Publ. Mat. **65** (2021), 319–333
DOI: 10.5565/PUBLMAT6512111

UNIFORM A PRIORI ESTIMATES FOR POSITIVE SOLUTIONS OF HIGHER ORDER LANE–EMDEN EQUATIONS IN $\mathbb{R}^n$

Wei Dai and Thomas Duyckaerts

Abstract: In this paper we study the existence of uniform a priori estimates for positive solutions to Navier problems of higher order Lane–Emden equations

$$(-\Delta)^m u(x) = u^p(x), \quad x \in \Omega, \tag{0.1}$$

for all large exponents p, where $\Omega \subset \mathbb{R}^n$ is a star-shaped or strictly convex bounded domain with C^{2m-2} boundary, $n \geq 4$, and $2 \leq m \leq \frac{n}{2}$. Our results extend those of previous authors for second order $m = 1$ to general higher order cases $m \geq 2$.

2010 Mathematics Subject Classification: Primary: 35B45; Secondary: 35J40, 35J91.

Key words: uniform a priori estimates, higher order Lane–Emden equations, Navier problems, positive solutions, blow up.

1. Introduction

In this paper we investigate the following higher order Lane–Emden equations in a bounded domain with Navier boundary conditions:

$$\begin{cases} (-\Delta)^m u(x) = u^p(x), \quad u(x) > 0, & x \in \Omega, \\ u(x) = (-\Delta)u(x) = \cdots = (-\Delta)^{m-1}u(x) \equiv 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p < +\infty$, $n \geq 2$, $1 \leq m \leq \frac{n}{2}$, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^{2m-2} boundary $\partial\Omega$. We assume the positive solutions u belong to $C^{2m}(\Omega) \cap C^{2m-2}(\overline{\Omega})$.

The Lane–Emden equations of type (1.1) have numerous important applications in conformal geometry and Sobolev inequalities. They also model many phenomena in mathematical physics and in astrophysics (see [**3, 18**]). We say equations (1.1) have critical order if $m = \frac{n}{2}$ and

W. Dai is supported by the NNSF of China (No. 11971049), the Fundamental Research Funds for the Central Universities, and the State Scholarship Fund of China (No. 201806025011). T. Duyckaerts is supported by the Institut Universitaire de France and the Labex MME-DII.

subcritical order if $m < \frac{n}{2}$. The nonlinear term in (1.1) is called critical if $p = p_c := \frac{n+2m}{n-2m}$ ($:= \infty$ if $m = \frac{n}{2}$) and subcritical if $1 < p < p_c$.

When $m = 1$, Ambrosetti and Rabinowitz ([**2**]) derived the existence of *least energy* positive solution to (1.1) for $1 < p < p_c$ via variational minimization methods. When $m \geq 2$, a priori estimates and existence results for positive solutions to (1.1) were derived by Sirakov in subcritical order cases [**25**]. For more literature on Liouville-type theorems, a priori estimates and existence results for solutions to (1.1), please refer to Chen, Fang, and Li [**5**], Dai, Peng, and Qin [**6**], Dai and Qin [**7**] (for $m < \frac{n}{2}$), Chen, Dai, and Qin [**4**], Dai and Qin [**8**] (for $m \geq \frac{n}{2}$). In [**7**] (for $m < \frac{n}{2}$) and [**8**] (for $m = \frac{n}{2}$), besides Liouville-type theorems, Dai and Qin established a priori estimates for classical solutions to generalized higher order equations (possibly sign-changing solutions) and existence of positive solutions to (1.1) for all $p \in (1, p_c)$. Moreover, the positive solution u to (1.1) derived in [**7, 8**] satisfies

$$\|u\|_{L^\infty(\overline{\Omega})} \geq \left(\frac{\sqrt{2n}}{\operatorname{diam}\Omega}\right)^{\frac{2m}{p-1}}. \tag{1.2}$$

The lower bounds (1.2) on the L^∞-norm of positive solutions u indicate that if $\operatorname{diam}\Omega < \sqrt{2n}$, then the L^∞-norm must blow up as $p \to 1^+$. For more general results and references related to boundary value problems for poly-harmonic equations, please refer to the book of Gazzola, Grunau, and Sweers [**14**].

1.1. The subcritical order cases $1 \leq m < \frac{n}{2}$. We first consider the subcritical order cases $1 \leq m < \frac{n}{2}$. When $m = 1$, using what are now known as Pohozaev identities (see also [**12, 20, 21, 22**]), Pohozaev ([**20**]) has shown that there are no positive solutions to (1.1) in the range $p_c < p < +\infty$ provided Ω is star-shaped. Han ([**16**]) and Rey ([**24**]) proved that the L^∞-norm of positive solutions of (1.1) with $m = 1$ blows up as $p \to {p_c}^-$. In addition, they have also obtained the precise asymptotic behaviour for the least-energy solutions. Di ([**11**]) established similar results as in [**16, 24**] for the bi-harmonic case $m = 2$ and strictly convex domain Ω.

Consider the following generalized higher order Navier problems:

$$\begin{cases} (-\Delta)^m u(x) = f(u(x)), \quad u(x) \geq 0, & x \in \Omega, \\ u(x) = (-\Delta)u(x) = \cdots = (-\Delta)^{m-1}u(x) \equiv 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $n \geq 2$, $1 \leq m < \frac{n}{2}$, the function $f\colon \overline{\mathbb{R}_+} \to \overline{\mathbb{R}_+}$ is continuous, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^{2m-2} boundary $\partial\Omega$. Van der Vorst ([**27**]) established Pohozaev-type identities for higher order

Navier problems (1.3) (see Lemma 3.9 in [**27**]). As a consequence of the Pohozaev-type identities, the author also deduced in [**27**] a Liouville theorem for higher order Navier problems (1.3). Suppose that the function f satisfies $n\int_0^t f(s)\,ds - \frac{n-2m}{2}f(t)t \leq 0$ and Ω is star-shaped, then the Navier problem (1.3) possesses no nontrivial nonnegative solutions (see Theorem 3.10 in [**27**]). In particular, if we take $f(t) := t^p$, then Theorem 3.10 in [**27**] implies immediately that there are no nontrivial nonnegative solutions for the Navier problem (1.1) in both the critical and super-critical cases $p_c := \frac{n+2m}{n-2m} \leq p < +\infty$ provided Ω is star-shaped. Liouville-type results for fractional and higher order Hénon–Hardy equations in balls with Dirichlet or Navier boundary conditions have been established in [**7**] by developing the method of scaling spheres. For Liouville theorems on higher order Dirichlet problems via Pohozaev-type variational identities, please also refer to [**12, 20, 21, 22**].

Since Theorem 3.10 in [**27**] indicates that there are no positive solutions to the Navier problem (1.1) in star-shaped domain Ω when $p = p_c$, we can infer from the existence of positive solutions for $p < p_c$ that the L^∞-norm of solutions to (1.1) in star-shaped domain Ω must blow up as $p \to {p_c}^-$. Otherwise, one could derive a positive solution in the critical case $p = p_c$ via compactness arguments. Therefore, we have the following immediate corollary of Theorem 3.10 in [**27**].

Corollary 1.1. *Assume Ω is a star-shaped domain with $\partial\Omega \in C^{2m-2}$. Then any sequence of solutions $\{u_{p_k}\}$ to the Navier problem* (1.1) *with $p = p_k \to p_c$ must blow up in L^∞-norm, that is,*

$$\|u_{p_k}\|_{L^\infty(\overline{\Omega})} \to +\infty \quad as \quad k \to \infty.$$

1.2. The critical order cases $m = \frac{n}{2}$ with $n \geq 2$ even. Next, we consider the critical order cases $m = \frac{n}{2}$ with $n \geq 2$ even.

In contrast with the subcritical order cases, when $n = 2$ and $m = 1$, Ren and Wei ([**23**]) showed that the least-energy solutions of (1.1) stay uniformly bounded as $p \to +\infty$. Subsequently, Kamburov and Sirakov ([**19**]) proved that positive solutions of (1.1) with $m = 1$ in a 2D smooth bounded domain Ω are uniformly bounded for all large exponents $p_0 \leq p < +\infty$. For asymptotic description of positive solutions to (1.1) in the case $m = 1$ and $n = 2$ as $p \to +\infty$, please refer to [**1, 9, 10**].

In this paper, by using the methods from Kamburov and Sirakov [**19**] of employing the Green's representation formula, we will establish uniform a priori estimates for positive solutions to the critical order Navier problem (1.1) (with general $m = \frac{n}{2}$ and $n \geq 4$ even) for all large exponents p in strictly convex domain Ω with C^{n-2} boundary $\partial\Omega$.

We have the following uniform a priori estimates for the critical order Navier problems (1.1).

Theorem 1.2. *Assume $n \geq 4$ is even, $m = \frac{n}{2}$, $\Omega \subset \mathbb{R}^n$ is a strictly convex domain with $\partial\Omega \in C^{n-2}$, and let $p_0 > 1$. There exists a constant C depending only on p_0, n, and Ω such that for all $p_0 \leq p < +\infty$, any solution $u_p \in C^n(\Omega) \cap C^{n-2}(\overline{\Omega})$ to the critical order problem* (1.1) *satisfies:*

$$\|u_p\|_{L^\infty(\overline{\Omega})} \leq C.$$

Remark 1.3. Under the strict convexity assumption on Ω, Theorem 1.2 extends the uniform a priori estimates derived in [**19, 23**] for the second order case $m = 1$ and $n = 2$ to the general critical order cases $m = \frac{n}{2}$ and $n \geq 4$ is even.

Remark 1.4. Being essentially different from the second order case $m = 1$ and $n = 2$, the information and estimates on $-\Delta u$ play a crucial role in the proof of Theorem 1.2; please see Lemma 2.1 and 2.3. More precisely, we proved in Lemma 2.3 the following crucial property:

$$\max_{\overline{\Omega}}(-\Delta)^k u \sim \frac{[\max_{\overline{\Omega}} u]^{\frac{2k}{n}p+(1-\frac{2k}{n})}}{p^{1-\frac{2k}{n}}} \tag{1.4}$$

for any $k = 1, \dots, \frac{n}{2} - 1$. In particular, from the proof of Lemma 2.3 (more precisely, see (2.34)), one has the following pointwise estimates at the maximum x_0 of u in $\overline{\Omega}$:

$$C_k'' \frac{(u(x_0))^{\frac{2k}{n}p+(1-\frac{2k}{n})}}{p^{1-\frac{2k}{n}}} \leq (-\Delta)^k u(x_0) \leq C_k' \frac{(u(x_0))^{\frac{2k}{n}p+(1-\frac{2k}{n})}}{p^{1-\frac{2k}{n}}} \tag{1.5}$$

for any $k = 1, \dots, \frac{n}{2} - 1$. For related pointwise inequality in $\mathbb{R}^n$, we refer to Fazly, Wei, and Xu [**13**].

2. Proof of Theorem 1.2

In this section we will prove Theorem 1.2 by using the methods from Kamburov and Sirakov [**19**] of employing the Green's representation formula.

In the following, we will use C to denote a general positive constant that may depend on n, p_0, and Ω, and whose value may differ from line to line. In all the proof, we assume $p \geq p_0$.

For $n \geq 4$ even and $m = \frac{n}{2}$, assume $u = u_p$ is a positive solution to the critical order Navier problem (1.1). By the maximum principle, we inductively have $(-\Delta)^k u_p > 0$ in Ω for any $k = 0, 1, \dots, \frac{n}{2} - 1$. Furthermore, we can prove the following lemma.

Lemma 2.1. *Assume $n \geq 4$ is even, $m = \frac{n}{2}$, Ω is a strictly convex domain with $\partial\Omega \in C^{n-2}$, and let $p_0 > 1$. There exist positive constants δ depending only on Ω, and C depending only on n, p_0, and Ω such that*

(i) *The maximum of the solution $u = u_p$ in $\overline{\Omega}$ is attained in $\overline{\Omega^\delta} := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \geq \delta\}$. Moreover, the maximum of $(-\Delta)^k u_p$ ($k = 1, \ldots, \frac{n}{2} - 1$) in $\overline{\Omega}$ is attained in $\overline{\Omega^\delta}$.*

(ii) *For every $p \geq p_0$, the solution $u = u_p$ satisfies the uniform bound:*

$$\int_\Omega u^p(x)\, dx \leq C.$$

Proof: (i) By using the method of moving planes in a local way, we can get (see Lemma 4.1 in Troy [**26**], or p. 21 in [**4**]) that, for any $x^0 \in \partial\Omega$, there exists a $\delta_0 > 0$ depending only on x^0 and Ω such that $u(x)$ is monotone increasing along the inner normal direction in the region

$$\overline{\Sigma_{\delta_0}} := \{x \in \overline{\Omega} \mid 0 \leq (x - x^0) \cdot \nu^0 \leq \delta_0\}, \tag{2.1}$$

where ν^0 denotes the unit inner normal vector at the point x^0. Since $\partial\Omega$ is C^{n-2}, there exists a small enough $0 < r_0 < \frac{\delta_0}{8}$ depending only on x^0 and Ω such that, for any $x \in B_{r_0}(x^0) \cap \partial\Omega$, $u(x)$ is monotone increasing along the inner normal direction at x in the region

$$\overline{\Sigma_x} := \left\{ z \in \overline{\Omega} \mid 0 \leq (z - x) \cdot \nu_x \leq \frac{3}{4}\delta_0 \right\}, \tag{2.2}$$

where ν_x denotes the unit inner normal vector at the point x ($\nu_{x^0} := \nu^0$). Since $x^0 \in \partial\Omega$ is arbitrary and $\partial\Omega$ is compact, we can cover $\partial\Omega$ by finitely many balls $\{B_{r_k}(x^k)\}_{k=0}^K$ with centers $\{x^k\}_{k=0}^K \subset \partial\Omega$ (K depends only on Ω). For each $x^k \in \partial\Omega$, choose δ_k depending only on x^k and Ω in a similar way as δ_0. Let $\delta := \frac{3}{4}\min\{\delta_0, \delta_1, \ldots, \delta_K\}$ depending only on Ω. Then it is clear that for any $x \in \partial\Omega$,

$$u(x + s\nu_x) \quad \text{is monotone increasing with respect to } s \in [0, \delta], \tag{2.3}$$

and hence property (i) for $u = u_p$ follows from (2.3) immediately.

Moreover, it is also clear from the procedure of moving planes (see Troy [**26**], or pp. 18–21 in [**4**]) that $(-\Delta)^k u_p$ ($k = 1, \ldots, \frac{n}{2} - 1$) are also monotone increasing along the inner normal directions in the boundary layer $\overline{\Omega \setminus \overline{\Omega^\delta}}$, and hence the maximum of $(-\Delta)^k u_p$ ($k = 1, \ldots, \frac{n}{2} - 1$) in $\overline{\Omega}$ can (only) be attained in $\overline{\Omega^\delta}$.

(ii) Let $0 < \overline{\lambda_1} < \overline{\lambda_2} \leq \cdots \leq \overline{\lambda_k} \leq \cdots$ be eigenvalues for $-\Delta$ in Ω with Dirichlet boundary condition. Then, one can easily verify that, for every $k = 1, 2, \ldots,$ λ_k is the eigenvalue for $(-\Delta)^{\frac{n}{2}}$ in Ω with Navier boundary condition if and only if $\lambda_k = (\overline{\lambda_k})^{\frac{n}{2}}$. Let $\phi > 0$ be the eigenfunction (without loss of generality, we may assume $\|\phi\|_{L^\infty(\overline{\Omega})} = 1$) corresponding to the first eigenvalue $\overline{\lambda_1}$ for $-\Delta$ with Dirichlet boundary condition. It follows that ϕ is also the eigenfunction corresponding to the first eigenvalue λ_1 for $(-\Delta)^{\frac{n}{2}}$ with Navier boundary condition, i.e.,

$$\begin{cases} (-\Delta)^{\frac{n}{2}}\phi(x) = \lambda_1\phi(x) & \text{in } \Omega, \\ \phi(x) = (-\Delta)\phi(x) = \cdots = (-\Delta)^{\frac{n}{2}-1}\phi(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Then, since

$$\int_\Omega u^p\phi\,dx = \int(-\Delta)^{\frac{n}{2}}u\phi\,dx = \lambda_1\int_\Omega u\phi\,dx \leq \lambda_1\left(\int_\Omega u^p\phi\,dx\right)^{\frac{1}{p}}\left(\int_\Omega \phi\,dx\right)^{\frac{1}{p'}},$$

we obtain, as in Lemma 3.2 in p. 22 of [**4**],

$$\int_\Omega u^p(x)\phi(x)\,dx \leq \lambda_1^{p'}\int_\Omega \phi(x)\,dx \leq \lambda_1^{p'}|\Omega|. \tag{2.5}$$

Thus, for any $p \geq p_0$, we have the following uniform bound:

$$\int_\Omega u^p(x)\phi(x)\,dx \leq C(n, p_0, \Omega). \tag{2.6}$$

Let $x \in \partial\Omega$. Since Ω is at least C^1, there exists a small $\varepsilon_x > 0$ and a neighborhood V_x of x in $\partial\Omega$ such that

$$W_x := \{y + \sigma\nu_x \mid y \in V_x,\, 0 < \sigma < \varepsilon_x\} \subset \Omega.$$

Let

$$W'_x := \left\{y + \sigma\nu_x \mid y \in V_x,\, \frac{\varepsilon_x}{2} < \sigma < \varepsilon_x\right\} \subset \Omega.$$

Since $x \in \partial\Omega$ is arbitrary and $\partial\Omega$ is compact, we can find a finite subset $\{x^k\}_{k=0}^K$ of $\partial\Omega$ (K depends only on Ω) such that $\partial\Omega \subset \bigcup_{k=0}^K V_{x^k}$. Considering the boundary layer $\overline{\Omega}_{\bar{\delta}} := \{x \in \overline{\Omega} \mid \mathrm{dist}(x, \partial\Omega) \leq \bar{\delta}\}$ we see that, if $\bar{\delta} > 0$ is small enough,

$$\overline{\Omega}_{\bar{\delta}} \subset \bigcup_{k=0}^K W_{x^k}, \quad \bigcup_{k=0}^K W'_{x^k} \subset \Omega \setminus \overline{\Omega}_{\bar{\delta}}. \tag{2.7}$$

From the procedure of moving planes (see Lemma 4.1 in Troy [**26**], or pp. 18–21 in [**4**]), for all $y \in V_{x^k}$, $\sigma \mapsto u(y + \sigma\nu_{x^k})$ is monotone increasing on $(0,\varepsilon_{x^k})$. Thus, using the definitions of W_{x^k}, W'_{x^k}, and the second inclusion in (2.7), there exists a $C_0 \geq 2$ depending only on Ω such that

$$\int_{W_{x^k}} u^p(x)\,dx \leq C_0 \int_{W'_{x^k}} u^p(x)\,dx \leq C_0 \int_{\Omega\setminus\overline{\Omega}_{\bar{\delta}}} u^p(x)\,dx. \tag{2.8}$$

As a consequence, using the first inclusion in (2.7),

$$\begin{aligned}\int_{\Omega} u^p(x)\,dx &\leq \sum_{k=0}^{K} \int_{W_{x^k}} u^p(x)\,dx + \int_{\Omega\setminus\overline{\Omega}_{\bar{\delta}}} u^p(x)\,dx \\ &\leq [C_0(K+1)+1] \int_{\Omega\setminus\overline{\Omega}_{\bar{\delta}}} u^p(x)\,dx.\end{aligned} \tag{2.9}$$

Combining with the uniform bound (2.6), we arrive at

$$\begin{aligned}\int_{\Omega} u^p(x)\,dx &\leq [C_0(K+1)+1] \max_{x\in\Omega\setminus\overline{\Omega}_{\bar{\delta}}} \frac{1}{\phi(x)} \int_{\Omega\setminus\overline{\Omega}_{\bar{\delta}}} \phi u^p(x)\,dx \\ &\leq C(n,p_0,\Omega),\end{aligned} \tag{2.10}$$

which proves property (ii). This completes our proof of Lemma 2.1. $\square$

From now on, we will denote the solution u_p by u for the sake of simplicity.

Let

$$M := \max_{\overline{\Omega}} u = \|u\|_{L^\infty(\overline{\Omega})}. \tag{2.11}$$

We aim to prove that there exists a constant $C > 0$ depending only on n, p_0, and Ω, such that $M \leq C$ for any $p \geq p_0$. We may assume that $M > \max\{2^n, 2^{\frac{2n}{p_0-1}}\}$ hereafter, or else we are done.

We first rescale u so that $\Omega \subseteq B_{\frac{1}{4}}(0)$. Indeed, let $R := R(\Omega) > 1$ be the smallest radius such that $\Omega \subseteq B_{\frac{R}{4}}(0)$. Then $u_R(x) := R^{\frac{n}{p-1}} u(Rx)$ is a nonnegative solution of Navier problem (1.1) in $R^{-1}\Omega \subseteq B_{\frac{1}{4}}(0)$ and we only need to consider u_R instead. By Lemma 2.1, the maximum M is attained at some point $x_0 \in \overline{\Omega^\delta} := \{x \in \Omega \mid \operatorname{dist}(x,\partial\Omega) \geq \delta\}$. Without loss of generality, translating Ω if necessary, we may assume $0 \in \overline{\Omega^\delta}$ and $x_0 = 0$, that is, $u(0) = M$. Note that after this translation, we have

$$\Omega \subseteq B_{\frac{1}{2}}(0).$$

For arbitrarily given $x \in \overline{\Omega^\delta}$, let

$$G(x,y) := \int_\Omega \Bigg(\int_\Omega \Bigg(\cdots \Bigg(\int_\Omega \Bigg(\int_\Omega G_2(x,z^1) G_2(z^1,z^2)\, dz^1 \Bigg) G_2(z^2,z^3)\, dz^2 \Bigg) \cdots \Bigg) \times G_2(z^{\frac{n}{2}-2}, z^{\frac{n}{2}-1})\, dz^{\frac{n}{2}-2} \Bigg) G_2(z^{\frac{n}{2}-1}, y)\, dz^{\frac{n}{2}-1} \tag{2.12}$$

be the Green's function for $(-\Delta)^{\frac{n}{2}}$ with pole at x (for more details on Green's functions for poly-harmonic operators, please see [**14**]), where $G_2(x,y)$ is the Green's function for $-\Delta$ with Dirichlet boundary condition in Ω. Then, we have

$$\begin{cases} (-\Delta)^{\frac{n}{2}} G(x,y) = \delta(x-y), & y \in \Omega, \\ G(x,y) = (-\Delta) G(x,y) = \cdots = (-\Delta)^{\frac{n}{2}-1} G(x,y) = 0, & y \in \partial\Omega, \end{cases} \tag{2.13}$$

where Δ is the Laplace operator with respect to the variable y at fixed x. Consequently, we can rewrite the Green's function $G(x,y)$ as:

$$G(x,y) = C_n \ln\left(\frac{1}{|x-y|} \right) - h(x,y) \quad \forall\, y \in \overline{\Omega}, \tag{2.14}$$

where the $\frac{n}{2}$-harmonic function h satisfies

$$\begin{cases} (-\Delta)^{\frac{n}{2}} h(x,y) = 0, & y \in \Omega, \\ (-\Delta)^k h(x,y) = (-\Delta)^k \big(C_n \ln \frac{1}{|x-y|} \big), \quad k = 0, 1, \dots, \frac{n}{2}-1, & y \in \partial\Omega. \end{cases} \tag{2.15}$$

Here, again $x \in \overline{\Omega^\delta}$ is arbitrarily fixed, h and $(-\Delta)^k h$ are treated as functions of y only. In (2.15), all the boundary data are positive. Indeed, $(-\Delta)^k \big(C_n \ln \frac{1}{|x-y|} \big) = \frac{C_{n,k}}{|x-y|^{2k}}$, where the $C_{n,k}$ are positive constants.

Since $\delta \leq |x-y| \leq 1$ for all $y \in \partial\Omega$, we have

$$\begin{aligned} C \leq (-\Delta)^{\frac{n}{2}-1} h(x,y) &= (-\Delta)^{\frac{n}{2}-1} \left(C_n \ln \frac{1}{|x-y|} \right) \\ &= \frac{C}{|x-y|^{n-2}} \leq \frac{C}{\delta^{n-2}} \end{aligned} \tag{2.16}$$

for any $y \in \partial\Omega$, and hence the maximum principle implies

$$C \leq (-\Delta)^{\frac{n}{2}-1} h(x,y) \leq \frac{C}{\delta^{n-2}} \quad \forall\, y \in \Omega. \tag{2.17}$$

On the boundary $\partial\Omega$ we also have

$$
\begin{aligned}
C &\le (-\Delta)^{\frac{n}{2}-2} h(x,y) = (-\Delta)^{\frac{n}{2}-2}\left(C_n \ln \frac{1}{|x-y|}\right) \\
&= \frac{C}{|x-y|^{n-4}} \le \frac{C}{\delta^{n-4}}
\end{aligned}
\tag{2.18}
$$

for all $y \in \partial\Omega$. It follows from (2.17), (2.18), and the maximum principle that

$$
C \le (-\Delta)^{\frac{n}{2}-2} h(x,y) \le \frac{C}{\delta^{n-4}} + \frac{C}{\delta^{n-2}} \le \frac{C}{\delta^{n-2}} \quad \forall\, y \in \Omega. \tag{2.19}
$$

Continuing in this way, we finally get

$$
0 \le h(x,y) \le C \ln \frac{1}{\delta} + \frac{C}{\delta^{n-2}} =: C \quad \forall\, y \in \Omega. \tag{2.20}
$$

In conclusion, we have arrived at the following estimates: for any given $x \in \overline{\Omega^\delta}$ and $k = 0, \ldots, \frac{n}{2} - 1$, there exist constants $C_k', C_k'' \ge 0$ such that

$$
C_k' \le (-\Delta)^k h(x,y) \le C_k'' \quad \forall\, y \in \Omega. \tag{2.21}
$$

We have the following lemma on uniform bound of the solution $u = u_p$.

Lemma 2.2. *Assume $n \ge 4$ is even, $m = \frac{n}{2}$, Ω is strictly convex, and let $p_0 > 1$. For every $x \in \overline{\Omega^\delta}$ and $p \ge p_0$, the solution $u = u_p$ satisfies the uniform bound:*

$$
\frac{1}{M} \int_\Omega \ln\left(\frac{1}{|x-y|}\right) u^p(y)\, dy \le C,
$$

where M is defined by (2.11).

Proof: By (ii) in Lemma 2.1, (2.21), and Green's representation formula we have, for any $x \in \overline{\Omega^\delta}$ and $p \ge p_0$,

$$
\begin{aligned}
M \ge u(x) &= \int_\Omega G(x,y) u^p(y)\, dy \\
&= C_n \int_\Omega \ln\left(\frac{1}{|x-y|}\right) u^p(y)\, dy - \int_\Omega h(x,y) u^p(y)\, dy \\
&\ge C_n \int_\Omega \ln\left(\frac{1}{|x-y|}\right) u^p(y)\, dy - C \int_\Omega u^p(y)\, dy \\
&\ge C_n \int_\Omega \ln\left(\frac{1}{|x-y|}\right) u^p(y)\, dy - C.
\end{aligned}
\tag{2.22}
$$

As a consequence, we immediately get that

$$\frac{1}{M}\int_{\Omega}\ln\left(\frac{1}{|x-y|}\right)u^p(y)\,dy \leq \frac{M+C}{MC} \leq C. \tag{2.23}$$

This finishes our proof of Lemma 2.2. $\square$

Let $M_k := \max_{\overline{\Omega}}(-\Delta)^k u = \|(-\Delta)^k u\|_{L^\infty(\overline{\Omega})}$ for $k = 1, \ldots, \frac{n}{2}-1$. By Lemma 2.1, the maximum M_k can (only) be attained at some point $x_k \in \overline{\Omega^\delta} := \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) \geq \delta\}$, that is, $(-\Delta)^k u(x_k) = M_k$.

We have the following lemma which is crucial in our proof.

Lemma 2.3. *Assume $n \geq 4$ is even, $m = \frac{n}{2}$, Ω is strictly convex, and let $p_0 > 1$. For every $k = 1, \ldots, \frac{n}{2}-1$ and $p \geq p_0$, we have the following precise bound:*

$$C_k'' \frac{M^{\frac{2k}{n}p+(1-\frac{2k}{n})}}{p^{1-\frac{2k}{n}}} \leq M_k \leq C_k' \frac{M^{\frac{2k}{n}p+(1-\frac{2k}{n})}}{p^{1-\frac{2k}{n}}}. \tag{2.24}$$

Moreover, we have, for any $p \geq p_0$,

$$0 \leq M - u(x) \leq \frac{C}{p}M \quad \forall\, |x| \leq \frac{\delta}{\sqrt[n]{p}M^{\frac{p-1}{n}}}. \tag{2.25}$$

Proof: Since $M_k = (-\Delta)^k u(x_k)$ and $x_k \in \overline{\Omega^\delta}$, by Green's representation formula and (2.21), we have

$$\begin{aligned} M_k &= (-\Delta)^k u(x_k) \\ &= C_k \int_{\Omega} \frac{1}{|x_k-y|^{2k}} u^p(y)\,dy - \int_{\Omega} (-\Delta)^k h(x_k, y) u^p(y)\,dy \\ &\leq C_k \int_{\Omega} \frac{1}{|x_k-y|^{2k}} u^p(y)\,dy. \end{aligned} \tag{2.26}$$

Note that $B_\delta(x_k) \subseteq \Omega$. For every $p \geq p_0$,

$$\begin{aligned} &\int_{|x_k-y| \leq \frac{\delta}{p^{\frac{1}{n}} M^{\frac{p-1}{n}}}} \frac{1}{|x_k-y|^{2k}} u^p(y)\,dy \\ &\qquad \leq M^p \int_{|x_k-y| \leq \frac{\delta}{p^{\frac{1}{n}} M^{\frac{p-1}{n}}}} \frac{1}{|x_k-y|^{2k}}\,dy \\ &\qquad \leq C_k \frac{M^p}{p^{1-\frac{2k}{n}} M^{(1-\frac{2k}{n})(p-1)}} = C_k \frac{M^{\frac{2k}{n}p+(1-\frac{2k}{n})}}{p^{1-\frac{2k}{n}}} \end{aligned} \tag{2.27}$$

and, by (ii) in Lemma 2.1,

$$
\begin{aligned}
&\int_{\Omega\cap\{|x_k-y|\geq \frac{p^{\frac{1}{2k}-\frac{1}{n}}\delta}{M^{\frac{p}{n}+(\frac{1}{2k}-\frac{1}{n})}}\}} \frac{1}{|x_k-y|^{2k}}u^p(y)\,dy \\
&\qquad\qquad \leq \left(\frac{M^{\frac{p}{n}+(\frac{1}{2k}-\frac{1}{n})}}{p^{\frac{1}{2k}-\frac{1}{n}}\delta}\right)^{2k}\int_\Omega u^p(y)\,dy \\
&\qquad\qquad \leq C_k\frac{M^{\frac{2k}{n}p+(1-\frac{2k}{n})}}{p^{1-\frac{2k}{n}}}.
\end{aligned}
\tag{2.28}
$$

In the case $\frac{1}{p^{\frac{1}{n}}M^{\frac{p-1}{n}}} < \frac{p^{\frac{1}{2k}-\frac{1}{n}}}{M^{\frac{p}{n}+(\frac{1}{2k}-\frac{1}{n})}}$, we can also deduce from Lemma 2.2 that, for every $p\geq p_0$,

$$
\begin{aligned}
&\int_{\frac{\delta}{p^{\frac{1}{n}}M^{\frac{p-1}{n}}}\leq|x_k-y|\leq\frac{p^{\frac{1}{2k}-\frac{1}{n}}\delta}{M^{\frac{p}{n}+(\frac{1}{2k}-\frac{1}{n})}}} \frac{1}{|x_k-y|^{2k}}u^p(y)\,dy \\
&\leq \left[\frac{1}{M}\int_\Omega \ln\left(\frac{1}{|x_k-y|}\right)u^p(y)\,dy\right]\frac{M}{\left(\frac{\delta}{p^{\frac{1}{n}}M^{\frac{p-1}{n}}}\right)^{2k}\ln\left(\frac{M^{\frac{p}{n}+(\frac{1}{2k}-\frac{1}{n})}}{p^{\frac{1}{2k}-\frac{1}{n}}\delta}\right)} \\
&\leq C_k\frac{M^{1+\frac{2k}{n}(p-1)}p^{\frac{2k}{n}}}{\left(\frac{p}{n}+\frac{1}{2k}-\frac{1}{n}\right)\ln M}\leq C_k\frac{M^{\frac{2k}{n}p+(1-\frac{2k}{n})}}{p^{1-\frac{2k}{n}}},
\end{aligned}
\tag{2.29}
$$

where in the last line we have used $M>\max\{2^n,2^{\frac{2n}{p_0-1}}\}$. In order to derive the penultimate inequality in (2.29), we have also used the following inequality:

$$
\begin{aligned}
\left(\frac{1}{2k}-\frac{1}{n}\right)\ln p &< \left(\frac{1}{2k}-\frac{1}{n}\right)\ln 2\cdot p < \frac{\ln 2}{2}p \\
&< \frac{p}{2n}\ln M < \frac{1}{2}\left(\frac{p}{n}+\frac{1}{2k}-\frac{1}{n}\right)\ln M.
\end{aligned}
\tag{2.30}
$$

Combining (2.26), (2.27), (2.28), and (2.29), we get

$$
M_k=(-\Delta)^k u(x_k)\leq C_k'\frac{M^{\frac{2k}{n}p+(1-\frac{2k}{n})}}{p^{1-\frac{2k}{n}}}. \tag{2.31}
$$

Since $B_\delta(0) \subseteq \Omega$ and $u(0) = M$, by (2.31) with $k = 1$ and applying the inhomogeneous Harnack inequality (see Theorems 9.20 and 9.22 in [**15**] or Theorem 4.17 in [**17**]), we get

$$0 \leq u(0) - u(x) \leq C M_1 r^2 \leq C \frac{M^{\frac{2}{n}p+(1-\frac{2}{n})}}{p^{1-\frac{2}{n}}} r^2 \quad \forall\, x \in B_r(0) \tag{2.32}$$

and for all $r \in \left[0, \frac{\delta}{4}\right]$. Indeed, since $B_{4r}(0) \subseteq \Omega$, by Theorem 9.22 in [**15**], there exists a q depending only on n such that

$$\left(\frac{1}{|B_{2r}(0)|}\int_{B_{2r}(0)} (u(0) - u(x))^q \, dx\right)^{\frac{1}{q}} \leq C\left(\inf_{x \in B_{2r}(0)} (u(0) - u(x)) + r\|\Delta u\|_{L^n(B_{2r}(0))}\right).$$

Combining this with Theorem 9.20 in [**15**], we deduce that

$$\begin{aligned}\sup_{x \in B_r(0)} (u(0) - u(x)) &\leq C\left(\left(\frac{1}{|B_{2r}(0)|}\int_{B_{2r}(0)} (u(0) - u(x))^q \, dx\right)^{\frac{1}{q}} + r\|\Delta u\|_{L^n(B_{2r}(0))}\right) \\ &\leq C\left(\inf_{x \in B_{2r}(0)} (u(0) - u(x)) + r\|\Delta u\|_{L^n(B_{2r}(0))}\right),\end{aligned}$$

which yields (2.32) immediately.

The inequality (2.32) implies, for any $p \geq p_0$,

$$0 \leq M - u(x) \leq \frac{C}{p} M \quad \forall\, |x| \leq \frac{\delta}{\sqrt[n]{p} M^{\frac{p-1}{n}}}. \tag{2.33}$$

Combining (ii) in Lemma 2.1, Green's representation formula, (2.21), and (2.33) we obtain, for any $p \geq p_0$,

$$\begin{aligned} M_k \geq (-\Delta)^k u(0) &= C_k \int_\Omega \frac{1}{|y|^{2k}} u^p(y)\, dy - \int_\Omega (-\Delta)^k h(0,y) u^p(y)\, dy \\ &\geq C_k \int_{|x| \leq \frac{\delta}{p^{\frac{1}{n}} M^{\frac{p-1}{n}}}} \frac{1}{|x|^{2k}} \left(1 - \frac{C}{p}\right)^p M^p \, dx - \widetilde{C}_k \\ &\geq C_k M^p \int_0^{\frac{\delta}{p^{\frac{1}{n}} M^{\frac{p-1}{n}}}} r^{n-1-2k} \, dr - \widetilde{C}_k \\ &\geq C_k'' \frac{M^{\frac{2k}{n}p+(1-\frac{2k}{n})}}{p^{1-\frac{2k}{n}}}. \end{aligned} \tag{2.34}$$

This concludes our proof of Lemma 2.3. □

Since $0 \in \overline{\Omega^\delta}$, the combination od Lemma 2.2 and Lemma 2.3 yields that, for any $p \geq p_0$,

$$
\begin{aligned}
C &\geq \frac{1}{M} \int_{|x| \leq \frac{\delta}{M^{\frac{p-1}{n}} p^{\frac{1}{n}}}} \ln\left(\frac{1}{|x|}\right)\left(1 - \frac{C}{p}\right)^p M^p \, dx \\
&\geq C M^{p-1} \int_0^{\frac{\delta}{M^{\frac{p-1}{n}} p^{\frac{1}{n}}}} \ln\left(\frac{1}{r}\right) r^{n-1} \, dr \qquad (2.35)\\
&\geq \frac{C M^{p-1}}{M^{p-1} p} \ln\left(\frac{M^{\frac{p-1}{n}} p^{\frac{1}{n}}}{\delta}\right) \\
&\geq C \ln M,
\end{aligned}
$$

which implies immediately the desired uniform a priori estimate:

$$M \leq e^C. \qquad (2.36)$$

This concludes our proof of Theorem 1.2.

Acknowledgements

The authors are grateful to the referees for their careful reading and valuable comments and suggestions that improved the presentation of the paper.

References

[1] Adimurthi and M. Grossi, Asymptotic estimates for a two-dimensional problem with polynomial nonlinearity, *Proc. Amer. Math. Soc.* **132(4)** (2004), 1013–1019. DOI: 10.1090/S0002-9939-03-07301-5.

[2] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis* **14(4)** (1973), 349–381. DOI: 10.1016/0022-1236(73)90051-7.

[3] S. Chandrasekhar, *"An Introduction to the Study of Stellar Structure"*, Dover Publications, Inc., New York, N. Y. 1957.

[4] W. Chen, W. Dai, and G. Qin, Liouville type theorems, a priori estimates and existence of solutions for critical order Hardy–Hénon equations in $\mathbb{R}^n$, Preprint (2018). arXiv:1808.06609.

[5] W. Chen, Y. Fang, and C. Li, Super poly-harmonic property of solutions for Navier boundary problems on a half space, *J. Funct. Anal.* **265(8)** (2013), 1522–1555. DOI: 10.1016/j.jfa.2013.06.010.

[6] W. Dai, S. Peng, and G. Qin, Liouville type theorems, a priori estimates and existence of solutions for non-critical higher order Lane–Emden–Hardy equations, Preprint 2018. arXiv:1808.10771.

[7] W. Dai and G. Qin, Liouville type theorems for fractional and higher order Hénon–Hardy type equations via the method of scaling spheres, Preprint (2018). arXiv:1810.02752.

[8] W. Dai and G. Qin, Liouville type theorem for critical order Hénon–Lane–Emden type equations on a half space and its applications, Preprint (2018). arXiv:1811.00881.

[9] F. De Marchis, I. Ianni, and F. Pacella, Asymptotic analysis and sign-changing bubble towers for Lane–Emden problems, *J. Eur. Math. Soc. (JEMS)* **17(8)** (2015), 2037–2068. DOI: 10.4171/JEMS/549.

[10] F. De Marchis, I. Ianni, and F. Pacella, Asymptotic profile of positive solutions of Lane–Emden problems in dimension two, *J. Fixed Point Theory Appl.* **19(1)** (2017), 889–916. DOI: 10.1007/s11784-016-0386-9.

[11] G. Di, On blow-up of positive solutions for a biharmonic equation involving nearly critical exponent, *Comm. Partial Differential Equations* **24(11–12)** (1999), 2333–2370. DOI: 10.1080/03605309908821504.

[12] D. E. Edmunds, D. Fortunato, and E. Jannelli, Critical exponents, critical dimensions and the biharmonic operator, *Arch. Rational Mech. Anal.* **112(3)** (1990), 269–289. DOI: 10.1007/BF00381236.

[13] M. Fazly, J.-C. Wei, and X. Xu, A pointwise inequality for the fourth-order Lane–Emden equation, *Anal. PDE* **8(7)** (2015), 1541–1563. DOI: 10.2140/apde.2015.8.1541.

[14] F. Gazzola, H.-C. Grunau, and G. Sweers, *"Polyharmonic Boundary Value Problems"*, Positivity preserving and nonlinear higher order elliptic equations in bounded domains, Lecture Notes in Mathematics **1991**, Springer-Verlag, Berlin, 2010. DOI: 10.1007/978-3-642-12245-3.

[15] D. Gilbarg and N. S. Trudinger, *"Elliptic Partial Differential Equations of Second Order"*, Second edition, Grundlehren der Mathematischen Wissenschaften **224**, Springer-Verlag, Berlin, 1983. DOI: 10.1007/978-3-642-61798-0.

[16] Z.-C. Han, Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **8(2)** (1991), 159–174. DOI: 10.1016/S0294-1449(16)30270-0.

[17] Q. Han and F. Lin, *"Elliptic Partial Differential Equations"*, Second edition, Courant Lecture Notes in Mathematics **1**, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2011.

[18] G. P. Horedt, Exact solutions of the Lane–Emden equation in N-dimensional space, *Astronom. and Astrophys.* **160(1)** (1986), 148–156.

[19] N. Kamburov and B. Sirakov, Uniform a priori estimates for positive solutions of the Lane–Emden equation in the plane, *Calc. Var. Partial Differential Equations* **57(6)** (2018), Paper No.164, 8 pp. DOI: 10.1007/s00526-018-1435-6.

[20] S. I. Pohožaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$ (Russian), *Dokl. Akad. Nauk SSSR* **165(1)** (1965), 36–39. English translation: *Soviet Math. Dokl.* **6** (1965), 1408–1411.

[21] P. Pucci and J. Serrin, A general variational identity, *Indiana Univ. Math. J.* **35(3)** (1986), 681–703. DOI: 10.1512/iumj.1986.35.35036.

[22] P. Pucci and J. Serrin, Critical exponents and critical dimensions for polyharmonic operators, *J. Math. Pures Appl. (9)* **69(1)** (1990), 55–83.

[23] X. Ren and J. Wei, On a two-dimensional elliptic problem with large exponent in nonlinearity, *Trans. Amer. Math. Soc.* **343(2)** (1994), 749–763. DOI: 10.2307/2154740.

[24] O. Rey, The role of the Green's function in a non-linear elliptic equation involving the critical Sobolev exponent, *J. Funct. Anal.* **89(1)** (1990), 1–52. DOI: 10.1016/0022-1236(90)90002-3.

[25] B. Sirakov, Existence results and a priori bounds for higher order elliptic equations and systems, *J. Math. Pures Appl. (9)* **89(2)** (2008), 114–133. DOI: 10.1016/j.matpur.2007.10.003.

[26] W. C. Troy, Symmetry properties in systems of semilinear elliptic equations, *J. Differential Equations* **42(3)** (1981), 400–413. DOI: 10.1016/0022-0396(81)90113-3.

[27] R. C. A. M. Van der Vorst, Variational identities and applications to differential systems, *Arch. Rational Mech. Anal.* **116(4)** (1992), 375–398. DOI: 10.1007/BF00375674.

Wei Dai
School of Mathematical Sciences, Beihang University (BUAA), Beijing 100083, P. R. China, and LAGA, UMR 7539, Institut Galilée, Université Sorbonne Paris Nord, 93430 - Villetaneuse, France
E-mail address: weidai@buaa.edu.cn

Thomas Duyckaerts
Institut Universitaire de France, and LAGA, UMR 7539, Institut Galilée, Université Sorbonne Paris Nord, 93430 - Villetaneuse, France
E-mail address: duyckaer@math.univ-paris13.fr

Primera versió rebuda el 4 d'octubre de 2019,
darrera versió rebuda el 20 d'abril de 2020.