $F(i) := \{f^n(i) : n \geq 0\}$, and note that $f(F(i)) \subseteq F(i)$.

As an inductive hypothesis, suppose that we have an f-closed subset $S \subseteq \mathbb{N}_+$ so that $P(S)$ is aspherical, and that $P(S)$ satisfies two other properties analogous to those claimed for $P(f)$, as described below. Firstly, we suppose that $P(S)$ is contractible if and only if every forward orbit of $f|_S$ is eventually periodic of period at most 3. Secondly, we suppose that for each $i \in S$, the subgroup of $\pi_1(P(S))$ generated by a_i is either infinite or trivial, and is trivial iff the forward orbit $F(i)$ is eventually periodic of period at most 3.

Let i be the least element of $\mathbb{N}_+ - S$. There are two main cases to consider, depending on whether $F(i) \cap S$ is empty or non-empty.

If $F(i) \cap S \neq \emptyset$, let n be minimal so that $f^n(i) \in S$. In this case, $P(F(i) \cup S)$ is isomorphic to the union of $P(S)$ and the subspace X consisting of the 0-cell, the 1-cells indexed by $\{i, f(i), \ldots, f^n(i)\}$, and the 2-cells indexed by the set $\{i, f(i), \ldots, f^{n-1}(i)\}$. The subspace X is isomorphic to $B(n)$. The intersection of $P(S)$ and X is the circle consisting of the 0-cell and the 1-cell indexed by $f^n(i)$. If the forward orbit of i is eventually periodic of period at most 3, then so is the forward orbit of $f^n(i)$, and so the circle indexed by $f^n(i)$ is contractible in $P(S)$, and attaching a copy of $B(n)$ to this circle does not change the homotopy type, so $P(F(i) \cup S)$ is homotopy equivalent to $P(S)$. If on the other hand the forward orbit of i is infinite, or eventually periodic of period greater than 3, then the circle indexed by $f^n(i)$ represents an element of infinite order in the fundamental groups of both $P(S)$ and $X \cong B(n)$. The inductive hypothesis passes to $P(S \cup F(i)) = P(S) \cup_{f^n(i)} X$.

If $F(i) \cap S = \emptyset$, then $P(F(i) \cup S) = P(F(i)) \vee P(S)$, the 1-point union of $P(F(i))$ and $P(S)$, and $\pi_1(P(F(i) \cup S)$ is just the free product $\pi_1(P(F(i))) * \pi_1(P(S))$. Thus our inductive hypothesis will hold for $F(i) \cup S$ provided that we can show that it holds for $F(i)$. This splits into three further subcases. If $F(i)$ is infinite, then $P(F(i))$ is isomorphic to the complex discussed in the first paragraph, and the claims hold. If f is periodic of period n when restricted to $F(i)$, then $P(F(i))$ is isomorphic to the complex $\mathcal{H}(n)$, and the inductive hypotheses hold by Corollary 9. If neither of these cases holds, we take an intermediate step between S and $S \cup F(i)$. Pick the least $n > 0$ so that there exists $m > n$ with $f^m(i) = f^n(i)$, and let $j := f^n(i)$. We have that f is periodic on $F(j)$ and $F(j) \cap S = \emptyset$, so by the cases already covered we deduce that the inductive hypotheses hold for $S' := S \cup F(j)$. Note also that $F(i) \cap S' = F(j) \neq \emptyset$, so getting from S' to $S' \cup F(i) = S \cup F(i)$ reduces to the other main case considered in the previous paragraph. $\square$

Next we prove Corollary 3.

Proof: By Lemma 2, it suffices to show that any forward orbit of the Collatz function C that does not eventually join the periodic orbit $1, 4, 2, \ldots$ is either infinite or has eventual period greater than 3. So let $n_1, n_2, \ldots, n_l, \ldots$ be a periodic orbit of period l, and choose the starting point so that $n_1 < n_i$ for $2 \leq i \leq l$. This implies that n_1 is odd, and that $n_2 = C(n_1) = 3n_1 + 1$. Since $3n_1 + 1 > 2n_1$, we see that $C(n_2) > n_1$ and so $l > 2$. If $l = 3$, it must be that $C(C(n_2)) = n_1$, and hence $3n_1 + 1 = 4n_1$, which implies that $n_1 = 1$. $\square$

A generalized Collatz function is a function $g\colon \mathbb{N}_+ \to \mathbb{N}_+$ such that there exists an integer $m > 0$ and rationals a_i, b_i for $0 \leq i < m$ so that whenever x is congruent to i modulo m, $g(x) = a_i x + b_i$. In particular, the function C can be written in this way for $m = 2$, $a_0 = 1/2$, $b_0 = 0$, $a_1 = 3$, $b_1 = 1$. To prove Theorem 1, we quote a theorem of Kurtz and Simon [**6**], that strengthens a well known result due to Conway [**2, 7**]. They define GCP to be the problem of deciding, for each generalized Collatz function g, whether every forward orbit of g contains 1. They show [**6**, Theorem 3] that GCP is Π^0_2-complete; and hence in particular there can be no partial algorithm that can identify either the generalized Collatz functions that satisfy GCP or the ones that do not.

Given a function $f\colon \mathbb{N}_+ \to \mathbb{N}_+$, we define a new function $\widehat{f}\colon \mathbb{N}_+ \to \mathbb{N}_+$ as follows. Firstly, let $\phi\colon \mathbb{N}_+ \to \mathbb{N}_+ \times \mathbb{Z}/4$ be the function $\phi(n) = (\lfloor (n+3)/4 \rfloor, [n])$, where as before we use $[n]$ to denote the class $n + 4\mathbb{Z} \in \mathbb{Z}/4$, and $\lfloor q \rfloor$ is the greatest integer less than or equal to q. Note that ϕ is a bijection and that both ϕ and ϕ^{-1} are easily computable. Now define $\widetilde{f}\colon \mathbb{N}_+ \times \mathbb{Z}/4 \to \mathbb{N}_+ \times \mathbb{Z}/4$ by

$$\widetilde{f}(m, [i]) := \begin{cases} (f(m), [i+1]) & \text{for } m \neq 1, \\ (1, [i]) & \text{for } m = 1. \end{cases}$$

Finally, define $\widehat{f} := \phi^{-1} \circ \widetilde{f} \circ \phi$.

Lemma 10. *The following are equivalent, for any function $f\colon \mathbb{N}_+ \to \mathbb{N}_+$:*

- *Every forward orbit of f contains* 1.
- *Every forward orbit of $\widehat{f}$ is eventually periodic of period at most* 3.

Proof: Since ϕ is a bijection, the dynamics of $\widehat{f}$ and $\widetilde{f}$ are identical. It is clear that for any i, x, the forward orbit of $(x, [i])$ under $\widetilde{f}$ will be eventually constant (i.e., eventually periodic of period 1) if the forward orbit of x under f contains 1. Similarly, if the forward orbit of x under f does not contain 1, then for each $n \geq 0$ we see that $\widetilde{f}^n(x, [i]) = (f^n(x), [i+n])$, and so the forward orbit of $(x, [i])$ will be either infinite or eventually periodic of period divisible by 4. □

We are now ready to prove Theorem 1.

Proof: Let g be an arbitrary generalized Collatz function, and consider the question of whether $P(\widehat{g})$ is contractible. By Lemma 2, this is equivalent to the question of whether every forward orbit of $\widehat{g}$ is eventually periodic of period at most three. By Lemma 10, this is equivalent to the question of whether every forward orbit of g contains 1. But Kurtz and Simon showed that this question is Π^0_2-complete, which implies the claim. □

3. Closing remarks

Similar but simpler techniques can be used to prove undecidability results for the homology of infinite complexes, and we give two examples below.

Theorem 11. *There is no algorithm to decide whether a recursively described graph is connected. Moreover, there is no partial algorithm to find the connected ones, and no partial algorithm to find the disconnected ones.*

Proof: For a function $f\colon \mathbb{N}_+ \to \mathbb{N}_+$, define a graph $\Gamma(f)$ with vertex and edge set indexed by $\mathbb{N}_+$, where the vertices of the edge e_i are v_{i+1} and $v_{f(i+1)}$. This graph is connected if and only if every forward orbit of f contains 1.

By the Kurtz–Simon theorem, the question of whether the graph $\Gamma(g)$ is connected, for g any generalized Collatz function, is Π^0_2-complete, which implies the claim. □

A similar argument applies to the computation of the first homology of recursively described 2-complexes, because the group with presentation $\mathcal{Q}(f)$ given by

$$\mathcal{Q}(f) := \langle a_i,\, i \in \mathbb{N}_+ : a_1 = 1,\, a_{i+1} = a_{f(i+1)} \rangle$$

is free, and is trivial if and only if every forward orbit of f contains 1.

The top homology group (i.e., H_1 for graphs or H_2 for 2-complexes) is different however:

Proposition 12. *For each n, there is a partial algorithm that identifies the recursively described n-dimensional complexes with non-zero nth homology group.*

Proof: Fix an integer parameter $m > 0$. Now compute the boundaries of the first m of the n-cells, expressed as formal sums of $(n-1)$-cells. Use standard techniques of linear algebra to decide whether these m elements are linearly dependent; if so, then $H_n \neq 0$. If not, increase m and repeat. □

The functions that we have considered are of course far simpler than arbitrary recursive functions. If g is either a generalized Collatz function or $g = \widehat{f}$ for some generalized Collatz function f, then the sets $g^{-1}(i)$ are of bounded size and can be easily computed. From this one sees that the cellular cochain complexes for $P(g)$, $Q(g)$, and $\Gamma(g)$ are also recursively described, where $Q(g)$ denotes the presentation 2-complex for $\mathcal{Q}(g)$. The argument used in Proposition 12, when applied $H^0(\Gamma(g))$

for any g for which the cochain complex for $\Gamma(g)$ is recursively described, gives that there is a partial algorithm that will identify when $\Gamma(g)$ has a *finite* connected component.

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