

LOCALLY SOLVABLE AND SOLVABLE-BY-FINITE MAXIMAL SUBGROUPS OF $GL_n(D)$

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Abstract: This paper aims to study solvable-by-finite and locally solvable maximal subgroups of an almost subnormal subgroup of the general skew linear group $GL_n(D)$ over a division ring D . It turns out that in the case where D is non-commutative, if such maximal subgroups exist, then either it is abelian or $[D : F] < \infty$. Also, if F is an infinite field and $n \geq 5$, then every locally solvable maximal subgroup of a normal subgroup of $GL_n(F)$ is abelian.

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1. Introduction and statements of main results

The question of the existence of maximal subgroups in a division ring is difficult and it has not been settled completely (see [1], [3], [12]). Despite this, various aspects of maximal subgroups in division rings have been studied substantially by many authors. Let D be a division ring and $D^* = D \setminus \{0\}$ its multiplicative group. In [5] it was shown that every nilpotent maximal subgroup of D^* is abelian. In [23] this result was extended to nilpotent maximal subgroups of a subnormal subgroup G of D^* . However, we do not have an analogous result if the word “nilpotent” is substituted by “solvable”. In fact, the set $\mathbb{C}^* \cup \mathbb{C}^*j$ is a non-abelian metabelian maximal subgroup of the multiplicative group \mathbb{H}^* of the division ring of real quaternions \mathbb{H} (see [1]). More generally, it was shown that if a subnormal subgroup G of D^* contains a non-abelian solvable maximal subgroup, then D must be a cyclic algebra of prime degree over the center of D (see [9] or [6]). Recently, we have obtained parallel results in [15] for locally nilpotent and locally solvable maximal subgroups of an almost subnormal subgroup of D^* .

Continuing in this direction, we devote Section 2 of the present paper to examining solvable-by-finite maximal subgroups of an almost subnormal subgroup of the general skew linear group $\mathrm{GL}_n(D)$, for an integer $n \geq 1$. Solvable-by-finite skew linear groups were also considered by several authors before (see [11, Section 2], [23], [32]). In Section 2, among other corollaries, we prove the following theorem.

Theorem 1.1. *Let D be a non-commutative division ring with center F which contains at least five elements, and G an almost subnormal subgroup of $\mathrm{GL}_n(D)$ with $n \geq 1$. If M is a non-abelian solvable-by-finite maximal subgroup of G , then $[D : F] < \infty$. Furthermore, we have $F[M] = M_n(D)$, and there exists a maximal subfield K of $M_n(D)$ containing F such that $K^* \cap G$ is the Fitting subgroup of M , K/F is a Galois extension, $N_{\mathrm{GL}_n(D)}(K^*) \cap G = M$, $K^* \cap G \trianglelefteq M$, and $M/K^* \cap G \cong \mathrm{Gal}(K/F)$ is a finite simple group of order $n\sqrt{[D : F]}$.*

Section 3 is about studying locally solvable maximal subgroups of an almost subnormal subgroup of $\mathrm{GL}_n(D)$. As we have mentioned above, the case $n = 1$ has been investigated subsequently in [15]. Therefore, we shall focus on the remaining case $n \geq 2$ only. In [3], it was pointed out that if D is a non-commutative division ring and $n \geq 2$, then every solvable maximal subgroup of $\mathrm{GL}_n(D)$ is abelian. In [14] and [6], this result was generalized to solvable maximal subgroups of a normal subgroup of $\mathrm{GL}_n(D)$. To the best of our knowledge, the latest results regarding locally solvable maximal subgroups of $\mathrm{GL}_n(D)$ were obtained in [22]. Indeed, it was shown in Theorem 1.6 of [22] that if D is an infinite division ring and $n \geq 2$, then every locally nilpotent maximal subgroup of $\mathrm{GL}_n(D)$ is abelian. Also, a similar result for locally solvable maximal subgroups was presented in [22, Theorem 1.5] under some additional conditions. Here, we generalize these results to the case of locally solvable maximal subgroups of a normal subgroup of $\mathrm{GL}_n(D)$. More precisely, we prove in Section 3 the following result.

Theorem 1.2. *Let D be a non-commutative division ring with center F which contains at least five elements, and G a normal subgroup of $\mathrm{GL}_n(D)$ with $n \geq 2$. If M is a locally solvable maximal subgroup of G , then M is abelian.*

Let us return to the linear case; that is, to the case when $D = F$ is a field. It is worth noting that the two conditions “local solvability” and “solvability” are equivalent in this case. For an infinite field F and $n < 5$, the work [3] demonstrates that $\mathrm{GL}_n(F)$ always contains non-abelian solvable maximal subgroups. Furthermore, in [3] all solvable maximal subgroups of $\mathrm{GL}_2(F)$ for an arbitrary field F have been determined

completely. At the end of Section 3 we show that if $n \geq 5$ and F is infinite, then every solvable maximal subgroup of a normal subgroup of $\text{GL}_n(F)$ is abelian.

Theorem 1.3. *Let F be an infinite field, and G a normal subgroup of $\text{GL}_n(F)$ with $n \geq 5$. If M is a solvable maximal subgroup of G , then M is abelian.*

The motivation of the obtained results is [1, Conjecture 1], which states that the general skew linear group $\text{GL}_n(D)$ contains no solvable maximal subgroups if D is non-commutative and $n \geq 2$ or if D is commutative and $n \geq 5$. Additionally, we refer the reader to [3] and [6] for a full discussion of this conjecture. In view of the obtained results, this conjecture is reduced to the case of abelian maximal subgroups, which is exactly Conjecture 2 of [1].

Let R be a ring, S a subring of R , and G a subgroup of R^* normalizing S such that $R = S[G]$. Suppose that $N = G \cap S$ is a normal subgroup of G and $R = \bigoplus_{t \in T} tS$, where T is some (and hence any) transversal T of N to G . Then, we say that R is a crossed product of S by G/N (see [31] or [25, 1.4]). Let K/F be a cyclic extension of fields with the Galois group generated by an automorphism σ of order $s = \dim_F K$. Fixing a non-zero element $a \in F$ and a symbol x , we let

$$C = K \cdot 1 \oplus K \cdot x \oplus \dots \oplus K \cdot x^{s-1},$$

and multiply elements in C by using the distributive law and the two rules

$$x^s = a, \quad x \cdot b = \sigma(b)x,$$

for any $b \in K$. Then C is an F -algebra and is called the *cyclic algebra associated with $(K/F, \sigma)$ and $a \in F \setminus \{0\}$* (see [16, p. 218]).

Throughout this paper, we denote by D a division ring with center F and by D^* the multiplicative group of D . For a positive integer n , the symbol $M_n(D)$ stands for the matrix ring of degree n over D . We identify F with FI_n via the ring isomorphism $a \mapsto aI_n$, where I_n is the identity matrix of degree n . If S is a subset of $M_n(D)$, then $F[S]$ denotes the subring of $M_n(D)$ generated by the set $S \cup F$. Also, if S is a subset of D , then $F(S)$ is the division subring of D generated by $S \cup F$. Recall that a division ring D is *locally finite* if for every finite subset S of D , the division subring $F(S)$ is a finite-dimensional vector space over F . If A is a ring or a group, then $Z(A)$ denotes the center of A .

Let $V = D^n = \{(d_1, d_2, \dots, d_n) \mid d_i \in D\}$. If G is a subgroup of $\text{GL}_n(D)$, then V may be viewed as a D - G bimodule. Recall that a subgroup G of $\text{GL}_n(D)$ is *irreducible* (resp. *reducible*, *completely reducible*)

if V is irreducible (resp. reducible, completely reducible) as a D - G bi-module. If $F[G] = M_n(D)$, then G is *absolutely irreducible* over D . An irreducible subgroup G is *imprimitive* if there exists an integer $m \geq 2$ such that $V = \bigoplus_{i=1}^m V_i$ as left D -modules and for any $g \in G$ the mapping $V_i \rightarrow V_i g$ is a permutation of the set $\{V_1, \dots, V_m\}$. If G is irreducible and not imprimitive, then G is *primitive*.

2. Solvable-by-finite maximal subgroups

Let us recall the notion of an almost subnormal subgroup. Let G be a group and H a subgroup of G . Following Hartley [10], we say that H is an *almost subnormal* subgroup of G if there is a finite chain of subgroups

$$H = H_0 \leq H_1 \leq \dots \leq H_r = G,$$

such that either $[H_{i+1} : H_i]$ is finite or H_i is normal in H_{i+1} for $1 \leq i \leq r-1$. It is clear that if H is a subnormal subgroup of G , then it is almost subnormal. Let D be a division ring and n an integer number. It was shown in [20, Theorem 3.3] that if $n \geq 2$ and D is infinite, then every almost subnormal subgroup of $\text{GL}_n(D)$ is normal. On the other hand, this result does not hold in the case $n = 1$; that is, the case $\text{GL}_1(D) = D^*$. There are examples of division rings which contain almost subnormal subgroups that are not subnormal (see [2] and [20]). In addition, it is surprising that every non-central almost subnormal subgroup of D^* always contains a non-central subnormal subgroup. This fact allows us to extend naturally a lot of results for subnormal subgroups to those of almost subnormal subgroups. This interesting result was obtained previously in [8] by using certain properties of graphs. In this paper we give an alternative proof for this fact without using graphs (Proposition 2.2). For this purpose, we need the following lemma which may be adopted from [26, Lemma 4].

Lemma 2.1. *Let D be a division ring, and N a non-central subnormal subgroup of D^* . If G is a non-central subgroup of D^* , which is invariant under N , that is $x^{-1}Gx \subseteq G$ for all $x \in N$, then $G \cap N$ is non-central.*

Proposition 2.2. *Let D be a division ring with center F . Then, every non-central almost subnormal subgroup of D^* contains a non-central subnormal subgroup.*

Proof: If G is a non-central almost subnormal subgroup of D^* , then by definition, there exists a finite chain of subgroups

$$G = G_0 \leq G_1 \leq \dots \leq G_n = D^*,$$

for which either $[G_{i+1} : G_i]$ is finite or $G_i \trianglelefteq G_{i+1}$. We will prove that G contains a non-central subnormal subgroup by induction on n .

Step 1. To prove the proposition when $n = 1$. Assume that $[D^* : G]$ is finite. If we set $N = \text{Core}_{D^*}(G) = \bigcap_{x \in D^*} x^{-1}Gx$, then N is a normal subgroup of finite index in D^* . It is clear that D^* is radical over N . We assert that N is not contained in F . Assume by contradiction that $N \subseteq F$. We divide our situation into two cases:

Case 1.1. F is finite. Since F is finite, so is N . Additionally, we have that D^* is radical over N , from which it follows that every element of D^* is periodic. According to [13, Theorem 8], we conclude that D is commutative, a contradiction.

Case 1.2. F is infinite. If we set $s = [D^* : N]$, then for any $a, b \in D^*$ we have $a^s, b^s \in N \subseteq F$. This implies that $a^{-s}b^{-s}a^sb^s = 1$ for any $a, b \in D^*$. In other words, D^* satisfies the non-trivial group identity $x^{-s}y^{-s}x^sy^s = 1$. In view of [21, Theorem 2.2], we obtain that D is commutative, a contradiction.

Step 2. To prove the proposition when $n > 1$. It follows by the induction hypothesis that G_1 possesses a non-central subnormal subgroup N of D^* with a subnormal series $N = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_m = D^*$. There are two possible cases:

Case 2.1. $G = G_0 \trianglelefteq G_1$. In this case, we have $N \cap G \trianglelefteq N \cap G_1 = N$; recall that $N \subseteq G_1$. Since G is invariant under G_1 , it is also invariant under N . It follows from Lemma 2.1 that $N \cap G$ is non-central. Consequently, $N \cap G$ is a non-central subnormal subgroup of D^* contained in G , with the subnormal series

$$N \cap G \trianglelefteq N = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_m = D^*.$$

Case 2.2. $[G_1 : G] < \infty$. If $C = \text{Core}_{G_1}(G)$, then C is a normal subgroup of finite index in G_1 contained in G . We claim that C is non-central. Indeed, assume on the contrary that $C \subseteq F$. If F is finite, then C is a finite group. This yields that G_1 is also finite, and so is N . It follows from [13, Theorem 8] that N is central, a contradiction. In the remaining case where F is infinite, if $C \subseteq F$, then G_1 is a non-central almost subnormal subgroup of D^* satisfying the identity $x^{-k}y^{-k}x^ky^k = 1$, where $k = [G_1 : C]$, which contrasts with [20, Theorem 2.2]. Therefore C is non-central, as claimed. By the same arguments used in Case 2.1, we conclude that $C \cap N$ is a non-central subnormal subgroup of D^* . \square

Corollary 2.3. *Let D be a division ring with center F , and G be a non-central almost subnormal subgroup of D^* . If K is a division subring of D normalized by G , then either $K \subseteq F$ or $K = D$.*

Proof: By Proposition 2.2, G contains a non-central subnormal subgroup N , which also normalizes K . The result follows immediately from [26, Theorem 1]. \square

Lemma 2.4. *Let D be a division ring with center F , and G an almost subnormal subgroup of D^* . If G is (locally solvable)-by-finite, then $G \subseteq F$.*

Proof: Let A be a locally solvable normal subgroup of finite index in G . Since A is an almost subnormal subgroup of D^* , it follows from [15] that $A \subseteq F$. This implies that $G/Z(G)$ is finite and, by Schur's theorem ([21, Lemma 1.4, p. 115]), we conclude that G' is finite. This means that G' is a finite almost subnormal subgroup of D^* . If G' is non-central, then by Proposition 2.2, G' contains a non-central finite subnormal subgroup of D^* , which contradicts [13, Theorem 8]. Hence, $G' \subseteq F$, which means that G is solvable. Therefore, the result follows from [15]. \square

Lemma 2.5. *Let D be a division ring with center F , and G an almost subnormal subgroup of D^* . Assume that M is a non-abelian solvable-by-finite maximal subgroup of G . If A is a normal subgroup of M , then either A is abelian or $F(A) = D$.*

Proof: The condition $A \trianglelefteq M$ implies immediately that $F(A)$ is normalized by M and so $M \subseteq N_{D^*}(F(A)^*) \cap G \subseteq G$. By virtue of maximality of M , we have either $N_{D^*}(F(A)^*) \cap G = M$ or $N_{D^*}(F(A)^*) \cap G = G$. If the former case occurs, then $F(A)^* \cap G$ is contained in M , which shows that A is normal in $F(A)^* \cap G$. Consequently, A is an almost subnormal subgroup of $F(A)^*$ contained in M . Since M is solvable-by-finite, so is A . It follows from Lemma 2.4 that A is contained in the center of $F(A)$ and so A is abelian. In the latter case, the division subring $F(A)$ is normalized by G . It follows from Corollary 2.3 that either $A \subseteq F$ or $F(A) = D$. In either case, we always have that A is abelian or $F(A) = D$. Our proof is now finished. \square

Lemma 2.6 ([32, Proposition 4.1]). *Let $D = E(A)$ be a division ring generated by its metabelian subgroup A and its division subring E such that $E \leq C_D(A)$. Set $H = N_{D^*}(A)$, $B = C_A(A')$, $K = E(Z(B))$, $H_1 = N_{K^*}(A) = H \cap K^*$, and let $\tau(B)$ be the maximal periodic normal subgroup of B .*

- (1) *If $\tau(B)$ has a quaternion subgroup $Q = \langle i, j \rangle$ of order 8 with $A = QC_A(Q)$, then $H = Q^+AH_1$, where $Q^+ = \langle Q, 1 + j, -(1 + i + j + ij)/2 \rangle$. Also, Q is normal in Q^+ and $Q^+/\langle -1, 2 \rangle \cong \text{Aut } Q \cong \text{Sym}(4)$.*
- (2) *If $\tau(B)$ is abelian and contains an element x of order 4 not in the center of B , then $H = \langle x + 1 \rangle AH_1$.*
- (3) *In all other cases, $H = AH_1$.*

Theorem 2.7. *Let D be a division ring with center F , and G an almost subnormal subgroup of D^* . If M is a non-abelian solvable-by-finite maximal subgroup of G , then M is abelian-by-finite and $[D : F] < \infty$.*

Proof: It follows from Lemma 2.5 that $F(M) = D$. Let N be a solvable normal subgroup of finite index in M . In the case where N is abelian, then, of course, M is abelian-by-finite and $L = F(N)$ is a subfield of D normalized by M . Take a transversal $\{x_1, x_2, \dots, x_k\}$ of N and set

$$\Delta = Lx_1 + Lx_2 + \dots + Lx_k.$$

The displayed relation provides that Δ is a domain with $\dim_L \Delta \leq k$. The implication of this fact is that Δ is a centrally finite division ring. Since Δ contains both F and M , it must coincide with D .

To settle the whole theorem, it remains to examine the case where N is a non-abelian. Therefore, we may suppose that N is solvable with derived series of length $s \geq 2$. In other words, we have the following series:

$$1 = N^{(s)} \trianglelefteq N^{(s-1)} \trianglelefteq N^{(s-2)} \trianglelefteq \dots \trianglelefteq N' \trianglelefteq N \trianglelefteq M.$$

If we set $A = N^{(s-2)}$, then A is a non-abelian metabelian normal subgroup of M . As A is non-abelian, Lemma 2.5 again says that $F(A) = D$, from which it follows that $Z(A) = F^* \cap A$ and $F = C_D(A)$. Set $H = N_{D^*}(A)$, $B = C_A(A')$, $K = F(Z(B))$, $H_1 = H \cap K^*$, and $\tau(B)$ to be the maximal periodic normal subgroup of B . Then, it is a simple matter to check that H_1 is an abelian group and $\tau(B)$ is characteristic in B . It follows from [15, Lemma 3.8] that $\tau(B)$ is characteristic in A . In order to use Lemma 2.6, we divide our situation into three cases:

Case 1. $\tau(B)$ is not abelian.

Since $\tau(B)$ is characteristic in B and B is normal in M , we conclude that $\tau(B)$ is normal in M . By virtue of Lemma 2.5, we have $F(\tau(B)) = D$. In addition, as $\tau(B)$ is solvable and periodic, it is actually a locally finite group ([15, Lemma 2.12]). It follows that $D = F(\tau(B)) = F[\tau(B)]$ is a locally finite division ring. Since M is solvable-by-finite, it contains no non-cyclic free subgroups. With reference to [9, Theorem 3.1], we deduce that $[D : F] < \infty$ and M is abelian-by-finite.

Case 2. $\tau(B)$ is abelian and contains an element x of order 4 not in the center of $B = C_A(A')$.

Since $x \notin Z(B)$, it does not belong to F . The condition x is of finite order implies that the field $F(x)$ is an algebraic extension of F . Note that $\langle x \rangle$ is indeed a 2-primary component of $\tau(B)$ (see [32, Theorem 1.1, p. 132]); thus, it is a characteristic subgroup of $\tau(B)$. Consequently, $\langle x \rangle$ is a normal subgroup of M . Therefore, all elements of the set $x^M =$

$\{m^{-1}xm \mid m \in M\} \subseteq F(x)$ have the same minimal polynomial over F . As a result, x is an FC -element and so $[M : C_M(x)] < \infty$. Now, if we set $C = \text{Core}_M(C_M(x))$, then C is a normal subgroup of finite index in M . In view of Lemma 2.5, either $F(C) = D$ or C is abelian. The first case cannot occur since it implies that $x \in F$, which is impossible. Therefore C is abelian and, in consequence, M is abelian-by-finite. By the same arguments used in the first paragraph, we conclude that D is centrally finite.

Case 3. $H = AH_1$.

The fact that $A' \subseteq H_1 \cap A$ implies that $H/H_1 \cong A/A \cap H_1$ is abelian and so $H' \subseteq H_1$. Since H_1 is abelian, it follows that H' is abelian too. Because $M \subseteq H$, we know that M' is also abelian. In other words, M is a metabelian group; hence, the conclusions follow from [15, Proposition 3.7]. \square

Lemma 2.8 ([25, 4.5.1]). *Let D be a division ring that is not a locally finite field, and $n \geq 2$ an integer. If N is a non-central normal subgroup of $\text{GL}_n(D)$, then N contains a non-cyclic free subgroup.*

Lemma 2.9. *Let D be a division ring with center F containing at least four elements, and G a normal subgroup of $\text{GL}_n(D)$ with $n \geq 2$. Assume that M is a maximal subgroup of G . If A is an F -subalgebra of $\text{M}_n(D)$ normalized by M , then either $A^* \cap G \subseteq M$ or $A = \text{M}_n(D)$ provided that $A^* \cap G \not\subseteq F$.*

Proof: Since M normalizes A , we have $M \subseteq N_{\text{GL}_n(D)}(A^*) \cap G \subseteq G$. Since M is maximal in G , we have either $M = N_{\text{GL}_n(D)}(A^*) \cap G$ or $N_{\text{GL}_n(D)}(A^*) \cap G = G$. If the first case occurs, then $G \cap A^* \subseteq M$. Now, suppose that $N_G(A^*) = G$ and that $A^* \cap G \not\subseteq F$. It is clear for this case that $A^* \cap G$ is a non-central normal subgroup of $\text{GL}_n(D)$, from which it follows that $\text{SL}_n(D) \subseteq A^*$ ([18, Theorem 11]). The consequence of this fact is that A contains the subring $F[\text{SL}_n(D)]$, which is normalized by $\text{GL}_n(D)$. According to the Cartan–Brauer–Hua theorem for the matrix ring (see e.g. [3, Theorem D]), we obtain that $A = \text{M}_n(D)$. \square

Lemma 2.10. *Let D be an infinite division ring with center F , and $n \geq 1$. If $\text{SL}_n(D)$ is (locally solvable)-by-finite, then $n = 1$ and $D = F$.*

Proof: For a proof by contradiction, we assume that $n > 1$. Set $S = \text{SL}_n(D)$. From our hypothesis, S contains a locally solvable normal subgroup G such that S/G is finite. If G is not contained in F , then $G = S$ ([18, Theorem 11]), and so S is locally solvable. By Lemma 2.8, we deduce that D is a field, which means that S is indeed a solvable linear group; recall that every locally solvable linear group is solvable. But

this leads at once to the contradiction that S is a perfect group. We may therefore assume that $G \subseteq F$, from which it follows that $S/S \cap F^*$ is finite. A similar argument permits us to conclude that F is again a field. Then, we also have a contradiction since it is well known that for an infinite field F , the special linear group $\mathrm{SL}_n(F)$ contains no proper subgroups of finite index. These contradictions imply that $n = 1$ and, in consequence, we have D' is (locally solvable)-by-finite. With reference to Lemma 2.4, we conclude that D' is contained in F . As a result, D^* is solvable, yielding that $D = F$. The proof is completed. \square

Lemma 2.11. *Let D be an infinite division ring with center F , and G a normal subgroup of $\mathrm{GL}_n(D)$. Assume that M is a non-abelian maximal subgroup of G , and that N is a subnormal subgroup of M . If either*

- (1) M is (locally solvable)-by-finite, D is non-commutative, and $n \geq 2$,
or
- (2) M is solvable, D is a field, and $n \geq 5$,

then the following hold:

- (i) M is primitive,
- (ii) $C_{M_n(D)}(M)$ is a field,
- (iii) either N is abelian or $F[N]$ is a prime Goldie ring whose classical right quotient ring coincides with $M_n(D)$,
- (iv) $C_{M_n(D)}(N)$ is a simple Artinian ring.

Proof: We begin by proving that M is irreducible. Otherwise, it follows from [25, 1.1.1] that there exist a matrix $P \in \mathrm{GL}_n(D)$ and some integer $0 < m < n$ such that $PMP^{-1} \subseteq H$, where

$$H = \left(\begin{array}{cc} \mathrm{GL}_m(D) & * \\ 0 & \mathrm{GL}_{n-m}(D) \end{array} \right) \cap G.$$

The normality of G and the maximality of M imply that PMP^{-1} is a maximal subgroup of G contained in H , which yields either $H = G$ or $H = PMP^{-1}$. We observe that in the former event, we would have $\mathrm{SL}_n(D) \subseteq H$, which is impossible since $I_n + E_{n1}$ belongs to $\mathrm{SL}_n(D)$ but it is not an element of H (here E_{n1} is the matrix with 1 in the position $(n, 1)$ and 0 everywhere else). Therefore, we may assume that $H = PMP^{-1}$, which is conjugate to M . It follows that H is (locally solvable)-by-finite. Consequently, the group

$$\left(\begin{array}{cc} \mathrm{SL}_m(D) & 0 \\ 0 & \mathrm{SL}_{n-m}(D) \end{array} \right) \subseteq H,$$

which clearly contains a copy of $\mathrm{SL}_m(D)$ and of $\mathrm{SL}_{n-m}(D)$, is (locally solvable)-by-finite too. By virtue of Lemma 2.10, we must have $m =$

$n - m = 1$ and so $n = 2$ and $D = F$, a contradiction. As a result, the group M is irreducible, proving our claim.

As M is irreducible, it follows from [1, Lemma 8] that $F_1 = C_{M_n(D)}(M)$ is a division ring. Take an element $x \in F_1' \subseteq \mathrm{SL}_n(D)$; by [18, Theorem 11], we have $x \in G$. The maximality of M in G again means that either $\langle M, x \rangle \cap G = M$ or $G \subseteq \langle M, x \rangle$. If the first case occurs, then $\langle M, x \rangle = M$, or $x \in M$. It follows that $x \in Z(M)$. In the latter case, we have $F[G] = F[\langle M, x \rangle] = M_n(D)$ by the Cartan–Brauer–Hua theorem for the matrix ring and so $x \in F$. Thus, in either case we have $x \in F^*Z(M)$, from which we conclude that F_1' is abelian. A consequence of this fact is that F_1^* is solvable, and so F_1 is indeed a field. Assertion (ii) is now established.

For a proof of (i), we assume by contradiction that M is imprimitive. Then, the proof of [9, Lemma 2.5] says that M contains a copy of $\mathrm{SL}_r(D) \wr S_k$, the wreath product of $\mathrm{SL}_r(D)$ and the symmetric group S_k for some $r > 1$ and $n = rk$. Since M is (locally solvable)-by-finite, so is $\mathrm{SL}_r(D)$. By Lemma 2.10, we have $r = 1$, $k = n$, and D is a field, which contradicts the assumption of (1). Therefore M is primitive under assumption (1). If $n \geq 5$ and M is solvable, then the fact that $n = k$ implies that S_n is solvable, a contradiction. This contradiction indicates that M is primitive, and (i) is proved.

To prove (iii), we note that R is in fact Goldie ([29, Corollary 24]) and prime. Therefore, the classical right quotient ring, say Q , of R must be simple Artinian ([7, Theorem 6.18]), which is embedded in $M_n(D)$ by [25, 5.7.8]. Therefore, there exist a division F_1 -algebra E and an integer $m \geq 1$ such that $Q \cong M_m(E)$. Since M normalizes R , it also normalizes Q . It follows from Lemma 2.9 that either $Q^* \cap G \trianglelefteq M$ or $Q = M_n(D)$. The first case implies that $Q^* \cap G$ is a subnormal subgroup of Q^* contained in M . As a result, the subgroup $N \subseteq Q^* \cap G$ is contained in $Z(Q)$ by Lemmas 2.4 and 2.8. In other words, N is abelian, and (iii) is proved.

The confirmation of the final assertion (iv) follows from the same argument used in the proof of [3, Proposition 3.3]. \square

Lemma 2.12 ([17, Theorem 2]). *Let R be a prime ring with identity, $Z = Z(R)$ the center of R containing at least five elements, and \bar{U} the Z -subalgebra of R generated by R^* . Assume \bar{U} contains a non-zero ideal of R . If R^* has a solvable normal subgroup which is not central, then R is a domain.*

Lemma 2.13. *Let A, B, C be groups such that $A \trianglelefteq B \trianglelefteq C$ and that A is a solvable subgroup of finite index in B . Then A is contained in a solvable subgroup of B and such a subgroup is normal in C .*

Proof: For any $x \in C$, it is clear that $x^{-1}Ax$ is a solvable normal subgroup of B . If we set $H = \langle x^{-1}Ax \rangle$ where x runs over C , then $A \trianglelefteq H \subseteq B$ and $H \trianglelefteq C$. Since $[B : A]$ is finite, we conclude that $[H : A]$ is finite. Let $\{h_1, \dots, h_n\}$ be a transversal of A in H . Since $H = \langle x^{-1}Ax \rangle$, for each $1 \leq i \leq n$, the element h_i may be expressed in the form

$$h_i = x_{i_1}^{-1} a_{i_1} x_{i_1} \cdots x_{i_{k_i}}^{-1} a_{i_{k_i}} x_{i_{k_i}},$$

where $x_{i_j} \in C$ and $a_{i_j} \in A$. It is clear that

$$H = \langle x_{i_j}^{-1} A x_{i_j} : 1 \leq i \leq n, 1 \leq j \leq k_i \rangle.$$

Since there are only finitely many x_{i_j} 's, it follows that H is a solvable normal subgroup of C containing A . \square

Theorem 2.14. *Let D be a non-commutative division ring with center F which contains at least five elements, and G a normal subgroup of $\text{GL}_n(D)$ with $n \geq 2$. If M is a solvable-by-finite maximal subgroup of G such that $F[M] \neq M_n(D)$, then M is abelian.*

Proof: Let N be a solvable normal subgroup of finite index of M . We shall show first that N is abelian. For this purpose, we set $R = F[N]$ and Q to be the classical right quotient ring of R . Then, Lemma 2.11(iii) ensures that R is prime and Goldie, and that either N is abelian or $Q = M_n(D)$. If the first case occurs, then we are done. Now, we shall show that the latter case cannot happen by contradiction. So, assume that $Q = M_n(D)$ and that N is non-abelian. In view of Lemma 2.9, we have $G \cap R^* \subseteq M$, from which it follows that $G \cap R^*$ is a normal subgroup of R^* contained in M . Now, we have $N \trianglelefteq G \cap R^* \trianglelefteq R^*$ and $[G \cap R^* : N] < \infty$. With reference to Lemma 2.13, we conclude that N is contained in a solvable normal subgroup, say H , of R^* . Since $Z(R)$ contains F , it has at least five elements. Additionally, the fact that $N \subseteq H$ ensures us that H is not contained in $Z(R)$. Therefore, we may apply Lemma 2.12 to obtain that R is a domain. Now, R is both a domain and Goldie, and actually an Ore domain, and so $Q = M_n(D)$ is a division ring. But this leads to the contradiction that $n > 1$. Therefore, we have that N is abelian, and so M is abelian-by-finite.

Next, we assert that M is indeed abelian. Again, Lemma 2.11(iii) shows that $S = F[M]$ is a prime ring. Because M is abelian-by-finite, we may apply [21, Lemma 11, p. 176] to conclude that the group ring FM is a PI-ring. Thus, as a homomorphic image of FM , the ring S is also a PI-ring. Since M normalizes S , by Lemma 2.9, we deduce that $S^* \cap G \subseteq F$ or $S^* \cap G \subseteq M$. The first case yields that M is abelian, and we are done. The latter illustrates that $M = S^* \cap G$. In view of Lemma 2.11(i), we obtain that $F_1 = C_{M_n(D)}(M)$ is a field. There are two possible cases:

Case 1. $F_1 \subseteq M$. In this case, the field F_1 is the center of the prime ring S . In view of [24, Corollary 1.6.28], we conclude that S is a simple ring. Now S is a simple PI-ring, so it is a simple Artinian ring by Kaplansky's theorem. Therefore, S coincides with its classical right quotient Q and thus we may apply Lemma 2.11(iii) to get that M is abelian.

Case 2. $F_1 \not\subseteq M$. Set $M_1 = F_1^*M$ and $N_1 = F_1^*N$. If $M_1 = N_1$, then M_1 contains $\mathrm{SL}_n(D)$, which is impossible since $\mathrm{SL}_n(D)$ cannot be abelian-by-finite. If $M_1 \neq N_1$, then M_1 is a maximal subgroup of N_1 . In the same way, we conclude that M_1 is abelian, and so is M . This completes the proof of the theorem. \square

Lemma 2.15. *Let D be a division ring with center F , and M a subgroup of $\mathrm{GL}_n(D)$ with $n \geq 1$. If $M/M \cap F^*$ is a locally finite group, then $F[M]$ is a locally finite-dimensional vector space over F .*

Proof: Pick a finite subset $S = \{x_1, x_2, \dots, x_k\}$ of $F[M]$. Then, for each $1 \leq i \leq k$, we can find elements $f_{i_1}, f_{i_2}, \dots, f_{i_s}$ in F and $m_{i_1}, m_{i_2}, \dots, m_{i_s}$ in M such that

$$x_i = f_{i_1}m_{i_1} + f_{i_2}m_{i_2} + \dots + f_{i_s}m_{i_s}.$$

Let G be the subgroup of M generated by the m_{i_j} 's. We know that $M/M \cap F^* \cong MF^*/F^*$ and so MF^*/F^* is locally finite, from which it may be concluded that GF^*/F^* is finite. If $\{y_1, y_2, \dots, y_t\}$ is a transversal of F^* in GF^* , then

$$R = Fy_1 + Fy_2 + \dots + Fy_t$$

is a ring containing S . The displayed relation also means R is finite-dimensional over F . Since S is chosen to be arbitrary, our result follows. \square

Lemma 2.16. *Let R be a ring, and G a subgroup of R^* . Assume that F is a central subfield of R and that A is a maximal abelian subgroup of G such that $K = F[A]$ is normalized by G . Then $F[G]$ is a crossed product of K by G/A . In addition, if K is a field, then it is a maximal subfield of R .*

Proof: Since K is normalized by G , it follows that $F[G] = \sum_{g \in T} Kg$ for every transversal T of A in G . Thus, to establish that $F[G]$ is a crossed product of K by G/A , it suffices to prove that every finite subset $\{g_1, g_2, \dots, g_n\}$ of T is linearly independent over K . For purposes of contradiction, we assume that there exists such a non-trivial relation

$$k_1g_1 + k_2g_2 + \dots + k_n g_n = 0.$$

Clearly, we can suppose that all the k_i 's are non-zero and that n is minimal. The case where $n = 1$ is obviously trivial and so we suppose that $n > 1$. As the cosets Ag_1 and Ag_2 are disjoint, we know that $g_1^{-1}g_2 \notin A = C_G(A)$. Therefore, there exists an element $x \in A$ for which $g_1^{-1}g_2x \neq xg_1^{-1}g_2$. For each $1 \leq i \leq n$, if we set $x_i = g_i x g_i^{-1}$, then $x_1 \neq x_2$. Since G normalizes K , it follows that $x_i \in K$ for all $1 \leq i \leq n$. Now, we have

$$(k_1g_1 + \cdots + k_ng_n)x - x_1(k_1g_1 + \cdots + k_ng_n) = 0.$$

By definition of the x_i 's, we deduce that $x_i g_i = g_i x$, and so $x, x_i \in K$ for all i . By the fact that $K = F[A]$ is commutative, the last equality reveals

$$(x_2 - x_1)k_2g_2 + \cdots + (x_n - x_1)k_ng_n = 0,$$

which is a non-trivial relation (since $x_1 \neq x_2$) with less than n summands, contrasting with the minimality of n . As a result, we obtain the desired fact that T is linearly independent over K .

Regarding the last assertion of our lemma, we assume that $R = F[G]$ and that K is a field. If we set $L = C_R(K)$, then every element $y \in L$ may be written in the form

$$y = l_1m_1 + l_2m_2 + \cdots + l_tm_t,$$

where $l_1, l_2, \dots, l_t \in K$ and $m_1, m_2, \dots, m_t \in T$. Take an arbitrary element $a \in A$; by the normality of A in M , there exist $a_i \in A$ such that $m_i a = a_i m_i$ for all $1 \leq i \leq t$. Since $ya = ay$, it follows that

$$(l_1a_1 - l_1a)m_1 + (l_2a_2 - l_2a)m_2 + \cdots + (l_ta_t - l_ta)m_t = 0.$$

As $\{m_1, m_2, \dots, m_t\}$ is linearly independent over K , the outcome is that $a = a_1 = \cdots = a_t$. Consequently, $m_i a = a m_i$ for all $a \in A$; thus, $m_i \in C_M(A) = A$ for all $1 \leq i \leq t$. The consequence of this fact is that $y \in K$, yielding $L = K$. This implies that K is a maximal subfield of R , and our proof is now completed. \square

Lemma 2.17 ([31, 3.2]). *Let R be a ring, J a subring of R , and $H \leq K$ subgroups of the group of units of R normalizing J such that R is the ring of right quotients of $J[H] \leq R$ and $J[K]$ is a crossed product of $J[B]$ by K/B for some normal subgroup B of K . Then $K = HB$.*

Theorem 2.18. *Let D be a non-commutative division ring with center F which contains at least four elements, and G a normal subgroup of $\text{GL}_n(D)$ with $n \geq 2$. If M is a non-abelian solvable-by-finite maximal subgroup of G such that $F[M] = M_n(D)$, then $[D : F] < \infty$.*

Proof: First, we observe that M is abelian-by-locally finite ([28, Theorem 1]). As a result, there exists in M a maximal subgroup A with respect to the property: A is an abelian normal subgroup of M and M/A is locally finite. In view of [25, 1.2.12], we conclude that $F[A]$ is a semisimple Artinian ring; thus, the Wedderburn–Artin theorem implies that

$$F[A] \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_s}(D_s),$$

where D_i are division F -algebras, $1 \leq i \leq s$. Since $F[A]$ is abelian, it follows that the n_i 's are equal to 1 and D_i 's are fields that contain F . Consequently,

$$F[A] \cong K_1 \times K_2 \times \cdots \times K_s.$$

With reference to Lemma 2.11(iii), we conclude that $F[A]$ is an integral domain and so $s = 1$. It follows that $K := F[A]$ is a subfield of $M_n(D)$ containing F .

If we set $L = C_{M_n(D)}(K)$, then by Lemma 2.11(iv), one has $L \cong M_m(E)$ for some division F -algebra E and some integer $m \geq 1$. Since M normalizes K , it also normalizes L ; hence, either $L^* \cap G \subseteq F$, or $L = M_n(D)$ or $L^* \cap G \subseteq M$ (Lemma 2.9). The first case implies that $M_n(D) = K$, which is impossible since $n > 1$. If the second case occurs, then $A \subseteq F$, from which it follows that $M/M \cap F^*$ is locally finite. According to Lemma 2.15, we deduce that D is a locally finite division ring. Now, we can use [9, Theorem 3.1] to conclude that $[D : F] < \infty$. It remains only to consider the third case, $L^* \cap G \subseteq M$, from which we have $L^* \cap G$ is a solvable-by-finite normal subgroup of $\mathrm{GL}_m(E)$. By applying both Lemmas 2.4 and 2.8 to this situation, we obtain that $L^* \cap G \subseteq Z(E)$. Thus, $L^* \cap G$ is an abelian normal subgroup of M and $M/L^* \cap G$ is locally finite. The maximality of A in M yields $A = L^* \cap G = L^* \cap M = C_M(A)$. These equalities imply that A is actually a maximal abelian subgroup of M . In view of Lemma 2.16, we conclude that $F[M] = M_n(D)$ is a crossed product of K by M/A , and that K is a maximal subfield of $M_n(D)$.

Our next step is to prove that M/A is simple. To do so, assume that B is an arbitrary normal subgroup of M properly containing A . Note that by the maximality of A in M , we may assume further that N is non-abelian. If we set $R = F[B]$ and Q to be its quotient ring, then Lemma 2.11(iii) says that $Q = M_n(D)$. Now, we may apply Lemma 2.17 to conclude that $M = AB = B$; recall that $A \subseteq B$. It follows that M/A is simple.

Since M is solvable-by-finite, it contains a solvable normal subgroup N such that M/N is finite. As AN is also a normal subgroup of M , the

simplicity of M/A shows that $AN = M$ or $AN = A$. The first case implies that M is solvable, which also means that M/A is solvable. Now, M/A is simple and solvable, and one has $M/A \cong \mathbb{Z}_p$ for some prime number p . By Lemma 2.16, it follows that $\dim_K M_n(D) = |M/A| = p$, which forces $n = 1$, an obvious contradiction. As a result, we must have $AN = A$, from which it follows that $[M : A] < \infty$. Again by Lemma 2.16, one has $[M_n(D) : K] = |M/A| < \infty$ and so $[D : F] < \infty$. \square

Corollary 2.19. *Let D be a non-commutative division ring with center F which contains at least five elements, and G a normal subgroup of $\text{GL}_n(D)$ with $n \geq 2$. If M is a solvable maximal subgroup of G , then M is abelian.*

Proof: If $F[M] \neq M_n(D)$, then the result follows from Theorem 2.14. In the case $F[M] = M_n(D)$, if M is non-abelian, then the last paragraph of the proof of the above theorem means that $n = 1$, a contradiction. \square

Here now is the proof of the main theorem of this section.

Proof of Theorem 1.1: Combining Theorems 2.7, 2.14, and 2.18, we get $[D : F] < \infty$. By hypothesis, we conclude that M contains no non-cyclic free subgroups. Therefore, most of the conclusions follow from [9, Theorem 3.1]. From the maximality of K , we deduce that $C_{M_n(D)}(K) = K$. By the centralizer theorem ([4, (vii), p. 42]) one has $[K : F]^2 = [M_n(D) : F] = n^2[D : F]$. It follows that $|M/K^* \cap G| = |\text{Gal}(K/F)| = [K : F] = n\sqrt{[D : F]}$. \square

Other results concerning solvable-by-finite subgroups of $\text{GL}_n(D)$, where D is a centrally finite division ring and $n \geq 1$, were obtained by Wehrfritz in [32]. In fact, he proved that if M is a solvable-by-finite subgroup of $\text{GL}_n(D)$, then it contains an abelian normal subgroup of index dividing $b(n)[D : F]^n$, where $b(n)$ is an integer-valued function that depends only on n . In view of Theorem 1.1, if M is supposed further to be a maximal subgroup of an almost subnormal subgroup of $\text{GL}_n(D)$, then M possesses an abelian normal subgroup of very explicit index.

Theorem 1.1 also gives some interesting corollaries that are closely related to the results obtained in [3], [11], [23], and [32]. More precisely, the authors of [11] asked whether a division D is a crossed product if the multiplicative group D^* contains an absolutely irreducible solvable-by-finite subgroup M (see [11, Question 2.5]). By definition, a centrally finite division D is called a *crossed product* if it contains a maximal subfield that is a Galois extension over the center of D . The following corollary, which follows immediately from Theorem 1.1, shows that the question has a positive answer if M is a non-abelian solvable-by-finite maximal subgroup of an almost subnormal subgroup of D^* .

Corollary 2.20. *Let D be a division ring, and G an almost subnormal subgroup of D^* . If M is a non-abelian solvable-by-finite maximal subgroup of G , then D is a crossed product.*

Polycyclic-by-finite maximal subgroups of $\mathrm{GL}_n(D)$ have already been studied in [23]. One of the main results of [23] states that $\mathrm{GL}_n(D)$ contains no polycyclic-by-finite maximal subgroups if $n = 1$ or the center of D contains at least five elements ([23, Theorem B]). In the next corollary, we extend this result to polycyclic-by-finite maximal subgroups of an almost subnormal subgroup of $\mathrm{GL}_n(D)$.

Corollary 2.21. *Let D, F, G as in Theorem 1.1. If M is a finitely generated solvable-by-finite maximal subgroup of G , then M is abelian. In particular, if M is polycyclic-by-finite, then it is abelian.*

Proof: With reference to Theorem 1.1, we have $[D : F] < \infty$. But then M cannot be finitely generated in view of [19, Corollary 3]. The rest of our corollary follows immediately from the fact that every polycyclic-by-finite group is finitely generated. \square

3. Locally solvable maximal subgroups

In this section we study locally solvable maximal subgroups of an almost subnormal subgroup of $\mathrm{GL}_n(D)$, with $n \geq 1$. We note that in the case $n = 1$, the following results were obtained in [15].

Theorem 3.1 ([15, Theorem 3.6]). *Let D be a division ring with center F , and G an almost subnormal subgroup of D^* . If M is a locally nilpotent maximal subgroup of G , then M is abelian.*

Theorem 3.2 ([15, Theorem 3.7]). *Let D be a division ring with center F , and G an almost subnormal subgroup of D^* . If M is a non-abelian locally solvable maximal subgroup of G , then the following hold:*

- (1) *There exists a maximal subfield K of D such that K/F is a finite Galois extension with $\mathrm{Gal}(K/F) \cong M/K^* \cap G \cong \mathbb{Z}_p$ and $[D : F] = p^2$, for some prime number p .*
- (2) *The subgroup $K^* \cap G$ is the FC-center. Also, $K^* \cap G$ is the Hirsch–Plotkin radical of M . For any $x \in M \setminus K$, we have $x^p \in F$ and $D = F[M] = \bigoplus_{i=1}^p Kx^i$.*

For any group G , we denote by $\tau(G)$ the unique maximal locally finite normal subgroup of G , $\eta(G)$ the Hirsch–Plotkin radical of G , and $\alpha(G)$ and $\beta(G)$ the two subgroups of G which are defined by

$$\beta(G)/\tau(G) = \eta(G/\tau(G)) \quad \text{and} \quad \alpha(G)/\tau(G) = Z(\beta(G)/\tau(G)).$$

Lemma 3.3 ([30]). *Let G be a locally solvable primitive subgroup of $\mathrm{GL}_n(D)$ with $n \geq 1$. Then the F -subalgebra $F[G]$ of the full matrix ring $M_n(D)$ generated by G is a crossed product over the (locally finite)-by-abelian normal subgroup $\alpha(G)$ of G .*

Proof of Theorem 1.2: For purposes of contradiction, assume that M is not abelian. The crucial step in our proof is to point out that $[D : F] < \infty$. For, since M is primitive (Lemma 2.11(i)), it follows from the previous lemma that $F[M]$ is a crossed product over the (locally finite)-by-abelian normal subgroup $\alpha(M)$ of M . Let A be the maximal (locally finite)-by-abelian normal subgroup of M containing $\alpha(M)$. If B is a normal subgroup of M properly containing A , then by the same arguments used in the second paragraph of the proof of Theorem 2.18 we conclude that $M = B\alpha(M) = B$ and, in consequence, the group M/A is simple.

Since M/A is simple and locally solvable, it is a finite group of prime degree. Let T be a locally finite normal subgroup of A such that A/T is abelian. Then, Lemma 2.11(iii) implies that $F[T]$ is prime. Because T is locally finite, we conclude that $F[T]$ is simple Artinian ([25, 1.1.14]). Thus, Lemma 2.11(iii) again says that either $F[T] = M_n(D)$ or T is abelian. If the first case occurs, then Lemma 2.15 implies that D is locally finite, and thus $[D : F] < \infty$ by [9, Theorem 3.1]; we are done. If the second case occurs, then A is solvable, and M is thus solvable-by-finite. With reference to Theorem 1.1, we conclude that $[D : F] < \infty$.

Set $k := [D : F]$. By viewing M as a (linear) subgroup of $\mathrm{GL}_{kn}(F)$, we conclude that it is solvable. It follows from Corollary 2.19 that M is abelian, and we arrive at the desired contradiction. The proof is now completed. \square

The proof of Theorem 1.2 depends strongly on the assumption that D is non-commutative. To deal with the commutative case, we need some different approaches. The rest of the present paper aims to consider Theorem 1.2 in the case where D is a field.

Remark 1. Let D be a division ring and $n \geq 1$ an integer. As mentioned in the introduction, a subgroup G of $\mathrm{GL}_n(D)$ is called an absolutely irreducible skew linear group over D if $F[G] = M_n(D)$. For the linear case, there are various equivalent definitions of this concept. To be more specific, suppose that G is a subgroup of $\mathrm{GL}_n(F)$ for some field F . Then G is said to be absolutely irreducible if one of the following equivalent conditions holds: $F[G] = M_n(F)$; G is irreducible over every field extension of F ; the centralizer of G in $M_n(F)$ is F (see [25, p. 10]). Therefore, the following lemma is immediate.

Lemma 3.4. *Let F be a field, \bar{F} the algebraic closure of F , and M a subgroup of $\mathrm{GL}_n(F)$ with $n \geq 1$. If M is absolutely irreducible over F , then it is absolutely irreducible over \bar{F} .*

Proof of Theorem 1.3: In order to arrive at a contradiction, we assume that M is non-abelian. If we set $R = F[M]$, then either $R^* \cap G \subseteq M$ or else $R = M_n(F)$ by Lemma 2.9. If the first case occurs, by the same arguments used (for N) in the proof of Theorem 2.14 we find that M is abelian, an obvious contradiction. We may therefore assume that $R = M_n(F)$, which means that M is absolutely irreducible over F . It follows from [27, Lemma 3.5(ii)] that M is abelian-by-finite. Let A be a maximal subgroup of M with respect to the property: A is an abelian normal subgroup of M such that M/A is finite. If $K = F[A]$, then the primitivity (see Lemma 2.11(i)) of M yields that K is a prime ring, and so this is actually an integral domain. Since $\dim_F K < \infty$, we conclude that K is a subfield of $M_n(F)$. If we set $L = C_{M_n(F)}(A)$, then there exist a division F -algebra E and an integer $m \geq 1$ such that $L \cong M_m(E)$ (Lemma 2.11(iv)). It follows from Lemma 2.9 that either $L = M_n(F)$ or $L^* \cap G \subseteq M$. There are two possible cases.

Case 1. $L = M_n(F)$. It is clear that $A \subseteq K \subseteq F$. Since we are in the case $F[M] = M_n(F)$, we conclude that $Z(M) = M \cap F^*$. Thus, the condition $[M : A] < \infty$ and $A \subseteq F$ imply that $[M : Z(M)] < \infty$. As a result, we can take a transversal $\{x_1, \dots, x_k\}$ of $Z(M)$ in M . Pick an element $x \in G \setminus M$ and set $H = \langle x, x_1, \dots, x_k \rangle$. The maximality of M in G ensures us that $G = HZ(M)$. Since $G' = (HZ(M))' = H' \subseteq G$, it follows that H is normal in G , from which it follows that H is a finitely generated subnormal subgroup of $\mathrm{GL}_n(F)$. By virtue of [19], we have $H \subseteq F$, and so M is abelian, which is a desired contradiction.

Case 2. $L^* \cap G \subseteq M$. In this case, one has $L^* \cap G$ is a solvable normal subgroup of $\mathrm{GL}_m(E)$, from which it follows that $L^* \cap G$ is an abelian normal subgroup of M . The maximality of A in M shows that $L^* \cap G = L^* \cap M = A$. Consequently, A is indeed a maximal abelian subgroup of M . Therefore, we may apply Lemma 2.16 to conclude that $F[M] = M_n(F)$ is a crossed product of K by M/A , and that K is a maximal subfield of $M_n(F)$. By the same arguments used in the second paragraph of the proof of Theorem 2.18, we conclude that M/A is a simple group.

Since M/A is solvable and simple, it is of prime degree, say p . Consequently, we have $n = p$ and $[K : F] = p$. Let \bar{F} be the algebraic closure of F . In view of Lemma 3.4, we conclude that $\bar{F}[M] = M_n(\bar{F})$. It follows that $Z(M) = M \cap \bar{F}^*$. The condition $A \subseteq K^* \subseteq \bar{F}^*$ implies that

$A \subseteq Z(M)$; hence, we deduce $A = Z(M)$ by the maximality of A . Also, the fact that $F[M] = M_n(F)$ yields that $Z(M) = M \cap F^*$, from which it follows that $A \subseteq F$. This being the case, we obtain that $K = F[A] = F$, contrasting with the fact that $[K : F] = p$ is a prime number. \square

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