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ON GROUPS OF FINITE RANK

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Abstract: We study the structure of groups of finite (Prüfer) rank in a very wide class of groups and also of central extensions of such groups. As a result we are able to improve, often substantially, on a number of known numerical bounds, in particular on bounds for the rank (resp. Hirsch number) of the derived subgroup of a group in terms of the rank (resp. Hirsch number) of its central quotient and on bounds for the rank of a group in terms of its Hirsch number.

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1. Introduction

Let G be any group. If G is finitely generated, then $d(G)$ denotes the minimal number of generators of G. In general the rank $\mathrm{rk}(G)$ of G is the least upper bound of the $d(X)$ as X ranges over all the finitely generated subgroups of G. (Alternative names for rank are Prüfer rank and Mal'cev special rank.) Since there exist infinite simple groups of rank 2 all of whose proper subgroups are cyclic, for our study here of finite rank we choose to avoid such groups.

The encompassing class of groups with which we work in this paper is $\langle P', L\rangle(\mathbf{AF})$, or equivalently $\langle P, L\rangle(\mathbf{AF})$: here P' denotes the ascending series operator, L the local operator, P the poly operator, $\mathbf{A}$ the class of abelian groups, and $\mathbf{F}$ the class of finite groups. (Groups in this class are sometimes said to be elementary amenable.) Thus we are considering the smallest class of groups containing all abelian groups and all finite groups, which is closed with respect to both the ascending series and the local operators. Also for any class $\mathbf{X}$ (of groups) $\langle P', L\rangle\mathbf{X} = \bigcup_\alpha (P'L)^\alpha\mathbf{X}$, where the union is over all ordinals α, not just the finite ones. For any real number x, $[x]$ denotes the largest integer not exceeding x, so $-[-x]$ is the least integer not less than x. We use this notation below.

Theorem 1. (a) *If G is a $\langle P', L\rangle(\mathbf{AF})$ group of finite rank, then every finitely generated subgroup of G is soluble-by-finite and minimax. Further G has normal subgroups $H \leq S$ with H hypercentral and periodic with each of its primary components soluble and Chernikov, S/H soluble, S locally soluble, and G/S finite.*

(b) *Let Z be a central subgroup of a group G such that G/Z is a $\langle P', L\rangle(\mathbf{AF})$ group of finite rank r. Then*

$$\mathrm{rk}(G' \cap Z) \leq r(3r+1)/2 + 2\sigma(r) \text{ and } \mathrm{rk}(G') \leq 3r(r+1)/2 + 2\sigma(r),$$

where if $m = r(1 - [-\log_2 r])$, then $m \leq r^2$ and

$$\sigma(r) \leq r(r-1)/2 + r^2 + m(m-1)/2 \leq r(r^3 + 2r - 1)/2 \leq r^4.$$

There are two obvious subclasses of $\langle P', L\rangle(\mathbf{AF})$ namely $\langle P', L\rangle\mathbf{A}$, where part (a) becomes a (substantial) theorem of D. J. S. Robinson (see [**6**, 10.38]) and $\langle P', L\rangle\mathbf{F}$. The latter class is just the class of locally finite groups, where much of part (a) becomes obvious. Part (b) looks complicated but its conclusion is actually very useful and has many corollaries. The function $\sigma(r)$ is defined solely in terms of properties of finite p-groups for all primes p (see Lemma 3.1 and comments below) and keeps appearing throughout this paper. Hence it is useful to have a special notation for it.

Special cases of part (b) are known of which, as far as I know, the most general is 7.2.35 of [**2**]. This concerns locally generalized radical groups; see [**2**, pp. 14 and 15] for definitions. In terms of our notation here a locally generalized radical group lies in the class $LP'LP(\mathbf{AF})$ and hence in $\langle P', L\rangle(\mathbf{AF})$. Also the bound derived in [**2**] for the ranks of the derived subgroups of these groups is substantially larger (and much more complicated) than those given in part (b) above. (The bound is $3s_4(r)+1$, where s_4 is defined low down on p. 190 in terms of s_3, s_3 is defined low down on p. 186 in terms of s_2, s_2 is defined on p. 184 in terms of s_1, and finally $s_1(r)$ is the bound we have stated above for $\sigma(r)$. Fitting all these together produces a very complicated expression indeed.)

Further notation. For any group G, $\tau(G)$ is its unique maximal locally finite normal subgroup, $\{G^{(m)}\}_{m\geq 0}$ its derived series, $\{\gamma^i G\}_{i\geq 1}$ its lower central series, and $\{\zeta_i(G)\}_{i\geq 0}$ its upper central series.

Corollary. *Let $\mathbf{X}$ be one of the five group classes labelled (a) to (e) below. If G is a group and $i \geq 0$ an integer with $G/\zeta_i(G) \in \mathbf{X}$, then $\gamma^{i+1}G \in \mathbf{X}$.*

(a) $\mathbf{X}_{\mathrm{fr}}$*, the class of all $\langle P', L\rangle(\mathbf{AF})$ groups of finite rank.*

(b) $\mathbf{X}_{\mathrm{mm}}$*, the class of all minimax $\langle P', L\rangle(\mathbf{AF})$ groups.*

(c) $\mathbf{X}_{\text{fh}}$, *the class of all groups with finite Hirsch number.*

(d) $\mathbf{X}_{\text{fh}} \cap \bigcap_{\text{primes } p} \text{Min-}p$, *where* Min-$p$ *is the class of all groups whose poset of p-subgroups satisfies the minimal condition.*

(e) *The class of finite extensions of soluble FAR-groups.*

Clearly (a) and (b) are related to Theorem 1, a minimax group being one with a finite series each factor of which satisfies either the minimal or the maximal condition on subgroups. For the locally generalized radical case of (a) and indirectly (b), see [**2**, 7.3.16]. A group G has Hirsch number (or torsion-free rank) $h < \infty$ if G has an ascending series exactly h factors of which are infinite cyclic, the remaining factors being locally finite. In this case set $\text{hn}(G) = h$. Such a G has a very restricted structure. In fact G has normal subgroups $T \leq N \leq M$, where $T = \tau(G)$ is locally finite, N/T is torsion-free nilpotent with $\text{hn}(N/T) \leq h$, M/N is free abelian of rank at most h, and G/M is finite with order bounded in terms of h. Clearly $\mathbf{X}_{\text{fh}} \leq \langle P', L\rangle(\mathbf{AF})$. For the Corollary in the case of (c), see also [**2**, 7.1.18].

The classes of finite extensions of soluble FAR groups and soluble FATR groups are defined and studied in [**5**], especially in Chapter 5. In terms of our notation here, the soluble FAR groups are exactly the soluble members of $\mathbf{X}_{\text{fh}} \cap \bigcap_{\text{primes } p} \text{Min-}p$, so (e) of the Corollary is effectively a special case of (d). Indeed this is our reason for including (d). The soluble FATR groups are exactly the soluble $\mathbf{X}_{\text{fh}}$ groups G for which $\tau(G)$ satisfies the minimal condition on subgroups.

A number of other classes $\mathbf{X}$ are well known to satisfy the conclusion of the Corollary, examples being the class of polycyclic-by-finite groups and the class of Chernikov groups; see [**6**, p. 115 of Part 1]. The class of soluble FATR groups is excluded from [**6**] and the Corollary above in this context and for good reason. The main problem with this class is that it is not quotient-closed and an example below shows that in fact it cannot be added to the list of classes in the Corollary. However we do have the following. (It was a study of rank in linear groups and a need for Theorem 2 and (e) of the Corollary that led to this paper.)

Theorem 2. *Let Z be a central subgroup of the subgroup G of* $\text{GL}(n, F)$, *n a positive integer, and F a field. If G/Z is a finite extension of a soluble FATR group, then so is G'.*

Finally we improve upon some published bounds relating to groups with finite Hirsch number. Theorem 1(b) can be viewed as a quantitative version of (a) and indirectly (b) of the Corollary. Theorem 3b) below can be viewed as a quantitative version of (c) and indirectly (d) and (e) of the Corollary.

Theorem 3. (a) *If G is a group with finite Hirsch number h, then $G/\tau(G)$ has finite rank at most $3h + [h/2] + 1$.*

(b) *Let Z be a central subgroup of a group G and assume that G/Z has finite Hirsch number h. Then G' has Hirsch number at most $h(h+1)/2$.*

Theorem 3(a) corresponds to [**2**, 7.1.24], where the bound is $5h^2(h+1)/2$ and Theorem 3(b) to [**2**, 7.1.18], where the bound is $h(5h^2+5h-1)/2$.

2. Proof of Theorem 1(a)

Lemma 2.1. *Let G be a locally soluble-by-finite group of finite rank r and finite exponent b. Then $|G|$ divides $b^r(b^r!)$.*

The point here is that $|G|$ is bounded in terms of r and b only, not the actual form of the above bound.

Proof: Let X be a finitely generated subgroup of G. Then X is soluble-by-finite and each of its abelian sections has order dividing b^r. In particular X is finite. Let A be a maximal abelian normal subgroup of a Sylow p-subgroup P of X. Then $|A|$ divides b^r, $C_P(A) = A$, and $(P : A)$ divides $b^r!$ and is a power of p. Therefore $|X|$ divides $b^r(b^r!)$. The lemma follows. □

Lemma 2.2. *A torsion-free locally nilpotent group G of rank at most r has derived length at most r.*

The existence of a bound is undoubtedly well-known. To prove Lemma 2.2 firstly one may assume that G is finitely generated and hence nilpotent. If A is a maximal abelian normal subgroup of G, then A is free abelian of rank at most r and $G/A = G/C_G(A)$ embeds into $\mathrm{Tr}_1(r, \mathbb{Z})$, which has derived length less than r, in fact at most $-[-\log_2 r]$.

Lemma 2.3. *Let G be a soluble-by-finite minimax group of rank at most r. Then there exists an integer $b = b(r)$ of r only such that G has a finite normal subgroup $K \le (G^b)'$ with $(G^b)'/K$ nilpotent and $(G^b)^{(r+1)} \le \tau(G)$.*

Of course it follows that G is nilpotent-by-abelian-by-finite, but this holds much more generally (see e.g. [**5**, 5.2.2]).

Proof: Now G has a normal series $\langle 1\rangle = G_0 \le G_1 \le \cdots \le G_t \le G$ such that for some $1 \le s \le t$, each G_i/G_{i-1} for $0 < i < s$ is a divisible abelian primary group of rank at most r, G_s/G_{s-1} and G/G_t are finite, and each G_i/G_{i-1} for $s < i \le t$ is torsion-free abelian of rank at most r. Set

$C_i = C_G(G_i/G_{i-1})$. If $i \neq s$, then G/C_i is isomorphic to a linear group of degree r and characteristic zero and hence there is an integer b depending only on r such that G/C_i has a triangularizable normal subgroup of finite index dividing b (see e.g. [**7**, 10.11 and 3.6]).

Set $L = (G^b)'$. Then L acts unipotently on each G_i/G_{i-1} for $i \neq s$; that is, $[G_i, {}_rL] \leq G_{i-1}$ for each such i. Using [**6**, 4.25 and 4.21] we may move finite factors up and then down a central series of finite length sometimes doubling its length. We conclude that there exists K a finite normal subgroup of L with L/K nilpotent, say of class c. Replacing K by $\gamma^{c+1}L$ shows that we may choose such a K normal in G. If $T = \tau(L)$, then $K \leq T$ and L/T is torsion-free nilpotent of rank at most r. Hence $L^{(r)} \leq T$ by Lemma 2.2 and consequently $(G^b)^{(r+1)} \leq T$, which is locally finite. The proof is complete. □

Lemma 2.4. *A finitely generated, soluble-by-finite group of finite rank is minimax.*

This is a very special case of Robinson's Theorem 10.38 of [**6**]; indeed Lemma 2.4 only requires part (a) of the proof of that result.

The classes **A** and **F** have been defined above. In addition we denote the class of nilpotent groups by **N** and the class of soluble groups by **S**.

Lemma 2.5. *Let G be a locally soluble-by-finite group of finite rank at most r. If $b = b(r)$ is as in Lemma 2.3, then $L = (G^b)^{(r+1)}$ is locally finite and $G \in L(\mathbf{N} \cap \mathbf{F})\mathbf{SF}$.*

Proof: If X is a finitely generated subgroup of G, then X is soluble-by-finite and hence minimax by Lemma 2.4. Therefore $(X^b)^{(r+1)}$ is locally finite by Lemma 2.3. If Y is any finitely generated subgroup of L, there exists X as above with $Y \leq (X^b)^{(r+1)}$. Consequently Y is finite and L is locally finite. By the Belyaev–Kargapolov Theorem (see [**1**, 3.5.15 and 3.2.3]) there is a characteristic subgroup M of L of finite index with M' locally nilpotent. Now G/G^b is finite by Lemma 2.1. Consequently G/M' is soluble-by-finite. □

Proof of Theorem 1(a): Let $\mathbf{X} = \langle P', L\rangle(\mathbf{AF}) = \bigcup_\alpha \mathbf{X}_\alpha$, where α is any ordinal and $\mathbf{X}_\alpha = (P'L)^\alpha(\mathbf{AF})$. If possible choose α minimal such that $\mathbf{X}_\alpha$ contains a group G of finite rank that is not in $L(\mathbf{SF})$. Clearly α is not a limit ordinal, so $\mathbf{X}_\alpha = P'L\mathbf{X}_{\alpha-1}$. Then G has an ascending series $\{G_\beta\}$ whose factors lie in $L\mathbf{X}_{\alpha-1}$. These factors also have finite rank, so by the choice of α they are $L(\mathbf{SF})$-groups. Suppose β is the least ordinal for which $G_\beta \notin L(\mathbf{SF})$. Then β is also not a limit ordinal and $G_{\beta-1}$ and $G_\beta/G_{\beta-1}$ both lie in $L(\mathbf{SF})$.

Let H be a finitely generated subgroup of G_β. Then $H/(H \cap G_{\beta-1})$ is soluble-by-finite and $K = H \cap G_{\beta-1} \in L(\mathbf{N} \cap \mathbf{F})\mathbf{SF}$ by Lemma 2.5. There exists therefore a periodic locally nilpotent normal subgroup N of H with $N \leq K$ and K/N soluble-by-finite. Then H/N' is soluble-by-finite and hence minimax by Lemma 2.4. In particular N/N' involves only finitely many primes. But N is a direct product of its primary components, each of which, by finite rank, is a soluble Chernikov group. Thus N is Chernikov and H is soluble-by-finite. Therefore no such β exists and thus no such α exists. Consequently any group G in $\mathbf{X}$ of finite rank lies in $L(\mathbf{SF})$. But if G is also finitely generated, then G is soluble-by-finite and hence is minimax by Lemma 2.4 again.

Now let G be any $\langle P', L\rangle(\mathbf{AF})$-group of finite rank. By the above G is locally (soluble-by-finite and minimax). By Lemma 2.5 there are normal subgroups $H \leq S$ of G with H periodic locally nilpotent, S/H soluble, and G/S finite. Since H has finite rank, its primary components are Chernikov, hypercentral, and soluble. In particular H is hypercentral. If X is a finitely generated subgroup of S, then $X/(H \cap X)$ is soluble. Also X is minimax, so $H \cap X$ involves only finitely many primes. Hence $H \cap X$ is soluble, so X is soluble and S is locally soluble. $\square$

3. Proof of Theorem 1(b)

Lemma 3.1 (see [**2**, 7.2.6]). *Let G be a finite p-group (always p is a prime) and Z a central subgroup of G. Suppose G/Z has finite rank at most r. Then G' has finite rank bounded by a function of r only.*

Let $\sigma(r)$ denote the minimal function of r only such that in Lemma 3.1 for all choices of p, G, and Z we have $\mathrm{rk}(G' \cap Z) \leq \sigma(r)$. Let $\sigma^+(r)$ be the similar function but bounding $\mathrm{rk}(G')$. Clearly $\sigma(r) \leq \sigma^+(r) \leq r + \sigma(r)$. The proof of 7.2.6 in [**2**] yields

$$\sigma(r) \leq \sigma^+(r) \leq r(r-1)/2 + r^2 + m(m-1)/2 \text{ for } m = r(1 - [-\log_2 r]).$$

Lemma 3.2. *Let Z be a central subgroup of the group G with $\mathrm{rk}(G/Z) \leq r$. Then $\mathrm{rk}((G' \cap Z)\gamma^i G/\gamma^i G) \leq \sigma(r)$ and $\mathrm{rk}(G'/\gamma^i G) \leq \sigma^+(r)$ for all $i \geq 2$. In particular, if G is nilpotent, then $\mathrm{rk}(G' \cap Z) \leq \sigma(r)$ and $\mathrm{rk}(G') \leq \sigma^+(r)$.*

Proof: If G is finite and nilpotent, the claims follow easily from Lemma 3.1. In general we may assume that G is nilpotent and finitely generated. Clearly $\mathrm{rk}((G' \cap Z)N/N) \leq \sigma(r)$ anf $\mathrm{rk}(G'N/N) \leq \sigma^+(r)$ for every normal subgroup N of G of finite index.

Let H be any (necessarily) finitely generated subgroup of G'. Then $d(H) = d(H/H')$ and there exists $H' \leq K \leq H$ with H/K finite and $d(H/H') = d(H/K)$. There exists a normal subgroup N of G of finite index with $H \cap N \leq K$; see e.g. [**9**, 2.9]. Then $d(H) = d(H/K) \leq d(HN/N) \leq \mathrm{rk}(G'N/N) \leq \sigma^+(r)$. Consequently $\mathrm{rk}(G') \leq \sigma^+(r)$. Choosing $H \leq G' \cap Z$ a similar proof yields that $\mathrm{rk}(G' \cap Z) \leq \sigma(r)$. □

Lemma 3.3. *Let Z be a central subgroup of the group G with G/Z locally finite and of finite rank at most r. Then $\mathrm{rk}(G' \cap Z) \leq \sigma(r)$ and $\mathrm{rk}(G') \leq r + \sigma(r)$.*

This lemma is 7.2.24 of [**2**] except that there the bound for $\mathrm{rk}(G')$ is $1+4r+3r^3+5r^4+\log_2(r!)(\log_2(r!)-1)/2+2r$(the bound given above for $\sigma^+(r)$).

Proof: Suppose G is finite and let P be a Sylow p-subgroup of G. Then $\mathrm{rk}(P' \cap Z) \leq \sigma(r)$; see Lemma 3.1 and comments. The transfer homomorphism of G into P/P' yields that $P \cap G' \cap \zeta_1(G) = P' \cap \zeta_1(G)$. Therefore $G' \cap Z$ is the direct product over primes p of the $P' \cap Z$. Hence $\mathrm{rk}(G' \cap Z) \leq \sigma(r)$ and $\mathrm{rk}(G') \leq r + \sigma(r)$.

In general we may assume that G is finitely generated. Then Z is finitely generated; also G' is finite (Schur's theorem). Hence there is a subgroup Y of finite index in Z with $G' \cap Y = \langle 1 \rangle$. Then $G' \cap Z \cong (G' \cap Z)Y/Y = (G'Y/Y) \cap (Z/Y)$ and the latter has rank at most $\sigma(r)$ by the finite case. Consequently $\mathrm{rk}(G' \cap Z) \leq \sigma(r)$ and $\mathrm{rk}(G') \leq r + \sigma(r)$. □

Lemma 3.4. *Let A be an abelian normal subgroup of the group G and Z a central subgroup of G.*

(a) *If G/A is finite and $G/Z \in \mathbf{X}$, where $\mathbf{X}$ is a class of groups satisfying $\mathbf{X} = S\mathbf{X} = Q\mathbf{X} = D_0\mathbf{X}$ (that is, $\mathbf{X}$ is subgroup, quotient group, and direct product of two groups closed), then $G' \in \mathbf{XF}$.*

(b) *If G/A is locally finite and $\mathrm{rk}(G/Z) \leq r$, then $\mathrm{rk}(G' \cap Z) \leq \sigma(r)$ and $\mathrm{rk}(G') \leq r + \sigma(r)$.*

(c) *If $Z \leq A$, $\mathrm{rk}(A/Z) \leq r$ and either $d(G/C_G(A)) \leq s$ or $\mathrm{rk}(G/C_G(A)) \leq s$, then $\mathrm{rk}([A, G]) \leq rs$.*

Proof: (a) For each g in G the map $a \mapsto [a, g]$ is an endomorphism of A with $A \cap Z$ in its kernel. Therefore $[A, g] \in QS\mathbf{X}$. If T is a transversal of A to G, then $B = [A, G] = \langle [A, t] : t \in T \rangle \leq A \cap G'$ and B lies in $QD_0\mathbf{X} = \mathbf{X}$. Clearly G/B is centre-by-finite, so G'/B is finite and $G' \in \mathbf{XF}$.

(b) Here we may assume that G and hence A and each subgroup of G is finitely generated. If N is any normal subgroup of G of finite index, then $\mathrm{rk}((G' \cap Z)N/N) \leq \sigma(r)$ by Lemma 3.3. Now $G' \cap Z$ is a finitely generated abelian group, so

$$\mathrm{rk}(G' \cap Z) = d(G' \cap Z) = d(G' \cap Z)/Y$$

for some subgroup Y of $G' \cap Z$ of finite index. Also there is a normal subgroup N of finite index in G with $G' \cap Z \cap N \leq Y$ (even equal if we wish). Consequently $\mathrm{rk}(G' \cap Z) \leq \sigma(r)$ and $\mathrm{rk}(G') \leq r + \sigma(r)$.

(c) This is 7.1.4 and 7.1.5 of [**2**] but the proofs there seem to me to need expanding. In particular in the proof of 7.1.4 that D is G-invariant needs proof. We give below an alternative approach.

If X is a finitely generated subgroup of $[A, G]$ there exist in both cases $H = \langle g_1, g_2, \ldots, g_s \rangle C_G(A) \leq G$ with $X \leq [A, H]$. Now $[A, g_i]$ is a homomorphic image of A/Z and hence has rank at most r. Therefore $B = \langle [A, g_i] : 1 \leq i \leq s \rangle \leq A$ has rank at most rs. If $x, y \in G$ and $a \in A$, then $[a, x]^y = [a, x][a, xy] = [a, x][[a, x]^y, y^{-1}]^{-1}$. Choosing $x = g_i$ and $y = g_j$ or g_j^{-1} shows that B is normalized by each g_j and therefore that B is normal in H. Finally A/B is central in H/B and $X \leq [A, H] \leq B \leq [A, H]$. Therefore $\mathrm{rk}([A, G]) \leq rs$ in both cases. □

Lemma 3.5. *Let Z be a central subgroup of the metabelian-by-finite group G with* $\mathrm{rk}(G/Z) \leq r$. *Then* $\mathrm{rk}(G' \cap Z) \leq r(3r-1)/2 + \sigma(r)$ *and* $\mathrm{rk}(G') \leq r(3r+1)/2 + \sigma(r)$.

Proof: G has normal subgroups $M \geq A \geq M' \geq A' = \langle 1 \rangle$ with $Z \leq A$ and G/M finite. Then $\mathrm{rk}([A, G]) \leq r^2$ by Lemma 3.4(c). Suppose $[A, G] = \langle 1 \rangle$. Then M is nilpotent of class at most 2 and $M/\zeta_1(M)$ is at most r-generator. Therefore $\mathrm{rk}(M') \leq r(r-1)/2$. If $M' = \langle 1 \rangle$, then $\mathrm{rk}(G' \cap Z) \leq \sigma(r)$ by Lemma 3.4(b). Thus in general

$$\mathrm{rk}(G' \cap Z) \leq r^2 + r(r-1)/2 + \sigma(r) = r(3r-1)/2 + \sigma(r)$$

and

$$\mathrm{rk}(G') \leq r(3r+1)/2 + \sigma(r). \qquad \square$$

Lemma 3.6. *Let Z be a central subgroup of the group G, where G/Z is soluble-by-finite, minimax, and of finite rank at most r. Then G' is minimax and*

$$\mathrm{rk}(G' \cap Z) \leq r(3r+1)/2 + 2\sigma(r) \text{ and } \mathrm{rk}(G') \leq 3r(r+1)/2 + 2\sigma(r).$$

Proof: Now G has normal subgroups $Z \leq N \leq M \leq G$ with N nilpotent, M/N abelian, and G/M finite. But $\mathrm{rk}(N') \leq r + \sigma(r)$ by Lemma 3.2. If $N' = \langle 1 \rangle$ then by Lemma 3.5

$$\mathrm{rk}(G' \cap Z) \leq r(3r-1)/2 + \sigma(r).$$

Thus in general

$$\mathrm{rk}(G' \cap Z) \leq r(3r+1)/2 + 2\sigma(r) \text{ and } \mathrm{rk}(G') \leq 3r(r+1)/2 + 2\sigma(r).$$

It remains to prove that G' is minimax. Now N' is minimax by Lemmas 9 and 10 of [**10**], so we may assume that N is abelian. Let $R/Z = \tau(G/Z)$. Then R/Z is periodic minimax and thus is a π-group for some finite set π of primes. Hence $R' \leq G'$ is a π-group ([**6**, 4.12]) and also has finite rank. Therefore R' is minimax and we may assume that $R' = \langle 1 \rangle$. Then each $[R, g]$, as an image of R/Z, is a π-group so $[R, G]$ is a π-group and minimax. If we now assume $[R, G] = \langle 1 \rangle$ and $R = Z$, then by [**5**, 5.2.3] we may choose N and M with M/N finitely generated, say $M = \langle g_1, g_2, \ldots, g_s \rangle N$. Then each $[N, g_i]$ is minimax and hence $[N, M] = \langle [N, g_1], \ldots, [N, g_s] \rangle \leq G'$ is minimax and we may assume that $[N, M] = \langle 1 \rangle$, so M is nilpotent of class at most 2. It follows that M' is finitely generated and hence minimax. Finally G'/M' is minimax by Lemma 3.4(a). Consequently G' is minimax. □

Proof of Theorem 1(b): Firstly G', indeed each subgroup of G, lies in $\langle P', L \rangle(\mathbf{AF})$. Secondly, if X is any finitely generated subgroup of G, then X is soluble-by-finite and minimax by Theorem 1(a). Hence by Lemma 3.6 we have

$$\mathrm{rk}(X' \cap Z) \leq r(3r+1)/2 + 2\sigma(r) \text{ and } \mathrm{rk}(X') \leq 3r(r+1)/2 + 2\sigma(r).$$

This is for all such X and hence

$$\mathrm{rk}(G' \cap Z) \leq r(3r+1)/2 + 2\sigma(r) \text{ and } \mathrm{rk}(G') \leq 3r(r+1)/2 + 2\sigma(r). \quad \square$$

4. Proof of the Corollary

Lemma 4.1. *Let* $\mathbf{X}$ *be a class of groups satisfying*

(a) $\mathbf{X} = S_N\mathbf{X} = Q\mathbf{X} = P\mathbf{X}$,
(b) $G' \in \mathbf{X}$ *whenever* $G/\zeta_1(G) \in \mathbf{X}$, *and*
(c) *the tensor product of two abelian* $\mathbf{X}$*-groups is an* $\mathbf{X}$*-group.*

If $G/\zeta_i(G) \in \mathbf{X}$ *for some* $i \geq 0$, *then* $\gamma^{i+1}G \in \mathbf{X}$.

Here S_N denotes the subnormal-subgroup closure operator and P the poly closure operator. Lemma 4.1 is immediate from Corollary 2 of [**6**, 4.21] and is essentially a result of P. Stroud.

Lemma 4.2. *For* $i = 1, 2$ *let* A_i *be an abelian group with finite Hirsch number* h_i. *Then the tensor product over the integers* $A_1 \odot A_2$ *has Hirsch number at most* h_1h_2. *If also each* $A_i \in$ Min-p *for some prime* p, *then* $A_1 \odot A_2 \in$ Min-p.

Proof: Set $T_i = \tau(A_i) = \bigoplus_{\text{primes } q} T_i(q)$, where $T_i(q)$ is a q-group. Also set $B_i = A_i/T_i$. There is an exact sequence

$$(A_1 \odot T_2) \oplus (T_1 \odot A_2) \to A_1 \odot A_2 \to B_1 \odot B_2 \to 0;$$

see [**3**, 60.3]. Now B_i has Hirsch number h_i and is torsion-free, $B_1 \odot B_2$ is torsion-free ([**3**, 61.4]), $\mathrm{rk}(B_i) = h_i$ and $\mathrm{hn}(B_1 \odot B_2) = \mathrm{rk}(B_1 \odot B_2) \leq h_1h_2$. Further $A_i \odot T_j$ is periodic for all choices of i and j. Thus $(A_1 \odot A_2)/\tau(A_1 \odot A_2) \cong B_1 \odot B_2$ and $\mathrm{hn}(A_1 \odot A_2) \leq h_1h_2$.

Now suppose each $A_i \in$ Min-p and set $r_i = \mathrm{rk}(T_i(p))$. Each r_i is finite. The sequence $0 \to T_1 \to A_1 \to B_1 \to 0$ is pure exact, so

$$0 \to T_1 \odot T_2(p) \to A_1 \odot T_2(p) \to B_1 \odot T_2(p) \to 0$$

is an exact sequence of p-groups ([**3**, 60.4]). The finitely generated subgroups of $B_1 \odot T_2(p)$ are at most h_1r_2-generator. Thus $\mathrm{rk}(B_1 \odot T_2(p)) \leq h_1r_2$. Also $T_1 \odot T_2(p) = T_1(p) \odot T_2(p)$ and hence has rank at most r_1r_2. Therefore $\mathrm{rk}(A_1 \odot T_2(p)) \leq h_1r_2 + r_1r_2$. Now $A_1 \odot T_2 = \bigoplus_q A_1 \odot T_2(q)$ and each $A_1 \odot T_2(q)$ is a q-group. Thus $A_1 \odot T_2(p)$ is the p-primary component of $A_1 \odot T_2$. Therefore $A_1 \odot T_2 \in$ Min-p. Similarly so does $T_1 \odot A_2$. Hence $(A_1 \odot T_2) \oplus (T_1 \odot A_2)$ and $\tau(A_1 \odot A_2)$ both lie in Min-p. Consequently so does $A_1 \odot A_2$. □

Notice that the proof above yields $\mathrm{rk}((A_1 \odot A_2)(p)) \leq h_1r_2 + 2r_1r_2 + r_1h_2$.

To prove the Corollary we have just to check that its five classes satisfy the conditions (a), (b), and (c) of Lemma 4.1. That they all satisfy Lemma 4.1(a) is very easy indeed. That $\mathbf{X}_{\mathrm{fr}}$ and $\mathbf{X}_{\mathrm{mm}}$ satisfy Lemma 4.1(c) is easy and in any case follows from Lemma 10 of [**10**]. The remaining three classes satisfy Lemma 4.1(c) by Lemma 4.2. That $\mathbf{X}_{\mathrm{fr}}$ satisfies Lemma 4.1(b) follows from Theorem 1(b) and that $\mathbf{X}_{\mathrm{mm}}$ does too follows from Lemma 3.6 (note that G here is soluble-by-finite by Theorem 1(a)). It remains to check that the remaining three do too. To see this we merely need to check that $\mathbf{X}_{\mathrm{fh}}$ and Min-p satisfy Lemma 4.1(b).

That $\mathbf{X}_{\mathrm{fh}}$ satisfies Lemma 4.1(b) is immediate from Theorem 3, but one can also see it easily at this stage as follows. Let G/Z have finite Hirsch number h. We may assume $\tau(G) = \langle 1 \rangle$. If $T/Z = \tau(G/Z)$, then T' is locally finite and hence $\langle 1 \rangle$. Then T is abelian and $[T, G]$ is periodic and hence $\langle 1 \rangle$. Thus assume $T = Z$. Hence G has normal subgroups $Z \leq N \leq M$ such that N/Z is torsion-free nilpotent of finite rank at most h, M/N is free abelian of rank at most h, and G/M is finite. Then G/Z has finite rank, so the torsion-free group $G' \cap Z$ does too by Theorem 1. Therefore G' has finite Hirsch number and $\mathbf{X}_{\mathrm{fh}}$ satisfies Lemma 4.1(b).

Let G be a locally finite group in Min-p. Then G contains a special type of maximal p-subgroup P (called a Sylow p-subgroup in [**4**]) with the following properties. P contains an isomorphic copy of every p-subgroup of G and a conjugate of every finite p-subgroup of G. Moreover all such P are isomorphic, indeed are sort of locally conjugate, and if N is a normal subgroup of G, then N and G/N also lie in Min-p with $P \cap N$ is a Sylow p-subgroup of N and PN/N is a Sylow p-subgroup of G/N. See [**4**, Chapter 3, Section A]. Set $r_p(G) = \mathrm{rk}(P)$. Clearly this is finite and does not depend on the choice of P.

Lemma 4.3. *Let Z be a central subgroup of the group G with $G/Z \in L\mathbf{F} \cap \text{Min-}p$. Then $G' \in L\mathbf{F} \cap \text{Min-}p$. Moreover $r_p(G' \cap Z) \leq \sigma(r_p(G/Z))$ and $r_p(G') \leq r_p(G/Z) + \sigma(r_p(G/Z))$.*

Proof: Set $r = r_p(G/Z)$. Let X be a finitely generated of G and set $Y = X \cap Z$. Then X/Y and X' are finite and Y is finitely generated abelian. If $Q = O_p(X' \cap Y)$, then there exists a power q of p such that $Q \cap Y^q = \langle 1 \rangle$.

Let ϕ denote the natural projection of X onto X/Y^q. Then $X\phi$ is finite. Pick a Sylow subgroup P of $X\phi$. Then $\mathrm{rk}(P/(P \cap Y\phi)) \leq r$. The transfer homomorphism of $X\phi$ into P/P' yields that $P' \cap Y\phi = X'\phi \cap Y\phi \cap P = O_p(X'\phi \cap Y\phi)$. Also $\mathrm{rk}(P' \cap Y\phi) \leq \sigma(r)$ by Lemma 3.1 and $Q \cong Q\phi \leq O_p(X'\phi \cap Y\phi)$. Therefore $\mathrm{rk}(Q) \leq \sigma(r)$. It follows that the p-subgroups of $G' \cap Z$ have rank at most $\sigma(r)$ and hence the p-subgroups of G' have rank at most $r + \sigma(r)$. Finally G' is locally finite, $G' \in \text{Min-}p$, $r_p(G' \cap Z) \leq \sigma(r)$, and $r_p(G') \leq r + \sigma(r)$. □

Lemma 4.4. *Let Z be a central subgroup of the group G with $G/Z \in \mathbf{X}_{\mathrm{fh}} \cap \text{Min-}p$. Then $G' \in \mathbf{X}_{\mathrm{fh}} \cap \text{Min-}p$.*

Proof: From the $\mathbf{X}_{\mathrm{fh}}$ case above $G' \in \mathbf{X}_{\mathrm{fh}}$. Set $T/Z = \tau(G/Z)$. Then G/T is soluble-by-finite of finite rank. Also $T/Z \in L\mathbf{F} \cap \text{Min-}p$. Consequently $T' \in L\mathbf{F} \cap \text{Min-}p$ by Lemma 4.3. We may now assume that $T' = \langle 1 \rangle$. In particular G is now soluble-by-finite.

Consider $X = \langle g_1, g_2, \ldots, g_r \rangle T \leq G$, where $r = \mathrm{rk}(G/T)$. Each $[T, g_i]$ is a homomorphic image of T/Z, so $r_p([T, g_i]) \leq r_p(T/Z)$ and $r_p([T, X]) \leq r \cdot r_p(T/Z)$. Hence $[T, G]$ is locally finite and contains a finite maximal elementary abelian p-subgroup (in fact of rank at most $r \cdot r_p(T/Z)$). Therefore $[T, G] \in L\mathbf{F} \cap \text{Min-}p$, by [**4**, 3.1]. Finally $G'[T, G]/[T, G]$ has finite rank by Theorem 1(b) and in particular lies in Min-p. Thus $G' \in \text{Min-}p$. □

It is a consequence of Lemma 4.4 that the class $\mathbf{X}_{\mathrm{fh}} \cap \bigcap_{\text{primes } p} \text{Min-}p$ and the class of finite extensions of soluble FAR groups both satisfy Lemma 4.1(b). (Note that in the latter case G/Z is soluble-by-finite, so G and hence G' are too.) This completes the proof of the Corollary.

5. Proof of Theorem 2

Lemma 5.1. *Let* $\mathbf{X}$ *be a subgroup-closed class of groups and suppose* Z *is a central subgroup of the subgroup* G *of* $\mathrm{GL}(n, F)$ *with* $G/Z \in \mathbf{X}$. *Then* $G'/u(G' \cap Z) \in \mathbf{FX}$.

Here and below n is a positive integer and F is a field. Also for any linear group H, $u(H)$ denotes the unique maximal unipotent normal subgroup of H (see e.g. [**7**] for basic facts about linear groups).

Proof: Clearly we may assume F is algebraically closed. If $X \leq \mathrm{GL}(n, F)$ is irreducible, then $X' \cap \zeta_1(X) \leq \mathrm{SL}(n, F) \cap F^* 1_n$. Hence $X' \cap \zeta_1(X)$ is cyclic of order dividing n. If X is just completely reducible, then $|X' \cap \zeta_1(X)|$ divides $\prod_i n_i$ for some n_j satisfying $\sum_i n_i = n$ and in any case is finite.

Set $U = u(G)$. Then G/U is isomorphic to a completely reducible subgroup of $\mathrm{GL}(n, F)$ and hence

$$(G' \cap Z)/u(G' \cap Z) \cong (G' \cap Z)U/U \leq (G'U/U) \cap \zeta_1(G/U),$$

which is finite. Clearly $G'/(G' \cap Z) \in \mathbf{SX} = \mathbf{X}$. Therefore $G' \in \mathbf{FX}$. □

Proof of Theorem 2: Trivially G is soluble-by-finite and by Lemma 5.1 the group $G'/u(G' \cap Z)$ is a finite extension of a soluble FATR-group. Further $\mathrm{rk}(G')$ is finite by Theorem 1(b). Now $u(G' \cap Z)$ either is torsion-free nilpotent or is a nilpotent p-group of finite exponent, where $p = \mathrm{char}\, F > 0$. Either way $u(G' \cap Z)$ is a nilpotent FATR-group. The theorem follows. □

Example. With $(e_{i,j})$ denoting the standard basis of a matrix ring, set $G = \mathrm{Tr}_1(3, \mathbf{Q})/(1 + \mathbb{Z}e_{3,1})$ (here $\mathbf{Q}$ denotes the rational numbers). Then G is nilpotent of class 2 and $G' = \zeta_1(G) = (1 + \mathbf{Q}e_{3,1})/(1 + \mathbb{Z}e_{3,1})$. Hence $G/\zeta_1(G) \cong \mathbf{Q}^+ \oplus \mathbf{Q}^+$, which is torsion-free of rank 2 and in particular is FATR. But G' is the direct product of Prüfer groups, one for each prime p and as such is not FATR. Thus we cannot omit G linear from Theorem 2. Further here $G/\zeta_1(G)$ is isomorphic to a linear group, so in Theorem 2 we cannot replace G linear by G/Z linear.

6. Hirsch number

Lemma 6.1. *Let* F *be a finite field extension of the rational field* $\mathbf{Q}$. *If* G *is a subgroup of* $\mathrm{GL}(n, F)$ *with a diagonal normal subgroup* A *of finite index and if* $\mathrm{hn}(G) \leq h < \infty$, *then* $\mathrm{rk}(G) \leq h + 2n + [n/2] + 1$.

Proof: Clearly we may assume that A is (Zariski) closed in G. Let E be the algebraic closure of F and H (resp. B) be the closure of G (resp. A) in $\mathrm{GL}(n, E)$. Then $H = XB$ for some finite subgroup X of H; see [**7**, 10.10]. Each Sylow subgroup of X has rank at most $n + [n/2]$ by [**8**] and hence $\mathrm{rk}(X) \le n + [n/2] + 1$ by the Lucchini–Guralnick Theorem (see e.g. [**2**, 6.3.15]). But $G \cap B = A$, so G/A is isomorphic to a section of X and consequently $\mathrm{rk}(G/A) \le n + [n/2] + 1$. Finally A embeds into $(F^*)^{(n)}$ and by [**3**, 127.2] the group F^* is the direct product of a free abelian group and a finite cyclic group. But $\mathrm{hn}(A) \le h$, so $\mathrm{rk}(A) \le h + n$. Therefore $\mathrm{rk}(G) \le h + 2n + 2n + [n/2] + 1$. □

Lemma 6.2. *Let Z be a central subgroup of the locally nilpotent group G with $\mathrm{hn}(G/Z) \le h < \infty$. Then $\mathrm{hn}(G') \le h(h-1)/2$.*

Proof: We may assume that $\tau(G) = \langle 1 \rangle$, so G is now torsion-free. Suppose first that G is finitely generated. The conclusion is trivial if G is abelian so assume otherwise. The upper central factors of G are torsion-free, so G has a normal subgroup $N \ge Z$ with G/N infinite cyclic. Then $\mathrm{hn}(N/Z) = h - 1$, so by induction $\mathrm{hn}(N') \le (h-1)(h-2)/2$. Suppose $N' = \langle 1 \rangle$. Now $G = \langle g \rangle N$ for some $g \in G$, so $G' = [N, g]$ and the latter is an image of N/Z. Thus $\mathrm{hn}(G') \le h - 1$ in this case. Therefore whenever G is finitely generated

$$\mathrm{hn}(G') \le (h-1)(h-2)/2 + (h-1) = h(h-1)/2.$$

In general we now have $\mathrm{hn}(X') \le h(h-1)/2$ for every finitely generated subgroup X of G. Also each X and indeed G itself is nilpotent of class at most $h + 1$. Choose X so that $\mathrm{hn}(X')$ is maximal. If $Y \ge X$ is also a finitely generated subgroup of G, then $X' \le Y'$ with X' subnormal in Y and $\mathrm{hn}(Y') = \mathrm{hn}(X')$. Therefore each y in Y' has a positive power in X'. Further X' is subnormal in G' and so there is a series of finite length from X' to G' with locally finite factors. Consequently $\mathrm{hn}(G') = \mathrm{hn}(X') \le h(h-1)/2$. □

Lemma 6.3. *Let G be a group with an abelian normal subgroup A of finite index and suppose G has a central subgroup Z with $\mathrm{hn}(G/Z) \le h < \infty$. Then $\mathrm{hn}(G') \le h$.*

Proof: We may assume that $\tau(G) = \langle 1 \rangle$ and that $Z = \zeta_1(G) \le A$. Then both A and A/Z are torsion-free, the latter of rank at most h. Let $X = G/C_G(A)$. Writing A additively $A \le \mathbf{Q}A = V$ (tensor product of $\mathbf{Q}$ and A over $\mathbb{Z}$), where V is completely reducible as $\mathbf{Q}X$-module (Maschke's theorem). Thus $V = W \oplus C_V(X)$ as $\mathbf{Q}X$-module for some W and clearly then $[V, X] \le W$ (actually $= W$).

Now $Z \leq C_V(X)$, so $W \cap \mathbf{Q}Z = \langle 0 \rangle$. Hence as a $\mathbb{Z}$-module W embeds into $\mathbf{Q}A/\mathbf{Q}Z \cong \mathbf{Q}(A/Z) \cong \mathbf{Q}^{\mathrm{hn}(A/Z)}$. Clearly $[A,G] \leq [V,X] \leq W$, so $\mathrm{rk}([A,G]) \leq h$. Finally $G/[A,G]$ is centre-by finite, so $G'/[A,G]$ is finite and $\mathrm{hn}(G') = \mathrm{hn}([A,G]) \leq h$. □

Proof of Theorem 3(a): We may assume that $\tau(G) = \langle 1 \rangle$. Then G is soluble-by-finite and G has a normal series $\langle 1 \rangle = G_0 \leq G_1 \leq \cdots \leq G_s \leq G$, where G/G_s is finite, each G_i/G_{i-1} is torsion-free abelian of finite rank, h_i say, $s \leq h$, and $\sum_i h_i = h$. If $C_i = C_G(G_i/G_{i-1})$, then G/C_i is isomorphic to a subgroup H_i of $\mathrm{GL}(h_i, \mathbf{Q})$. There is a finite field extension F of $\mathbf{Q}$ such that for all i the group H_i is triangularizable-by-finite actually over F. Thus there is a normal subgroup U_i/C_i of G/C_i acting unipotently on G_i/G_{i-1} such that G_i/U_i is isomorphic to a diagonal-by-finite subgroup of $\mathrm{GL}(h_i, F)$.

Set $U = \bigcap_i U_i$. Then U acts nilpotently on G_s, so $U \cap G_s$ lies in some finite indexed term of the upper central series of U. Hence U is finite-by-nilpotent (see [**6**, 4.21, Corollary 2]). But $\tau(G) = \langle 1 \rangle$. Therefore U is torsion-free nilpotent and G/U is isomorphic to a diagonal-by-finite subgroup of $\mathrm{GL}(h, F)$. If $\mathrm{hn}(U) = h_U$, then $\mathrm{rk}(U) \leq h_U \leq h$ and $\mathrm{hn}(G/U) \leq h - h_U$. Thus by Lemma 6.1 we have

$$\mathrm{rk}(G) \leq h_U + (h - h_U) + 2h + [h/2] + 1 = 3h + [h/2] + 1. \qquad \square$$

Proof of Theorem 3(b): Again we may assume that $\tau(G) = \langle 1 \rangle$. If $T/Z = \tau(G/Z)$, then $T' \leq \tau(G) = \langle 1 \rangle$. But then $[T,G]$ is also locally finite and hence $\langle 1 \rangle$. Consequently we may also assume that $Z = T$. Since $\mathrm{hn}(G/Z) \leq h$ is finite, there exist normal subgroups $Z \leq N \leq M$ of G with N/Z torsion-free nilpotent with $h_1 = \mathrm{hn}(N/Z)$ finite, M/N free abelian of finite rank h_2, G/M finite, and $h_1 + h_2 = \mathrm{hn}(G/Z) \leq h$; see [**5**, 5.2.3].

Clearly N is nilpotent, so $\mathrm{hn}(N') \leq h_1(h_1-1)/2$ by Lemma 6.2. Pass to G/N' and factor by $\tau(G/N')$. Equivalently just assume N is abelian. If $g \in G$, then $[N,g]$ is an image of N/Z and as such has Hirsch number at most h_1. Further M/N is h_2-generator and abelian. Consequently $\mathrm{hn}([N,M]) \leq h_1 h_2$. Now assume $[N,M] = \langle 1 \rangle$. Then M is nilpotent of class at most 2 with $\mathrm{hn}(M') \leq h_2(h_2-1)/2$. Finally if $M' = \langle 1 \rangle$, then $\mathrm{hn}(G') \leq \mathrm{hn}(G/Z) = h = h_1 + h_2$ by Lemma 6.3. Fitting these pieces together we find

$$\begin{aligned} \mathrm{hn}(G') &\leq h_1(h_1-1)/2 + h_1 h_2 + h_2(h_2-1)/2 + h_1 + h_2 \\ &= (h_1+h_2)^2/2 + (h_1+h_2)/2, \end{aligned}$$

so $\mathrm{hn}(G') \leq h(h+1)/2$. □

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