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ALGEBRAIC REFLEXIVITY OF DIAMETER-PRESERVING LINEAR BIJECTIONS BETWEEN $C(X)$-SPACES

A. Jiménez-Vargas and Fereshteh Sady

Abstract: We prove that if X and Y are first countable compact Hausdorff spaces, then the set of all diameter-preserving linear bijections from $C(X)$ to $C(Y)$ is algebraically reflexive.

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1. Introduction and statement of the result

This paper is concerned with the algebraic reflexivity of the set of all diameter-preserving linear bijections between $C(X)$-spaces. We shall denote by $C(X)$ the Banach algebra of all continuous complex-valued functions on a compact Hausdorff space X with the usual supremum norm.

Our interest focuses on the local behaviour of linear maps on $C(X)$ which preserve the diameter of the ranges of functions in $C(X)$. Let us recall that, for compact Hausdorff spaces X and Y, a map T from $C(X)$ into $C(Y)$ is said to be diameter-preserving if $\operatorname{diam}(T(f)) = \operatorname{diam}(f)$ for all $f \in C(X)$, where $\operatorname{diam}(f)$ denotes the diameter of $f(X)$.

Győry and Molnár ([**11**]) introduced this kind of maps and stated the general form of diameter-preserving linear bijections of $C(X)$, when X is a first countable compact Hausdorff space. Cabello Sánchez ([**4**]) and, independently, González and Uspenskij ([**9**]) extended this description by removing the hypothesis of first countability. Namely, they proved the following:

Theorem 1 ([**4, 9, 11**])**.** *Let X and Y be compact Hausdorff spaces. A linear bijection $T\colon C(X) \to C(Y)$ is diameter-preserving if and only if there exist a homeomorphism $\phi\colon Y \to X$, a linear functional $\mu\colon C(X) \to \mathbb{C}$, and a number $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $\lambda \neq -\mu(1_X)$ such that*

$$T(f)(y) = \lambda f(\phi(y)) + \mu(f)$$

for every $y \in Y$ and $f \in C(X)$.

The statement of Theorem 1 also holds for the algebra of continuous real-valued functions on X.

A problem addressed by different authors is the Banach–Stone type representation of diameter-preserving maps between function spaces. See, for example, the papers by Aizpuru and Rambla [**1**], Barnes and Roy [**2**], Font and Hosseini [**7**], Győry [**10**], Jamshidi and Sady [**13**], and Rao and Roy [**20**].

On the other hand, a linear map T of $C(X)$ into itself is called a local isometry (respectively, local automorphism) if for every $f \in C(X)$, there exists a surjective linear isometry (respectively, automorphism) T_f of $C(X)$, depending on f, such that $T(f) = T_f(f)$.

It is said that the set of all surjective linear isometries (respectively, automorphisms) of $C(X)$ is algebraically reflexive if every local isometry (respectively, local automorphism) of $C(X)$ is a surjective linear isometry (respectively, automorphism) of $C(X)$.

The algebraic reflexivity of both sets of surjective linear isometries and automorphisms of $C(X)$ was stated by Molnár and Zalar in [**18**, Theorem 2.2] whenever X is a first countable compact Hausdorff space. Furthermore, Cabello and Molnár ([**5**]) gave an example where that reflexivity fails even if X lacks first countability at only one point. Afterwards, the algebraic reflexivity of some function spaces has been studied by Botelho and Jamison [**3**], Cabello Sánchez and Molnár [**5**], Dutta and Rao [**6**], Jarosz and Rao [**14**], and Oi [**19**], among others.

Motivated by the precedent considerations, we introduce the following concept.

Definition 1. Let X and Y be compact Hausdorff spaces. A linear map $T\colon C(X) \to C(Y)$ is local diameter-preserving if for every $f \in C(X)$, there exists a diameter-preserving linear bijection $T_f\colon C(X) \to C(Y)$, depending on f, such that $T(f) = T_f(f)$.

We say that the set of all diameter-preserving linear bijections from $C(X)$ to $C(Y)$ is algebraically reflexive if every local diameter-preserving linear map from $C(X)$ to $C(Y)$ is a diameter-preserving bijection.

Our main result is the following.

Theorem 2. *Let X and Y be first countable compact Hausdorff spaces. Then the set of all diameter-preserving linear bijections from $C(X)$ to $C(Y)$ is algebraically reflexive.*

Our proof consists of showing that every local diameter-preserving linear map T from $C(X)$ to $C(Y)$ can be expressed in the form

$$T(f)(y) = \lambda f(\phi(y)) + \mu(f) \quad (y \in Y,\ f \in C(X)),$$

with λ, ϕ, and μ being as in the statement of Theorem 1. Using this representation, it is easily proven that T is surjective.

Our approach is in the line of the proof of the known Holsztyński Theorem [**12**], which provides a Banach–Stone type representation for non-surjective linear isometries of $C(X)$ with the supremum norm. However, the adaptation of the Holsztyński's method to the setting of diameter-preserving linear maps is far from being immediate due to the representation of the diameter-preserving linear bijections from $C(X)$ to $C(Y)$ as sum of a weighted composition operator from $C(X)$ to $C(Y)$ and a linear functional on $C(X)$.

We shall apply the known Gleason–Kahane–Żelazko Theorem [**8, 16, 21**] to prove our main result. A similar strategy was used in the study of local isometries between complex-valued Lipschitz algebras [**15**] or uniform algebras [**5**]. Recently, Li, Peralta, Wang, and Wang ([**17**]) established a spherical variant of the Gleason–Kahane–Żelazko Theorem to analyse weak-local isometries on uniform algebras and Lipschitz algebras.

2. Proof of Theorem 2

Before proving our result, we fix some notation and recall the existence of certain peaking functions. Given a set X, the notation $|X|$ denotes the cardinality of X and 1_X denotes the constant function on X which takes the value 1. For a set X with $|X| \geq 2$, we put

$$\widetilde{X} = \{(x_1, x_2) \in X \times X : x_1 \neq x_2\},$$
$$X_2 = \{\{x_1, x_2\} : (x_1, x_2) \in \widetilde{X}\}.$$

As usual, $\mathbb{T}$ stands for the set of all unimodular complex numbers. We also denote

$$\mathbb{T}^+ = \{e^{it} : t \in [0, \pi[\}.$$

An application of Urysohn's lemma shows that if X is a first countable compact Hausdorff space and $(x_1, x_2) \in \widetilde{X}$, then there exists a continuous function $h_{(x_1,x_2)} : X \to [0, 1]$ with $h^{-1}_{(x_1,x_2)}(\{1\}) = \{x_1\}$ and $h^{-1}_{(x_1,x_2)}(\{0\}) = \{x_2\}$. Hence

$$h_{(x_1,x_2)}(x_1) - h_{(x_1,x_2)}(x_2) = 1 = \operatorname{diam}(h_{(x_1,x_2)})$$

and

$$\{(x, y) \in \widetilde{X} : h_{(x_1,x_2)}(x) - h_{(x_1,x_2)}(y) = 1\} = \{(x_1, x_2)\}.$$

Therefore, given a first countable compact Hausdorff space X and any $\{x_1, x_2\} \in X_2$, we may consider the nonempty sets:

$$\mathcal{F}_{\{x_1,x_2\}} = \{f \in C(X) : |f(x_1) - f(x_2)| = 1 = \operatorname{diam}(f)\},$$

$$\mathcal{F}'_{\{x_1,x_2\}} = \{f \in \mathcal{F}_{\{x_1,x_2\}} : \{\{x,y\} \in X_2 : |f(x) - f(y)| = 1\} = \{\{x_1, x_2\}\}\}.$$

We should note that, since the range of a local diameter-preserving linear map is a subspace without any additional (separating) property, the standard reasoning does not work here in some steps of the proof. Indeed, we need the next two lemmas. The first one provides some functions in $\mathcal{F}'_{\{x_1,x_2\}}$ satisfying an additional condition and the second one shows, in particular, that $C(X)$ is the linear span of $\bigcup_{\{x_1,x_2\}\in X_2} \mathcal{F}_{\{x_1,x_2\}}$.

Lemma 1. *Let X be a first countable compact Hausdorff space and let x_1, x_2, x_3, x_4 be pairwise distinct points in X. Then there exists a function $f \in \mathcal{F}'_{\{x_1,x_2\}}$ for which $f(x_3) = f(x_4)$.*

Proof: We construct the function f in several stages:

(1) Choose $f_0 \in \mathcal{F}'_{\{x_1,x_2\}}$ with values in $[0, 1]$ such that $f_0^{-1}(\{1\}) = \{x_2\}$ and $f_0^{-1}(\{0\}) = \{x_1\}$. If $f_0(x_3) = f_0(x_4)$, then f_0 is the desired function. So we assume that $f_0(x_3) \neq f_0(x_4)$. Put $a = f_0(x_3)$ and $b = f_0(x_4)$, and assume without loss of generality that $a < b$. Clearly, $0 < a < b < 1$.

(2) Let U and V be neighbourhoods of x_3 and x_4, respectively, with $\overline{U} \cap \overline{V} = \emptyset$ and $x_2 \notin \overline{U} \cup \overline{V}$. Choose $g_0 \in C(X)$ satisfying $g_0 \leq 0$, $g_0(x_3) = \ln(b)$, $g_0(x_4) = \ln(a)$, and $g_0 = 0$ on $X\backslash(U \cup V)$. For such a function it suffices to take $h_0, h_1 \in C(X)$ with values in $[0, 1]$ such that $h_0(x_3) = 1$ and $\operatorname{supp}(h_0) \subseteq U$ and similarly $h_1(x_4) = 1$ and $\operatorname{supp}(h_1) \subseteq V$. Then $g_0 = \ln(b)h_0 + \ln(a)h_1$ has the desired properties.

(3) Put $g = e^{g_0}$. Since $g_0 \leq 0$, we have $0 < g \leq 1$. Clearly, $g(x_3) = b$, $g(x_4) = a$, and $g(x_2) = 1$.

(4) Take $f = f_0 g$. Then $f(x_2) = 1$, $f(x_1) = 0$, and

$$f(x_3) = f_0(x_3)g(x_3) = ab = f_0(x_4)g(x_4) = f(x_4).$$

Moreover, for any $x \in X$, we have $f(x) = 1$ only if $f_0(x) = 1$, i.e., $x = x_2$. Similarly, $f(x) = 0$ if and only if $f_0(x) = 0$, i.e., $x = x_1$. Hence $0 < f(x) < 1$ for all $x \notin \{x_1, x_2\}$. This implies that $f \in \mathcal{F}'_{\{x_1,x_2\}}$ and this completes the proof. □

Lemma 2 ([**13**, Lemma 2.1(i)])**.** *Let X be a compact Hausdorff space and $x_1, x_2 \in X$ be distinct. If $f \in C(X)$ such that $0 \leq f \leq 1$ and $f(x_1) = f(x_2)$, then there exists a function $g \in C(X)$ such that both g and $h := \frac{1}{2}f + g$ satisfy $g(x_1) - g(x_2) = 1 = \operatorname{diam}(g)$ and $h(x_1) - h(x_2) = 1 = \operatorname{diam}(h)$. In particular, we have $g, h \in F_{\{x_1,x_2\}}$.*

Let T be a local diameter-preserving linear map from $C(X)$ to $C(Y)$. We have divided the proof of Theorem 2 in several steps.

Step 1. *T is diameter-preserving.*

Proof: Let $f \in C(X)$. By hypothesis, there is a diameter-preserving linear bijection T_f from $C(X)$ to $C(Y)$ such that $T(f) = T_f(f)$. Hence $\operatorname{diam}(T(f)) = \operatorname{diam}(T_f(f)) = \operatorname{diam}(f)$. □

Step 2. *For every $f \in C(X)$, there exist a homeomorphism $\phi_f : Y \to X$, a linear functional μ_f on $C(X)$, and a number $\lambda_f \in \mathbb{T}$ with $\lambda_f \neq -\mu_f(1_X)$ such that*

$$T(f)(y) = \lambda_f f(\phi_f(y)) + \mu_f(f)$$

for all $y \in Y$.

Proof: This follows immediately from Definition 1 and Theorem 1. □

Step 2 will be frequently applied without any explicit mention along the paper. By Step 2, there exists a homeomorphism from Y onto X. Hence $|Y| = |X|$. Since Theorem 2 is easy to verify when $|Y| = 1$, we shall suppose $|Y| \geq 2$ from now on.

Step 3. *For every $(x_1, x_2) \in \widetilde{X}$, the set*

$$\mathcal{B}_{(x_1,x_2)} = \bigcap_{f \in \mathcal{F}_{\{x_1,x_2\}}} \mathcal{B}_{(x_1,x_2),f}$$

is nonempty, where

$$\mathcal{B}_{(x_1,x_2),f} = \{((y_1,y_2),\lambda) \in \widetilde{Y} \times \mathbb{T} : T(f)(y_1) - T(f)(y_2) = \lambda(f(x_1) - f(x_2))\} \qquad (f \in \mathcal{F}_{\{x_1,x_2\}}).$$

Proof: Let $(x_1, x_2) \in \widetilde{X}$. We shall first prove that for each $f \in \mathcal{F}_{\{x_1,x_2\}}$, the set $\mathcal{B}_{(x_1,x_2),f}$ is a nonempty closed subset of $\widetilde{Y} \times \mathbb{T}$. Fix $f \in \mathcal{F}_{\{x_1,x_2\}}$ and take $y_1, y_2 \in Y$ such that $\phi_f(y_1) = x_1$ and $\phi_f(y_2) = x_2$. Clearly, $y_1 \neq y_2$. We have

$$T(f)(y_1) - T(f)(y_2) = \lambda_f(f(\phi_f(y_1)) - f(\phi_f(y_2))) = \lambda_f(f(x_1) - f(x_2)),$$

and thus $((y_1, y_2), \lambda_f) \in \mathcal{B}_{(x_1,x_2),f}$. Therefore $\mathcal{B}_{(x_1,x_2),f}$ is nonempty, and to prove that it is closed in $\widetilde{Y} \times \mathbb{T}$, assume that $\{((y_i, z_i), \lambda_i)\}_{i \in I}$ is a net in $\mathcal{B}_{(x_1,x_2),f}$ converging to $((y_1, y_2), \lambda)$ in $\widetilde{Y} \times \mathbb{T}$ equipped with the product topology. We have

$$T(f)(y_i) - T(f)(z_i) = \lambda_i(f(x_1) - f(x_2))$$

for all $i \in I$. Since $T(f) \in C(Y)$, we infer that

$$T(f)(y_1) - T(f)(y_2) = \lambda(f(x_1) - f(x_2)),$$

and thus $((y_1, y_2),\lambda)\in\mathcal{B}_{(x_1,x_2),f}$. A similar reasoning shows that $\mathcal{B}_{(x_1,x_2),f}$ is a nonempty closed subset of $Y^2 \times \mathbb{T}$.

We shall prove next that the family $\{\mathcal{B}_{(x_1,x_2),f} : f \in \mathcal{F}_{\{x_1,x_2\}}\}$ has the finite intersection property. Let $n \in \mathbb{N}$ and $f_1, \dots, f_n \in \mathcal{F}_{\{x_1,x_2\}}$. Define the function $g\colon X \to \mathbb{C}$ by

$$g(x) = \frac{1}{n}\sum_{i=1}^{n}(\overline{f_i(x_1) - f_i(x_2)})f_i(x).$$

It is clear that $g \in C(X)$ with $g(x_1) - g(x_2) = 1$. By Step 2, consider $\lambda_g \in \mathbb{T}$ and take $y_1, y_2 \in Y$ such that $\phi_g(y_1) = x_1$ and $\phi_g(y_2) = x_2$. Clearly, $y_1 \neq y_2$. We have

$$T(g)(y_1)-T(g)(y_2) = \lambda_g(g(\phi_g(y_1))-g(\phi_g(y_2))) = \lambda_g(g(x_1)-g(x_2))=\lambda_g.$$

Using the linearity of T, we can write

$$\lambda_g = T(g)(y_1)-T(g)(y_2)=\frac{1}{n}\sum_{i=1}^{n}(\overline{f_i(x_1) - f_i(x_2)})(T(f_i)(y_1)-T(f_i)(y_2)).$$

By Step 1, note that

$$\begin{aligned}|(\overline{f_i(x_1) - f_i(x_2)})(T(f_i)(y_1) - T(f_i)(y_2))| &= |T(f_i)(y_1) - T(f_i)(y_2)| \\ &\leq \operatorname{diam}(T(f_i)) = \operatorname{diam}(f_i) = 1\end{aligned}$$

for every $i \in \{1, \dots, n\}$. By the strict convexity of $\mathbb{C}$, it follows that

$$T(f_i)(y_1) - T(f_i)(y_2) = \lambda_g(f_i(x_1) - f_i(x_2))$$

for all $i \in \{1, \dots, n\}$, and thus $((y_1, y_2), \lambda_g) \in \bigcap_{i=1}^{n} \mathcal{B}_{(x_1,x_2),f_i}$, as desired.

Hence $\{\mathcal{B}_{(x_1,x_2),f} : f \in \mathcal{F}_{\{x_1,x_2\}}\}$ is a family of closed subsets of the compact space $Y^2 \times \mathbb{T}$ with the finite intersection property. Therefore there exists $((y_1, y_2), \lambda) \in Y^2 \times \mathbb{T}$ such that

$$T(f)(y_1) - T(f)(y_2) = \lambda(f(x_1) - f(x_2))$$

for any $f\in\mathcal{F}_{\{x_1,x_2\}}$. This implies $y_1 \neq y_2$ and thus $((y_1, y_2), \lambda) \in \mathcal{B}_{(x_1,x_2)}$. □

Step 4. *For every $(x_1, x_2) \in \widetilde{X}$, there exist $(y_1, y_2) \in \widetilde{Y}$ and $\lambda \in \mathbb{T}^+$ such that*

$$\mathcal{B}_{(x_1,x_2)} = \{((y_1, y_2), \lambda), ((y_2, y_1), -\lambda)\}.$$

Proof: Let $(x_1,x_2)\in\widetilde{X}$. By Step 3, the set $\mathcal{B}_{(x_1,x_2)}$ is nonempty. Hence we can take an element $((y_1,y_2),\lambda)\in\mathcal{B}_{(x_1,x_2)}$. Note that $((y_2,y_1),-\lambda)\in\mathcal{B}_{(x_1,x_2)}$. Let $((z_1,z_2),\beta)\in\mathcal{B}_{(x_1,x_2)}$ be arbitrary. We have

$$T(f)(y_1)-T(f)(y_2)=\lambda(f(x_1)-f(x_2)),$$
$$T(f)(z_1)-T(f)(z_2)=\beta(f(x_1)-f(x_2)),$$

for all $f\in\mathcal{F}_{\{x_1,x_2\}}$. Fix any $f\in\mathcal{F}'_{\{x_1,x_2\}}$. Using Step 2, we deduce

$$\lambda_f(f(\phi_f(y_1))-f(\phi_f(y_2)))=\lambda(f(x_1)-f(x_2)),$$
$$\lambda_f(f(\phi_f(z_1))-f(\phi_f(z_2)))=\beta(f(x_1)-f(x_2)).$$

Since $f\in\mathcal{F}'_{\{x_1,x_2\}}$ and

$$|f(\phi_f(y_1))-f(\phi_f(y_2))|=|f(\phi_f(z_1))-f(\phi_f(z_2))|=1,$$

we derive from above that

$$\{(\phi_f(y_1),\phi_f(y_2)),(\phi_f(z_1),\phi_f(z_2))\}\subseteq\{(x_1,x_2),(x_2,x_1)\}.$$

We have four possibilities:

(1) $x_1=\phi_f(y_1),\ x_2=\phi_f(y_2),\ x_1=\phi_f(z_1),\ x_2=\phi_f(z_2)$.
(2) $x_1=\phi_f(y_1),\ x_2=\phi_f(y_2),\ x_1=\phi_f(z_2),\ x_2=\phi_f(z_1)$.
(3) $x_1=\phi_f(y_2),\ x_2=\phi_f(y_1),\ x_1=\phi_f(z_2),\ x_2=\phi_f(z_1)$.
(4) $x_1=\phi_f(y_2),\ x_2=\phi_f(y_1),\ x_1=\phi_f(z_1),\ x_2=\phi_f(z_2)$.

Using the injectivity of ϕ_f, we deduce that

$$((z_1,z_2),\beta)\in\{((y_1,y_2),\lambda),((y_2,y_1),-\lambda)\}.$$

Therefore

$$\mathcal{B}_{(x_1,x_2)}=\{((y_1,y_2),\lambda),((y_2,y_1),-\lambda)\}.$$

Finally, notice that either $\lambda\in\mathbb{T}^+$ or $-\lambda\in\mathbb{T}^+$. □

Step 5. *For every* $(x_1,x_2)\in\widetilde{X}$*, the set*

$$\mathcal{A}_{(x_1,x_2)}=\{(y_1,y_2)\in\widetilde{Y}\mid\exists\lambda\in\mathbb{T}^+:((y_1,y_2),\lambda)\in\mathcal{B}_{(x_1,x_2)}\}$$

is a singleton. Let $\Gamma\colon\widetilde{X}\to\widetilde{Y}$ *be the map given by*

$$\{\Gamma(x_1,x_2)\}=\mathcal{A}_{(x_1,x_2)}.$$

Furthermore, $(y_2,y_1)=\Gamma(x_2,x_1)$ *if* $(y_1,y_2)=\Gamma(x_1,x_2)$.

Proof: Given $(x_1,x_2)\in\widetilde{X}$, the set $\mathcal{A}_{(x_1,x_2)}$ is a singleton by Step 4, say $\mathcal{A}_{(x_1,x_2)}=\{(y_1,y_2)\}$. Hence $\Gamma(x_1,x_2)=(y_1,y_2)\in\widetilde{Y}$. Let (x_1,x_2),$(x_3,x_4)\in\widetilde{X}$ be such that $(x_1,x_2)=(x_3,x_4)$. Let $\Gamma(x_1,x_2)=(y_1,y_2)\in\widetilde{Y}$. Hence $(y_1,y_2)\in\mathcal{A}_{(x_1,x_2)}$ and therefore there exists $\lambda\in\mathbb{T}^+$ such that $((y_1,y_2),\lambda)\in\mathcal{B}_{(x_1,x_2)}$. It follows that $((y_1,y_2),\lambda)\in\mathcal{B}_{(x_3,x_4)}$, hence $(y_1,y_2)\in\mathcal{A}_{(x_3,x_4)}$

and so $(y_1, y_2) = \Gamma(x_3, x_4)$. Consequently, $\Gamma(x_1, x_2) = \Gamma(x_3, x_4)$. This justifies that the map $\Gamma\colon \widetilde{X} \to \widetilde{Y}$ is well-defined.

For the last statement, let $(y_1, y_2) = \Gamma(x_1, x_2)$. Then $(y_1, y_2) \in \mathcal{A}_{(x_1,x_2)}$ and therefore there exists $\lambda \in \mathbb{T}^+$ such that $((y_1, y_2), \lambda) \in \mathcal{B}_{(x_1,x_2)}$. It follows that $((y_2, y_1), \lambda) \in \mathcal{B}_{(x_2,x_1)}$, hence $(y_2, y_1) \in \mathcal{A}_{(x_2,x_1)}$ and thus $(y_2, y_1) = \Gamma(x_2, x_1)$, as required. □

Step 6. *If* $(x_1, x_2) \in \widetilde{X}$ *and* $(y_1, y_2) = \Gamma(x_1, x_2)$, *then*

$$T(f)(y_1) = T(f)(y_2)$$

for all $f \in C(X)$ *such that* $f(x_1) = f(x_2)$.

Proof: Let $(x_1, x_2) \in \widetilde{X}$ and $(y_1, y_2) = \Gamma(x_1, x_2)$. By Step 5, there is a $\beta(x_1, x_2) \in \mathbb{T}^+$ such that

$$T(h)(y_1) - T(h)(y_2) = \beta(x_1, x_2)(h(x_1) - h(x_2))$$

for all $h \in \mathcal{F}_{\{x_1,x_2\}}$. Let f be in $C(X)$ with $f(x_1) = f(x_2)$. Assume first that $0 \le f \le 1$. By Lemma 2, we can take a function $g \in \mathcal{F}_{\{x_1,x_2\}}$ such that $(1/2)f + g \in \mathcal{F}_{\{x_1,x_2\}}$. Therefore

$$T\left(\frac{1}{2}f + g\right)(y_1) - T\left(\frac{1}{2}f + g\right)(y_2) = \beta(x_1, x_2)\left(\left(\frac{1}{2}f + g\right)(x_1) - \left(\frac{1}{2}f + g\right)(x_2)\right).$$

Using the linearity of T and the equality

$$T(g)(y_1) - T(g)(y_2) = \beta(x_1, x_2)(g(x_1) - g(x_2)),$$

we get

$$T(f)(y_1) = T(f)(y_2).$$

If f is arbitrary, consider the decomposition

$$f = (\operatorname{Re} f)^+ - (\operatorname{Re} f)^- + i[(\operatorname{Im} f)^+ - (\operatorname{Im} f)^-],$$

apply the previous case to each one of the four functions of the decomposition $f/(1+||f||_\infty)$, and the same conclusion is achieved by using the linearity of T. □

Some arguments used in Steps 7, 8, and 9 below appear for the first time in the papers [**4, 5**].

Step 7. *For every* $(x_1, x_2) \in \widetilde{X}$, *there exists a number* $\lambda(x_1, x_2) \in \mathbb{T}^+$ *such that*

$$T(f)(y_1) - T(f)(y_2) = \lambda(x_1, x_2)(f(x_1) - f(x_2))$$

for all $f \in C(X)$, *where* $(y_1, y_2) = \Gamma(x_1, x_2)$. *Furthermore,* $\lambda(x_1, x_2) = \lambda(x_2, x_1)$.

Proof: Let $(x_1, x_2) \in \widetilde{X}$ and $(y_1, y_2) = \Gamma(x_1, x_2)$. Let $\lambda(x_1, x_2)$ be the number given by

$$\lambda(x_1, x_2) = T(f)(y_1) - T(f)(y_2),$$

where f is any function in $C(X)$ which satisfies $f(x_1) - f(x_2) = 1$. The number $\lambda(x_1, x_2)$ does not depend on such a function f by Step 6, and it is well-defined. Using the homogeneity of T, we may deduce easily that

$$T(f)(y_1) - T(f)(y_2) = \lambda(x_1, x_2)(f(x_1) - f(x_2))$$

for all $f \in C(X)$.

Since $(y_1, y_2) = \Gamma(x_1, x_2)$, Step 5 gives a $\lambda \in \mathbb{T}^+$ such that

$$T(f)(y_1) - T(f)(y_2) = \lambda(f(x_1) - f(x_2))$$

for all $f \in \mathcal{F}_{\{x_1,x_2\}}$. In particular, taking $f = h_{(x_1,x_2)}$ yields

$$\lambda(x_1, x_2) = T(h_{(x_1,x_2)})(y_1) - T(h_{(x_1,x_2)})(y_2) = \lambda,$$

and so $\lambda(x_1, x_2) \in \mathbb{T}^+$.

Similarly, since $(y_2, y_1) = \Gamma(x_2, x_1)$ by Step 5, we have

$$T(f)(y_2) - T(f)(y_1) = \lambda(x_2, x_1)(f(x_2) - f(x_1))$$

for all $f \in C(X)$. Combining the equations obtained, we infer that

$$\begin{aligned}\lambda(x_1, x_2)(f(x_1) - f(x_2)) &= T(f)(y_1) - T(f)(y_2)\\ &= -(T(f)(y_2) - T(f)(y_1))\\ &= -\lambda(x_2, x_1)(f(x_2) - f(x_1))\\ &= \lambda(x_2, x_1)(f(x_1) - f(x_2))\end{aligned}$$

for all $f \in C(X)$, and taking $f = h_{(x_1,x_2)}$ yields $\lambda(x_1, x_2) = \lambda(x_2, x_1)$. □

Step 8. *The map* Γ *is a bijection from* $\widetilde{X}$ *to* $\bigcup_{(x_1,x_2)\in\widetilde{X}} \mathcal{A}_{(x_1,x_2)}$.

Proof: Let $(y_1, y_2) \in \bigcup_{(x_1,x_2)\in\widetilde{X}} \mathcal{A}_{(x_1,x_2)}$. Then $(y_1, y_2) \in \mathcal{A}_{(x_1,x_2)}$ for some $(x_1, x_2) \in \widetilde{X}$. By Step 5, $\mathcal{A}_{(x_1,x_2)} = \{(y_1, y_2)\}$ and thus $\Gamma(x_1, x_2) = (y_1, y_2)$ by the definition of Γ. This proves the surjectivity of Γ.

To prove its injectivity, let $(x_1, x_2), (x_3, x_4) \in \widetilde{X}$ be such that

$$(y_1, y_2) = \Gamma(x_1, x_2) = \Gamma(x_3, x_4),$$

where $(y_1, y_2) \in \bigcup_{(x_1,x_2)\in\widetilde{X}} \mathcal{A}_{(x_1,x_2)}$. By Step 7, we have

$$\lambda(x_1, x_2)(f(x_1)-f(x_2)) = T(f)(y_1)-T(f)(y_2) = \lambda(x_3, x_4)(f(x_3)-f(x_4))$$

for all $f \in C(X)$, with $\lambda(x_1, x_2), \lambda(x_3, x_4) \in \mathbb{T}^+$. Substituting any function $f \in \mathcal{F}'_{\{x_1,x_2\}}$, we deduce that $\{x_3, x_4\} = \{x_1, x_2\}$. This implies that either $(x_1, x_2) = (x_4, x_3)$ or $(x_1, x_2) = (x_3, x_4)$. In the former case, we would have

$$\begin{aligned}\lambda(x_1, x_2)(f(x_1) - f(x_2)) &= \lambda(x_3, x_4)(f(x_2) - f(x_1)) \\ &= -\lambda(x_3, x_4)(f(x_1) - f(x_2))\end{aligned}$$

for all $f \in C(X)$. In particular, for $f = h_{(x_1,x_2)}$ we would obtain $\lambda(x_1, x_2) = -\lambda(x_3, x_4)$, which is impossible. Therefore $(x_1, x_2) = (x_3, x_4)$. □

Step 9. *Let* $(x_1, x_2), (x_3, x_4) \in \widetilde{X}$, $(y_1, y_2) = \Gamma(x_1, x_2)$, *and* $(y_3, y_4) = \Gamma(x_3, x_4)$. *Then*

$$|\{x_1, x_2\} \cap \{x_3, x_4\}| = |\{y_1, y_2\} \cap \{y_3, y_4\}|.$$

In others words, if $\Lambda_X : \widetilde{X} \to X_2$ *and* $\Lambda_Y : \widetilde{Y} \to Y_2$ *are the maps defined by* $\Lambda_X(x_1, x_2) = \{x_1, x_2\}$ *and* $\Lambda_Y(y_1, y_2) = \{y_1, y_2\}$, *respectively, we have*

$$|\Lambda_X(x_1, x_2) \cap \Lambda_X(x_3, x_4)| = |\Lambda_Y(\Gamma(x_1, x_2)) \cap \Lambda_Y(\Gamma(x_3, x_4))|$$

for all $(x_1, x_2), (x_3, x_4) \in \widetilde{X}$.

Proof: Firstly, assume $|\{x_1, x_2\} \cap \{x_3, x_4\}| = 2$. Then $(x_1, x_2) \in \{(x_3, x_4), (x_4, x_3)\}$, hence $(y_1, y_2) \in \{\Gamma(x_3, x_4), \Gamma(x_4, x_3)\} = \{(y_3, y_4), (y_4, y_3)\}$ and thus

$$|\{y_1, y_2\} \cap \{y_3, y_4\}| = 2.$$

Secondly, if $|\{x_1, x_2\} \cap \{x_3, x_4\}| = 1$, then $|\{y_1, y_2\} \cap \{y_3, y_4\}| \leq 1$ by the injectivity of Γ, and therefore

$$|\{y_1, y_2\} \cap \{y_3, y_4\}| = 1.$$

Indeed, we can assume without loss of generality that $x_1 = x_3$ and $x_2 \neq x_4$, and assume on the contrary that $|\{y_1, y_2\} \cap \{y_3, y_4\}| = 0$. By Step 7, we have the equations

$$\begin{aligned}T(f)(y_1) - T(f)(y_2) &= \lambda(x_1, x_2)(f(x_1) - f(x_2)), \\ T(f)(y_3) - T(f)(y_4) &= \lambda(x_1, x_4)(f(x_1) - f(x_4)),\end{aligned}$$

for all $f \in C(X)$. Since the finite sets in a first countable compact space X are G_δ-sets, it is possible to take a continuous function $f : X \to [0, 1]$ such that $f^{-1}(\{1\}) = \{x_1\}$ and $f^{-1}(\{0\}) = \{x_2, x_4\}$ and, consequently,

$$\{\{x, y\} \in X_2 : |f(x) - f(y)| = 1\} = \{\{x_1, x_2\}, \{x_1, x_4\}\}.$$

From the equations, it follows that

$$\begin{aligned}\lambda_f(f(\phi_f(y_1)) - f(\phi_f(y_2))) &= \lambda(x_1, x_2)(f(x_1) - f(x_2)), \\ \lambda_f(f(\phi_f(y_3)) - f(\phi_f(y_4))) &= \lambda(x_1, x_4)(f(x_1) - f(x_4)),\end{aligned}$$

which imply that $\{\phi_f(y_1), \phi_f(y_2)\}, \{\phi_f(y_3), \phi_f(y_4)\} \in \{\{x_1, x_2\}, \{x_1, x_4\}\}$.

In any case, we deduce that $\phi_f(y_i) = x_1 = \phi_f(y_j)$ for some $i \in \{1,2\}$ and $j\in\{3,4\}$ with $i\neq j$. Since ϕ_f is injective, we get $y_i = y_j$, a contradiction.

Finally, assume $|\{x_1,x_2\}\cap\{x_3,x_4\}| = 0$. Then $|\{y_1,y_2\}\cap\{y_3,y_4\}| \leq 1$ by the injectivity of Γ, and we shall prove that

$$|\{y_1,y_2\}\cap\{y_3,y_4\}| = 0.$$

Assume on the contrary that $y_1 = y_3$ and $y_2 \neq y_4$ (the other cases are proved in a similar form). Then we have the two equations:

$$T(f)(y_1) - T(f)(y_2) = \lambda(x_1,x_2)(f(x_1) - f(x_2)),$$
$$T(f)(y_1) - T(f)(y_4) = \lambda(x_3,x_4)(f(x_3) - f(x_4)),$$

for all $f \in C(X)$. It follows that

$$T(f)(y_4)-T(f)(y_2) = \lambda(x_1,x_2)(f(x_1)-f(x_2))-\lambda(x_3,x_4)(f(x_3)-f(x_4))$$

for all $f \in C(X)$. Since $\{x_1,x_2\}\cap\{x_3,x_4\} = \emptyset$, Lemma 1 provides a function $f \in \mathcal{F}'_{\{x_1,x_2\}}$ satisfying $f(x_3) = f(x_4)$. Hence we have

$$\lambda_f(f(\phi_f(y_4)) - f(\phi_f(y_2))) = \lambda(x_1,x_2)(f(x_1) - f(x_2)),$$

which implies that $\{\phi_f(y_4),\phi_f(y_2)\} = \{x_1,x_2\}$. Using the first one of the above-mentioned equations, we also obtain $\{\phi_f(y_1),\phi_f(y_2)\} = \{x_1,x_2\}$. These equalities imply that $\phi_f(y_4) = \phi_f(y_1)$ and, since ϕ_f is injective, we get $y_4 = y_1$, hence $y_4 = y_3$, a contradiction. □

Step 10. *Assume $|X| \geq 3$. For each $x \in X$ and any $(x_1,x_2) \in \widetilde{X}$ with $x_1 \neq x \neq x_2$, there exists a unique point, depending only on x and denoted by $\varphi(x)$, in the intersection*

$$\Lambda_Y(\Gamma(x,x_1)) \cap \Lambda_Y(\Gamma(x,x_2)).$$

The map $\varphi\colon X \to Y$ defined in this way is injective and we have that $\{\varphi(x_1),\varphi(x_2)\} = \Lambda_Y(\Gamma(x_1,x_2))$ for every $(x_1,x_2) \in \widetilde{X}$.

Proof: Let $x \in X$ and let $x_1,x_2 \in X$ be with $x_1 \neq x_2$ and $x_1 \neq x \neq x_2$. Let y be the unique point of the set $\Lambda_Y(\Gamma(x,x_1)) \cap \Lambda_Y(\Gamma(x,x_2))$ (see Step 9).

We claim that $y \in \Lambda_Y(\Gamma(x,x_3))$ for every $x_3 \in X$ with $x_3 \neq x$, which shows that y does not depend on x_1 and x_2 and thus it depends only on x. Indeed, if $|X| = 3$, this is obvious. Assume $|X| \geq 4$. Pick $x_3 \in X\setminus\{x,x_1,x_2\}$ and suppose on the contrary that $y \notin \Lambda_Y(\Gamma(x,x_3))$. We can write $\Lambda_Y(\Gamma(x,x_1)) = \{y,y_1\}$ and $\Lambda_Y(\Gamma(x,x_2)) = \{y,y_2\}$ for some $y_1,y_2 \in Y$ with $y_1 \neq y \neq y_2$. In light of Step 9, we obtain $y_1 \neq y_2$. Since the cardinal of both sets $\Lambda_Y(\Gamma(x,x_3)) \cap \Lambda_Y(\Gamma(x,x_1))$ and $\Lambda_Y(\Gamma(x,x_3)) \cap \Lambda_Y(\Gamma(x,x_2))$ is one, we deduce that $\Lambda_Y(\Gamma(x,x_3)) = \{y_1,y_2\}$. This implies that $\Gamma(x,x_3) = (y_1,y_2)$ or $\Gamma(x,x_3) = (y_2,y_1)$. We shall only prove the first case and the other is similarly proven. Since $\lambda(x,x_3),\lambda(x,x_1),\lambda(x,x_2)\in\mathbb{T}^+$, an easy argument shows that

$$\begin{aligned}\lambda(x,x_3)(f(x)-f(x_3)) &= T(f)(y_1)-T(f)(y_2)\\ &= (T(f)(y_1)-T(f)(y))+(T(f)(y)-T(f)(y_2))\\ &= \lambda(x,x_1)(f(x)-f(x_1))+\lambda(x,x_2)(f(x)-f(x_2))\end{aligned}$$

for all $f\in C(X)$. Taking suitable functions $f\in C(X)$, we can deduce that

$$\lambda(x,x_3)=\lambda(x,x_1)=\lambda(x,x_2),$$

and so $f(x)=f(x_1)+f(x_2)-f(x_3)$ for all $f\in C(X)$, which is impossible. This proves our claim.

We shall next prove the injectivity of φ. Suppose first that $|X|=3$, say $X=\{x_1,x_2,x_3\}$. If $\varphi(x_1)=\varphi(x_2)=y_1$, then $y_1\in\Lambda_Y(\Gamma(x_1,x_2))\cap\Lambda_Y(\Gamma(x_1,x_3))\cap\Lambda_Y(\Gamma(x_2,x_3))$. As the cardinality of each one of the three sets in this intersection is 2, there are $y_2,y_3,y_4\in Y\setminus\{y_1\}$ such that $\Lambda_Y(\Gamma(x_1,x_2))=\{y_1,y_2\}$, $\Lambda_Y(\Gamma(x_1,x_3))=\{y_1,y_3\}$, and $\Lambda_Y(\Gamma(x_2,x_3))=\{y_1,y_4\}$. Applying Step 9 yields $y_2\neq y_3\neq y_4\neq y_2$, and thus $|Y|\geq 4$ which contradicts that $|X|=|Y|$.

Assume now $|X|\geq 4$. Let $x_1,x_2\in X$ be with $x_1\neq x_2$ and suppose $\varphi(x_1)=\varphi(x_2)=y_2$. Take $\{z_1,z_2\}\in X_2$ such that $\{z_1,z_2\}\cap\{x_1,x_2\}=\emptyset$. We have $y_2\in\Lambda_Y(\Gamma(x_1,z_1))\cap\Lambda_Y(\Gamma(x_2,z_2))$; but since $|\Lambda_X(x_1,z_1)\cap\Lambda_X(x_2,z_2)|=0$, we have $|\Lambda_Y(\Gamma(x_1,z_1))\cap\Lambda_Y(\Gamma(x_2,z_2))|=0$ by Step 9, a contradiction. This completes the proof that φ is injective.

For the second assertion, note that if $(x_1,x_2)\in\widetilde{X}$, then $\varphi(x_1)$ and $\varphi(x_2)$ are distinct and belong to $\Lambda_Y(\Gamma(x_1,x_2))$ (see Step 5). Hence $\{\varphi(x_1),\varphi(x_2)\}=\Lambda_Y(\Gamma(x_1,x_2))$. □

Step 11. *There exist a nonempty subset $Y_0\subseteq Y$ and a bijection $\phi_0\colon Y_0\to X$ such that $\{y_1,y_2\}=\Lambda_Y(\Gamma(\phi_0(y_1),\phi_0(y_2)))$ for all $y_1,y_2\in Y_0$ with $y_1\neq y_2$.*

Proof: Assume first that $|X|=2$. Then $|Y|=2$ by Step 2. Hence $X=\{x_1,x_2\}$ and $Y=\{y_1,y_2\}$ for certain $(x_1,x_2)\in\widetilde{X}$ and $(y_1,y_2)\in\widetilde{Y}$. Clearly, $\widetilde{X}=\{(x_1,x_2),(x_2,x_1)\}$ and $\widetilde{Y}=\{(y_1,y_2),(y_2,y_1)\}$. Since Γ is a map from $\widetilde{X}$ to $\widetilde{Y}$, we have $\Lambda_Y(\Gamma(x_1,x_2))=\{y_1,y_2\}$. Take $Y_0=Y$ and the bijection $\phi_0\colon Y_0\to X$ defined by $\phi_0(y_1)=x_1$ and $\phi_0(y_2)=x_2$, and the proof is finished if $|X|=2$.

Assume now that $|X|\geq 3$. Let $\varphi\colon X\to Y$ be the injective map defined in Step 10. Then $Y_0=\varphi(X)$ and $\phi_0=\varphi^{-1}\colon Y_0\to X$ satisfy the required conditions. □

Step 12. *There exists a number $\lambda \in \mathbb{T}$ such that*

$$T(f)(y_1) - T(f)(y_2) = \lambda(f(\phi_0(y_1)) - f(\phi_0(y_2)))$$

for all $y_1, y_2 \in Y_0$ and $f \in C(X)$.

Proof: Let $Y_0 \subseteq Y$ and $\phi_0 \colon Y_0 \to X$ be the set and the bijection given in Step 11. Let $y_1, y_2 \in Y_0$ with $y_1 \neq y_2$. By Step 11, $\{y_1, y_2\} = \Lambda_Y(\Gamma(\phi_0(y_1), \phi_0(y_2)))$. Hence either $\Gamma(\phi_0(y_1), \phi_0(y_2)) = (y_1, y_2)$ or else $\Gamma(\phi_0(y_1), \phi_0(y_2)) = (y_2, y_1)$. By Step 7, we have

$$T(f)(y_1) - T(f)(y_2) = \pm\lambda(\phi_0(y_1), \phi_0(y_2))(f(\phi_0(y_1)) - f(\phi_0(y_2)))$$

for all $f \in C(X)$, where $\lambda(\phi_0(y_1), \phi_0(y_2)) \in \mathbb{T}^+$. Put $\beta(\phi_0(y_1), \phi_0(y_2)) \in \{\pm\lambda(\phi_0(y_1), \phi_0(y_2))\}$.

We now claim that $\beta(\phi_0(y_1), \phi_0(y_2))$ does not depend on the variables y_1, y_2. This is clear when $|Y_0| = 2$ because $\beta(\phi_0(y_1), \phi_0(y_2)) = \beta(\phi_0(y_2), \phi_0(y_1))$ by Step 7. Otherwise, let $y_3 \in Y_0$ be with $y_3 \notin \{y_1, y_2\}$. We have the equation

$$\begin{aligned}
\beta(\phi_0(y_1), \phi_0(y_2))&(f(\phi_0(y_1)) - f(\phi_0(y_2))) \\
&= T(f)(y_1) - T(f)(y_2) \\
&= (T(f)(y_1) - T(f)(y_3)) + (T(f)(y_3) - T(f)(y_2)) \\
&= \beta(\phi_0(y_1), \phi_0(y_3))(f(\phi_0(y_1)) - f(\phi_0(y_3))) \\
&\quad + \beta(\phi_0(y_3), \phi_0(y_2))(f(\phi_0(y_3)) - f(\phi_0(y_2)))
\end{aligned}$$

for all $f \in C(X)$. For each $i \in \{1, 2\}$, consider the set

$$F_i = \{\phi_0(y_1), \phi_0(y_2), \phi_0(y_3)\} \setminus \{\phi_0(y_i)\}$$

and take a function $f_i \in C(X)$ satisfying $f_i(x) = 0$ for all $x \in F_i$ and $f_i(\phi_0(y_i)) = 1$. Taking $f = f_i$ for $i = 1, 2$ in the equation above, it follows that

$$\beta(\phi_0(y_1), \phi_0(y_3)) = \beta(\phi_0(y_1), \phi_0(y_2)) = \beta(\phi_0(y_3), \phi_0(y_2)),$$

as claimed. Indeed, by the arbitrariness of y_1, y_2, and y_3, the first equality in the preceding equation means that the function $\beta(\cdot, \cdot)$ does not depend on the second variable, while the second equality tells us that the same occurs with the first one. Hence there exists a constant $\lambda \in \mathbb{T}$ such that $\beta(\phi_0(y_1), \phi_0(y_2)) = \lambda$ for all $y_1, y_2 \in Y_0$ with $y_1 \neq y_2$.

Now we get

$$\begin{aligned}
T(f)(y_1) - T(f)(y_2) &= \beta(\phi_0(y_1), \phi_0(y_2))(f(\phi_0(y_1)) - f(\phi_0(y_2))) \\
&= \lambda(f(\phi_0(y_1)) - f(\phi_0(y_2)))
\end{aligned}$$

for all $f \in C(X)$ and $y_1, y_2 \in Y_0$. □

Step 13. *There exists a linear functional $\mu\colon C(X) \to \mathbb{C}$ such that*

$$T(f)(y) = \lambda f(\phi_0(y)) + \mu(f)$$

for every $y \in Y_0$ and $f \in C(X)$.

Proof: Define a functional $\mu\colon C(X) \to \mathbb{C}$ by

$$\mu(f) = T(f)(y) - \lambda f(\phi_0(y))$$

for all $f \in C(X)$, where y is an arbitrary point in Y_0. By Step 12, μ is well-defined. Since T is linear, so is μ. □

Step 14. $\lambda \neq -\mu(1_X)$.

Proof: By Step 2, we have

$$T(1_X)(y) = \lambda_{1_X} + \mu_{1_X}(1_X)$$

for all $y \in Y$, with $\lambda_{1_X} \in \mathbb{T}$ and $\lambda_{1_X} \neq -\mu_{1_X}(1_X)$. On the other hand, by Step 13 we have

$$T(1_X)(y) = \lambda + \mu(1_X)$$

for all $y \in Y_0$. Hence $\lambda + \mu(1_X) = \lambda_{1_X} + \mu_{1_X}(1_X) \neq 0$. □

Step 15. $\phi_0\colon Y_0 \to X$ *is a homeomorphism.*

Proof: We first prove that ϕ_0 is continuous. Let $y \in Y_0$ and let $\{y_i\}_i$ be a net in Y_0 which converges to y. Since X is compact, taking a subnet if necessary we can suppose that $\{\phi_0(y_i)\}_i$ converges to some $x_0 \in X$. We claim that $x_0 = \phi_0(y)$. Otherwise, we could find an open neighborhood U of x_0 in X such that $\phi_0(y) \in X \setminus U$. Take a function $f \in C(X)$ which satisfies $f(x_0) = 1$ and $f(x) = 0$ for all $x \in X \setminus U$. There exists $i_0 \in I$ such that $|f(\phi_0(y_i)) - f(x_0)| = |f(\phi_0(y_i)) - 1| < 1/3$ for all $i \geq i_0$ and, by Step 13, it follows that $|T(f)(y_i) - T(f)(y)| = |f(\phi_0(y_i))| > 2/3$ for all $i \geq i_0$, which contradicts the continuity of $T(f)$. This proves our claim and so ϕ_0 is continuous.

We next show that Y_0 is closed in Y. Since $Y_0 = Y$ in the case $|X| = 2$, we suppose that $|X| \geq 3$. Let $\{y_i\}_i$ be a net in Y_0 convergent to some point $y \in Y$. By the compactness of X, taking a subnet if necessary, we can suppose that $\{\phi_0(y_i)\}_i$ converges to some $x_1 \in X$. Given $x_2 \in X \setminus \{x_1\}$, there exists $y_2 \in Y_0$ such that $\phi_0(y_2) = x_2$. By Step 13, we have

$$T(f)(y_i) - T(f)(y_2) = \lambda(f(\phi_0(y_i)) - f(\phi_0(y_2))) = \lambda(f(\phi_0(y_i)) - f(x_2))$$

for each $f \in C(X)$ and all $i \in I$. Since f and $T(f)$ are continuous, taking limits in i above, it follows that

$$T(f)(y) - T(f)(y_2) = \lambda(f(x_1) - f(x_2))$$

for all $f \in C(X)$. Note that $y \neq y_2$ since $C(X)$ separates the points of X. In particular, we get

$$T(f)(y) - T(f)(y_2) = \lambda(f(x_1) - f(x_2))$$

for all $f \in \mathcal{F}_{\{x_1,x_2\}}$. Hence $((y, y_2), \lambda) \in \mathcal{B}_{(x_1,x_2)}$. By Steps 4 and 5, we have either $(y, y_2) \in \mathcal{A}_{(x_1,x_2)}$ or else $(y_2, y) \in \mathcal{A}_{(x_1,x_2)}$. Hence either $\Gamma(x_1, x_2) = (y, y_2)$ or else $\Gamma(x_1, x_2) = (y_2, y)$ by Step 5. Therefore $\{y, y_2\} = \Lambda_Y(\Gamma(x_1, x_2)) = \{\varphi(x_1), \varphi(x_2)\}$ by Step 10, and so $y \in \varphi(X) = Y_0$.

Finally, since $\phi_0\colon Y_0 \to X$ is bijective and continuous, Y_0 is compact, and X is Hausdorff, we see that ϕ_0 is a homeomorphism. □

We have $|Y| = |X|$ since for any $f \in C(X)$, $\phi_f\colon X \to Y$ is a bijection by Step 2. Since, by Step 11, $\phi_0 = \varphi^{-1}\colon Y_0 \to X$ is also a bijection, it follows that $|Y_0| = |X|$. Hence $|Y| = |X| = |Y_0|$. If Y is finite, then $Y_0 = Y$ since $Y_0 \subseteq Y$, and we would obtain Steps 16 and 17 taking $\phi = \phi_0$. Suppose that Y is not finite henceforth.

Step 16. *There exists a continuous map $\phi\colon Y \to X$ such that*

$$T(f)(y) = \lambda f(\phi(y)) + \mu(f)$$

for all $f \in C(X)$ and $y \in Y$.

Proof: For each $y \in Y$, define the linear functional $S_y\colon C(X) \to \mathbb{C}$ by

$$S_y(f) = T(f)(y) - \mu(f) \quad (f \in C(X)),$$

with $\mu\colon C(X) \to \mathbb{C}$ being as in Step 13. Note that $T(1_X)(y_0) = \lambda+\mu(1_X)$ for each $y_0 \in Y_0$ by Step 13. Since $T(1_X)$ is a constant function by Step 1, it follows that $T(1_X) = (\lambda + \mu(1_X))1_Y$. Hence $S_y(1_X) = \lambda$.

We shall now prove that $\lambda^{-1}S_y$ is multiplicative. By the Gleason–Kahane–Żelazko Theorem, it suffices to show that for each non-vanishing function $f \in C(X)$, we have $S_y(f) \neq 0$. For this, let $f \in C(X)$ be with $f(x) \neq 0$ for all $x \in X$ and assume on the contrary that $T(f)(y) = \mu(f)$. Being $\phi_0\colon Y_0 \to X$ a bijective map, there exists $y_0 \in Y_0$ such that

$$\phi_0(y_0) = \phi_f(y).$$

In the same way we can find a sequence $\{y_i\}_{i=0}^{\infty}$ in Y_0 satisfying

$$\phi_0(y_{i+1}) = \phi_f(y_i) \quad (i \in \mathbb{N} \cup \{0\}).$$

Since Y is a compact (first countable) space, passing through a subsequence we may assume that $\{y_i\}_i \to z_0$ for some $z_0 \in Y_0$. Hence, letting $i \to \infty$ in the above equality, we get

$$\phi_0(z_0) = \phi_f(z_0).$$

For each $i \in \mathbb{N} \cup \{0\}$, since $z_0, y_i \in Y_0$, Step 12 yields

$$\begin{aligned}\lambda(f(\phi_0(z_0)) - f(\phi_0(y_i))) &= T(f)(z_0) - T(f)(y_i)\\ &= \lambda_f(f(\phi_f(z_0)) - f(\phi_f(y_i)))\\ &= \lambda_f(f(\phi_0(z_0)) - f(\phi_0(y_{i+1}))).\end{aligned}$$

Hence, for each $i \in \mathbb{N} \cup \{0\}$ we have

$$f(\phi_0(z_0)) - f(\phi_0(y_i)) = \lambda^{-1}\lambda_f(f(\phi_0(z_0)) - f(\phi_0(y_{i+1}))).$$

For each $i \in \mathbb{N} \cup \{0\}$, it follows by induction on n that

$$f(\phi_0(z_0)) - f(\phi_0(y_i)) = (\lambda^{-1}\lambda_f)^n(f(\phi_0(z_0)) - f(\phi_0(y_{i+n})))$$

for all $n \in \mathbb{N}$. Now, letting $n \to \infty$, we get

$$f(\phi_0(z_0)) = f(\phi_0(y_i)) \quad (i \in \mathbb{N} \cup \{0\}).$$

Therefore, we have

$$T(f)(z_0) = \lambda f(\phi_0(z_0)) + \mu(f) = \lambda f(\phi_0(y_0)) + \mu(f).$$

On the other hand, since $f(\phi_f(y)) = f(\phi_0(y_0)) = f(\phi_0(z_0))$ and $\phi_f(z_0) = \phi_0(z_0)$, we also get

$$\begin{aligned}T(f)(y) &= \lambda_f f(\phi_f(y)) + \mu_f(f)\\ &= \lambda_f f(\phi_0(z_0)) + \mu_f(f)\\ &= \lambda_f f(\phi_f(z_0)) + \mu_f(f)\\ &= T(f)(z_0).\end{aligned}$$

Now, since $T(f)(y) = \mu(f)$, we deduce that $f(\phi_0(y_0)) = 0$, which is a contradiction.

Hence, for each $y \in Y$, $\lambda^{-1}S_y$ is a nonzero complex homomorphism on $C(X)$. This easily implies that the map $S\colon C(X) \to C(Y)$ defined by

$$S(f)(y) = \lambda^{-1}S_y(f) = \lambda^{-1}(T(f)(y) - \mu(f)) \quad (f \in C(X),\, y \in Y)$$

is a unital homomorphism and, consequently, it is continuous as well. Thus the restriction of its adjoint map to the maximal ideal space of $C(Y)$ induces a continuous map $\phi\colon Y \to X$ satisfying

$$S(f)(y) = f(\phi(y)) \quad (f \in C(X),\, y \in Y).$$

Hence $T(f)(y) = \lambda f(\phi(y)) + \mu(f)$ for all $f \in C(X)$ and $y \in Y$. □

Step 17. *$\phi\colon Y \to X$ is a homeomorphism.*

Proof: First we show that ϕ is injective. For this, let y_1, y_2 be in Y and assume that $\phi(y_1) = \phi(y_2)$. Clearly, the function ϕ is not constant since otherwise, $T(f)$ would be a constant function on Y for each $f \in C(X)$.

Then, since T is diameter preserving, it follows that each $f \in C(X)$ is constant on X, while Y is infinite and $|X| = |Y| \neq 1$. Hence we can find a point $y_3 \in Y$ such that $\phi(y_3) \neq \phi(y_1)$. Put $x_k = \phi(y_k)$ for $k = 1, 2, 3$. Choose $f \in \mathcal{F}'_{\{x_1,x_3\}}$ and then, using Step 16, we deduce that

$$T(f)(y_1) - T(f)(y_3) = \lambda(f(x_1) - f(x_3)),$$

which implies that

$$\lambda_f(f(\phi_f(y_1)) - f(\phi_f(y_3))) = \lambda(f(x_1) - f(x_3)).$$

Thus $\{\phi_f(y_1), \phi_f(y_3)\} = \{x_1, x_3\}$. A similar argument shows that $\{\phi_f(y_2), \phi_f(y_3)\} = \{x_2, x_3\}$. Since $x_1 = x_2$, these equalities imply that $\phi_f(y_1) = \phi_f(y_2)$ and, consequently, $y_1 = y_2$.

Now we show that ϕ is surjective. Assume on the contrary that there exists a point $x_0 \in X \backslash \phi(Y)$. Being $\phi(Y)$ compact, we can choose a function $f \in C(X)$ satisfying $f(x_0) = 1$ and $f = 0$ on $\phi(Y)$. Then, using Step 16, we get $T(f)(y) = \mu(f)$ for all $y \in Y$, that is, $T(f)$ is a constant function, a contradiction since Y is infinite.

It follows immediately that ϕ is a homeomorphism from Y onto X. □

Step 18. *T is bijective.*

Proof: We first prove that T is injective. Let $f \in C(X)$ and assume $T(f) = 0$. By Step 1, $\operatorname{diam}(f) = \operatorname{diam}(T(f)) = 0$ and thus f is a constant function. Hence $f = \alpha 1_X$ for some $\alpha \in \mathbb{C}$. Since $0 = T(f) = T(\alpha 1_X) = \alpha T(1_X)$ and $T(1_X) = (\lambda + \mu(1_X))1_Y$, and also $\lambda \neq -\mu(1_X)$ by Step 14, we obtain $\alpha = 0$ and thus $f = 0$.

On the other hand, given $g \in C(Y)$, the function

$$f = \overline{\lambda} g \circ \phi^{-1} - \frac{\overline{\lambda}\mu(g \circ \phi^{-1})}{\lambda + \mu(1_X)} 1_X$$

belongs to $C(X)$ and $T(f) = g$. Indeed, we have

$$\begin{aligned}
T(f) &= T\left(\overline{\lambda} g \circ \phi^{-1} - \frac{\overline{\lambda}\mu(g \circ \phi^{-1})}{\lambda + \mu(1_X)} 1_X\right) \\
&= \lambda\left(\overline{\lambda} g \circ \phi^{-1} \circ \phi - \frac{\overline{\lambda}\mu(g \circ \phi^{-1})}{\lambda + \mu(1_X)} 1_X \circ \phi\right) \\
&\quad + \mu\left(\overline{\lambda} g \circ \phi^{-1} - \frac{\overline{\lambda}\mu(g \circ \phi^{-1})}{\lambda + \mu(1_X)} 1_X\right) \\
&= g - \frac{\mu(g \circ \phi^{-1})}{\lambda + \mu(1_X)} 1_Y + \overline{\lambda}\mu(g \circ \phi^{-1}) 1_Y - \frac{\overline{\lambda}\mu(g \circ \phi^{-1})\mu(1_X)}{\lambda + \mu(1_X)} 1_Y = g.
\end{aligned}$$

Hence T is surjective. This completes the proof of Theorem 2. □

For a compact Hausdorff space X, let $C_\rho(X)$ denote the quotient space of $C(X)$ by the constant functions and let $\pi_X : C(X) \to C_\rho(X)$ be the canonical quotient surjection. Then,

$$\|\pi_X(f)\|_\rho = \operatorname{diam}(f) \quad (f \in C(X))$$

defines a norm on $C_\rho(X)$. Clearly, for compact Hausdorff spaces X and Y, any diameter-preserving linear map $T\colon C(X) \to C(Y)$ induces a linear isometry from $C_\rho(X)$ into $C_\rho(Y)$ which is surjective if and only if so is T. On the other hand, any linear isometry $S\colon C_\rho(X) \to C_\rho(Y)$ induces an injective diameter-preserving linear map $T\colon C(X) \to C(Y)$. Indeed, fixing two points $u_0 \in X$ and $w_0 \in Y$, if we consider the following linear bijections

$$\Psi_X\colon C(X) \to C_\rho(X) \oplus \mathbb{C}, \quad \Psi_X(f) = (\pi_X(f), f(u_0)) \quad (f \in C(X))$$

and

$$\Psi_Y\colon C(Y) \to C_\rho(Y) \oplus \mathbb{C}, \quad \Psi_Y(g) = (\pi_Y(g), g(w_0)) \quad (g \in C(Y)),$$

then the linear map $T\colon C(X) \to C(Y)$ defined by

$$T(f) = \Psi_Y^{-1}(S(\pi_X(f)), f(u_0)) \quad (f \in C(X))$$

is an injective diameter-preserving map. Moreover, T is surjective if and only if so is S. Hence, using Theorem 2, we easily get the following corollary.

Corollary 1. *Let X and Y be first countable compact Hausdorff spaces, then the set of all linear surjective isometries from $C_\rho(X)$ to $C_\rho(Y)$ is algebraically reflexive.*

Remark 1. By Example 2 in [**5**] for certain compact Hausdorff spaces X we can find a local automorphism $T\colon C(X) \to C(X)$ which is not surjective. Since any automorphism S on $C(X)$ is of the form $S(f) = f \circ \varphi$, for some homeomorphism $\varphi\colon X \to X$, it follows that S is a diameter-preserving linear bijection as well. Hence this example can be applied to show that Theorem 2 is not valid for arbitrary compact Hausdorff spaces.

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A. Jiménez-Vargas
Departamento de Matemáticas, Universidad de Almería, 04120, Almería, Spain
E-mail address: ajimenez@ual.es

Fereshteh Sady
Department of Pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Tehran 14115-134, Iran
E-mail address: sady@modares.ac.ir

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