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HARDY INEQUALITIES IN FRACTIONAL ORLICZ–SOBOLEV SPACES

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Abstract: In this article we prove both norm and modular Hardy inequalities for class functions in one-dimensional fractional Orlicz–Sobolev spaces.

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1. Introduction

In the early 1920's, in the seminal article [**15**], G. H. Hardy obtained inequalities of the form

$$\int_0^\infty \frac{|u(x)|^p}{x^p}\,dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty |u'(x)|^p\,dx, \tag{1.1}$$

where $p>1$ and u is a nonnegative measurable function defined on $(0,\infty)$.

Throughout the decades that followed, many authors contributed to characterize the family of weights v, w and powers p, q for which inequalities of the type

$$\left(\int_0^\infty |u(x)|^p v(x)\,dx\right)^{\frac{1}{p}} \leq \mathbf{C}_H \left(\int_0^\infty |u'(x)|^q w(x)\,dx\right)^{\frac{1}{q}}$$

are valid for a suitable positive constant $\mathbf{C}_H$ independent of u. See for instance the pioneering works [**7, 30, 31**]. We also refer to the seminal books [**22, 26**].

Orlicz spaces play a fundamental role when describing phenomena with non-standard growth. See [**19, 21, 28**]. Generalizations of Hardy type inequalities to the Orlicz space structure provide for a family of inequalities admitting behaviors more general than powers. In recent years many authors have undertaken the task of characterizing the class of admissible weights v, w, a, b and Young functions P, Q for which an inequality as follows is valid:

$$Q^{-1}\left(\int_0^\infty Q(a(x)|Tu(x)|)v(x)\,dx\right) \leq \mathbf{C}_H P^{-1}\left(\int_0^\infty P(b(x)|u(x)|)w(x)\,dx\right),$$

where $Tu(x) = \int_0^x \mathcal{K}(x,y)u(y)\,dy$ is the generalized Hardy operator and $\mathbf{C}_H$ is a positive constant independent of u. For more details we refer to [**4, 6, 16, 23**] and references therein. In contrast with the L^p case, Hardy inequalities in integral form may differ from norm inequalities. See [**4, 8**].

Nonlocal Hardy inequalities have been a subject of study in recent years. The nonlocal counterpart of (1.1), for suitable values of $p > 1$ and $s \in (0,1)$, takes the form

$$\int_0^\infty \frac{|u(x)|^p}{x^{sp}}\,dx \leq \mathbf{C}_H \int_0^\infty \int_0^\infty \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}}\,dx\,dy$$

for functions in an appropriate Sobolev space. Hardy inequalities for the fractional p-Laplacian date back to the early 1960's and were derived independently in [**14**] and [**18**]. For the case $p = 2$ see also [**1, 3**]. See also [**20**] for a different approach.

To our knowledge, the only research to have been done on Hardy inequalities in a nonlocal framework with nonstandard growth is [**2**], for values of $s \in (0,1)$ close to 0. See also [**24**]. Therefore, the main scope of this paper is to study the validity of these inequalities in the fractional order Orlicz–Sobolev spaces introduced in [**2, 12**], in both integral and norm form for *large* values of $s \in (0,1)$, that is, when $s > 1/p^-$, where $p^- > 1$ is a fixed constant.

In order to state our results, recall that a *Young function* G is a continuous, nonnegative, strictly increasing, and convex function on $\mathbb{R}_+ := [0,\infty)$ (see Section 2), for which we assume the growth condition

$$\text{(L)} \qquad 1 < p^- \leq \tfrac{tg(t)}{G(t)} \leq p^+ < \infty \quad \text{for all } t > 0$$

for fixed constants $p^\pm$, where $g = G'$. Given a Young function G, we define

$$L^G(\mathbb{R}_+) := \{u\colon \mathbb{R}_+ \to \mathbb{R} \text{ measurable} : \Phi_G(u) < \infty\}.$$

The *modular* and the *Luxemburg norm* of $u \in L^G(\mathbb{R}_+)$, respectively, are defined as

$$\Phi_G(u) = \int_0^\infty G(|u(x)|)\,dx \quad \text{and} \quad \|u\|_G := \inf\left\{\lambda > 0\colon \Phi_G\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

Given $s \in (0,1)$, we define the *fractional Orlicz–Sobolev space* as (see [**12**])

$$W^{s,G}(\mathbb{R}_+) := \left\{u \in L^G(\mathbb{R}_+) \text{ such that } \Phi_{s,G}(u) < \infty\right\}.$$

Here the *fractional modular* of $u \in W^{s,G}(\mathbb{R}_+)$ is defined as

$$\Phi_{s,G}(u) := \int_0^\infty \int_0^\infty G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|}.$$

These spaces are endowed with the so-called *Luxemburg norm*, defined as

$$\|u\|_{s,G} := \|u\|_G + [u]_{s,G},$$

where the (s,G)-*Gagliardo semi-norm* reads as

$$[u]_{s,G} := \inf\left\{\lambda > 0\colon \Phi_{s,G}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

We now describe our results. As we will see, the proof of our nonlocal modular inequality is based on an astute application of the following modular Hardy inequality for the local Orlicz–Sobolev space $W^{1,G}(\mathbb{R}_+)$ (see Lemma 3.4 for details):

$$\int_0^\infty G\left(\left|\frac{1}{x^s}\int_0^x \frac{u(t)}{t}\,dt\right|\right)dx \le \mathbf{C}_H \int_0^\infty G\left(\frac{|u(x)|}{x^s}\right)dx. \tag{1.2}$$

Following [**5**], inequality (1.2) can be obtained from the local norm inequalities studied in [**25, 27**].

Finally, for simplicity in our notation, given a Young function G satisfying (L) we define the functions

$$\psi_G(x) = \begin{cases} x^{p^+} & \text{if } x \ge 1,\\ x^{p^-} & \text{if } x < 1,\end{cases} \quad \text{and} \quad \phi_G(x) = \begin{cases} x^{1/p^-} & \text{if } x \ge 1,\\ x^{1/p^+} & \text{if } x < 1.\end{cases} \tag{1.3}$$

With these preliminaries, our first result reads as follows.

Theorem 1.1. *Let G be a Young function satisfying* (L) *and let $s \in (0,1)$ be such that $sp^- > 1$. Then for all $u \in W^{s,G}(\mathbb{R}_+)$ such that $\lim_{x\to 0}\frac{1}{x}\int_0^x u(x)\,dx = u_0$, the following inequality holds:*

$$\int_0^\infty G\left(\frac{|u(x)-u(x_0)|}{x^s}\right)dx \le \mathbf{C}_H\Phi_{s,G}(u), \quad \mathbf{C}_H := \mathbf{C}(1+\mathbf{C}_H),$$

where $\mathbf{C}_H$ is given in (1.2) *and $\mathbf{C} := 2^{p+}$ is the doubling constant for G.*

Remark 1.2. The constant $\mathbf{C}_H$ in Theorem 1.1 can be computed explicitly as

$$\mathbf{C}_H = \mathbf{C}(1+\mathbf{C}_H) = 2^{p^+}\left(1+\psi_G\left(\frac{p^-}{sp^- - 1}\right)\right).$$

Remark 1.3. Observe that $u \in W^{s,G}(\mathbb{R}_+)$ is in fact continuous in a neighborhood $\mathcal{O}$ of the origin when $sp^- > 1$ due to the embedding of $W^{s,G}(\mathcal{O})$ into the space $C^{0,s-1/p^-}(\mathcal{O})$ of Hölder continuous functions (see Proposition 2.2).

From the modular inequality it is easy to deduce a norm inequality.

Corollary 1.4. *With the same assumptions of Theorem 1.1,*

$$\left\|\frac{u-u_0}{x^s}\right\|_G \le \phi_G(\mathbf{C}_H)[u]_{s,G},$$

where ϕ_G is given in (1.3) *and $\mathbf{C}_H$ is the constant given in Theorem 1.1.*

From Theorem 1.1 and Remark 1.3 the following consequence can easily be deduced.

Corollary 1.5. *Let G be a Young function satisfying* (L) *and let $s \in (0,1)$ be such that $sp^- > 1$. Then for all $u \in W^{s,G}(\mathbb{R}_+)$ such that $u(0) = 0$ it holds that*

$$\int_0^\infty G\left(\frac{|u(x)|}{x^s}\right)dx \le \mathbf{C}_H\Phi_{s,G}(u), \quad \left\|\frac{u}{x^s}\right\|_G \le \phi_G(\mathbf{C}_H)[u]_{s,G},$$

where ϕ_G is given in (1.3) *and $\mathbf{C}_H$ is the constant given in Theorem 1.1.*

Although a norm inequality can be deduced from Theorem 1.1, it can also be obtained independently of the modular inequality. In fact, the following result provides for a more accurate constant.

Theorem 1.6. *Let G be a Young function satisfying* (L) *and let $s \in (0,1)$ be such that $sp^- > 1$. Then*

$$\left\|\frac{u}{x^s}\right\|_G \le \frac{(1+s)p^- - 1}{sp^- - 1}[u]_{s,G}$$

for all $u \in W^{s,G}(\mathbb{R}_+)$.

Sharpness of the Hardy constant is known both in the local and nonlocal case when G is a power; see [**13, 17**] for instance. However, in the Orlicz setting it is unknown even in the local case.

2. Preliminary results

2.1. Young functions. A *Young function* is an application $G\colon \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous, nonnegative, strictly increasing, convex on $[0,\infty)$, and admits the integral formulation $G(t) = \int_0^t g(s)\,ds$. For some fixed constants $p^\pm$ we assume that G satisfies the growth condition (L).

The *complementary Young function* $\tilde{G}$ of a Young function G is defined as

$$\tilde{G}(t) = \sup\{tw - G(w) : w > 0\}.$$

We introduce some well-known results on Young functions. See [**21, 28**] for details.

Lemma 2.1. *Let G be a Young function satisfying* (L) *and $a, t \geq 0$. Then*

$$(G_1) \qquad \min\{a^{p^-}, a^{p^+}\}G(t) \leq G(at) \leq \max\{a^{p^-}, a^{p^+}\}G(t),$$

$$(G_2) \qquad G(a+t) \leq \mathbf{C}(G(a) + G(t)) \text{ with } \mathbf{C} := 2^{p^+}.$$

Condition (G_2) is known as the Δ_2 *condition* or *doubling condition.* It can be proved that (L) implies that both G and $\tilde{G}$ satisfy (G_2). See [**21**, Theorem 3.4.4 and Theorem 3.13.9].

Young functions include for instance powers (when $g(t) = t^{p-1}$, $p^\pm = p > 1$, and hence $G(t) = \frac{t^p}{p}$) and logarithmic perturbations of powers (when $g(t) = t\log(b + ct)$, where $p^- = 1 + a$, $p^+ = 2 + a$). See [**21**] for more examples.

2.2. Fractional Orlicz–Sobolev spaces. Given a Young function G, a fractional parameter $s \in (0,1)$, and an open interval $\Omega \subseteq \mathbb{R}$, we have already defined the fractional Orlicz–Sobolev space $W^{s,G}(\Omega)$ in the introduction. We also define the following related space:

$$W_0^{s,G}(\Omega) := \{u \in W^{s,G}(\mathbb{R}) : u = 0 \text{ a.e. in } \mathbb{R} \setminus \Omega\},$$

which coincides with the closure of C_c^∞ functions with respect to the $\|\cdot\|_{s,G}$ norm, and is the natural space for the well-posedness of Dirichlet problems.

For a further generalization of these spaces we refer to [**9**].

The following result characterizes continuous functions in fractional Orlicz–Sobolev spaces.

Proposition 2.2. *Let G be a Young function satisfying* (L) *and let $s \in (0,1)$ be such that $sp^- > 1$. Then, given an open and bounded interval $\Omega \subset \mathbb{R}$, it holds that $W^{s,G}(\Omega) \subset C^{0,s-1/p^-}(\Omega)$.*

Proof: From [**11**, Proposition 2.9 and Proposition 2.7], for any $u \in W^{s,G}(\Omega)$ we get

$$[u]_{W^{s,p^-}_{\mathrm{reg}}(\Omega)} + \|u\|_{L^{p^-}(\Omega)} \leq C([u]_{W^{s,G}(\Omega)} + \|u\|_{L^G(\Omega)}),$$

where

$$[u]_{W^{s,p^-}_{\mathrm{reg}}(\Omega)} = \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p^-}}{|x-y|^{n+sp^-}}\,dx\,dy\right)^{\frac{1}{p}}.$$

Moreover, by [**10**, Theorem 8.2], $\|u\|_{C^{0,s-1/p^-}(\Omega)} \leq C([u]_{W^{s,p^-}_{\mathrm{reg}}(\Omega)} + \|u\|_{L^{p^-}(\Omega)})$ and the result follows. □

3. One-dimensional Hardy inequalities

In this section we prove our main results. First, following [**5**], we link norm inequalities for linear operators with integral inequalities.

Proposition 3.1. *Let G be a Young function satisfying* (L). *Suppose that the inequality $\|Tu\|_{\varepsilon G} \leq C\|u\|_{\varepsilon G}$ holds for all $\varepsilon > 0$ with C independent of ε, where $\|u\|_{\varepsilon G} = \inf\{\lambda : \int_0^\infty G(\frac{|u(x)|}{\lambda})\varepsilon\,dx \leq 1\}$ and T is a linear operator. Then it holds that*

$$\int_0^\infty G(|Tu(x)|)\,dx \leq \int_0^\infty G(C|u(x)|)\,dx.$$

Proof: Given $u \in L^G(\mathbb{R}_+)$, define the number $\varepsilon = \left(\int_0^\infty G(|u|)\,dx\right)^{-1}$ and observe that $\|u\|_{\varepsilon G} \leq 1$. Therefore, $\|Tu\|_{\varepsilon G} \leq C\|u\|_{\varepsilon G} \leq C$, and then, by definition of the Luxemburg norm we get

$$\int_0^\infty G\left(\frac{|Tu|}{C}\right)dx \leq \frac{1}{\varepsilon}\int_0^\infty G\left(\frac{|Tu|}{\|Tu\|_{\varepsilon G}}\right)\varepsilon\,dx \leq \frac{1}{\varepsilon} = \int_0^\infty G(|u|)\,dx.$$

Finally, since T is linear, replacing u with Cu the result follows. □

In order to apply Proposition 3.1 we use the following inequality due to [**27**] (cf. also [**25**, Corollary 4]).

Proposition 3.2. *Given $I = (0,\ell)$, $0 < \ell \leq \infty$, if $\theta \in \mathbb{R}$ is such that $\theta < 1/(p^-)'$, then, for $x^\theta u(x) \in L^G(I)$,*

$$\left\| x^{\theta-1}\int_0^x u(t)\,dt \right\|_{L^G(I)} \leq \frac{(p^-)'}{1-\theta(p^-)'}\|x^\theta u(x)\|_{L^G(I)},$$

where $(p^-)'$ is the conjugated exponent of p^-.

Corollary 3.3. *Given $I = (0, \ell)$, $0 < \ell \leq \infty$, and $s \in (0,1)$,*

(i) *if $\theta = 1 - s$ and $x^{1-s}u(x) \in L^G(I)$, for $sp^- > 1$ it holds that*

$$\left\| \frac{1}{x^s} \int_0^x u(t)\, dt \right\|_{L^G(I)} \leq \frac{p^-}{sp^- - 1} \|x^{1-s}u(x)\|_{L^G(I)}, \tag{3.1}$$

(ii) *if $\theta = -s$ and $x^{-s}u(x) \in L^G(I)$, for $(1+s)p^- > 1$ it holds that*

$$\left\| \frac{1}{x^{1+s}} \int_0^x u(t)\, dt \right\|_{L^G(I)} \leq \frac{p^-}{(1+s)p^- - 1} \|x^{-s}u(x)\|_{L^G(I)}. \tag{3.2}$$

We now prove a key lemma for our arguments.

Lemma 3.4. *Let G be a Young function satisfying* (L) *and let $s \in (0,1)$ be such that $sp^- > 1$. Given u such that $x^{-s}u(x) \in L^G(\mathbb{R}_+)$, then we have that*

$$\int_0^\infty G\left(\left| \frac{1}{x^s} \int_0^x \frac{u(t)}{t}\, dt \right| \right) dx \leq \mathbf{C}_H \int_0^\infty G\left(\frac{|u(x)|}{x^s} \right) dx. \tag{3.3}$$

The constant $\mathbf{C}_H$ is given by $\mathbf{C}_H = \psi_G\big(\frac{p^-}{sp^- - 1}\big)$, where ψ_G is given in (1.3).

Proof: Given G satisfying (L), we define the Young function $G_\varepsilon := \varepsilon G$ for some $\varepsilon > 0$. It is immediate that G_ε satisfies (L) with the same constants as G.

Let u be a fixed function such that $x^{-s}u(x) \in L^G(\mathbb{R}_+)$, then also $x^{-s}u(x) \in L^{\varepsilon G}(\mathbb{R}_+)$. Hence, by applying (3.1) in Corollary 3.3 to $x^{-1}u(x)$ we find that

$$\left\| \frac{1}{x^s} \int_0^x \frac{u(t)}{t}\, dt \right\|_{G_\varepsilon} \leq \frac{p^-}{sp^- - 1} \left\| \frac{u}{x^s} \right\|_{G_\varepsilon} \quad \forall \varepsilon > 0. \tag{3.4}$$

Since (3.4) holds with constant independent of ε, from Proposition 3.1 we get that

$$\int_0^\infty G\left(\left| \frac{1}{x^s} \int_0^x \frac{u(t)}{t}\, dt \right| \right) dx \leq \int_0^\infty G\left(\frac{p^-}{sp^- - 1} \frac{|u(x)|}{x^s} \right) dx$$

and the result follows by using condition (G_1). □

We are ready to prove our main result.

Proof of Theorem 1.1: We assume without loss of generality that

$$\lim_{x \to 0} \frac{1}{x} \int_0^x u(x)\, dx = 0.$$

Then, the general case will follow by applying Theorem 1.1 to $u - u_0$.

Given $u \in W^{s,G}(\mathbb{R}_+)$ we consider the auxiliary function

$$v(x) = u(x) - \frac{1}{x}\int_0^x u(t)\,dt.$$

For $0 < a < b < \infty$, using integration by parts it is straightforward to see that

$$\begin{aligned}\int_a^b \frac{v(x)}{x}\,dx &= \int_a^b \frac{u(x)}{x}\,dx - \int_a^b \frac{1}{x^2}\int_0^x u(s)\,ds\,dx \\ &= \int_a^b \frac{u(x)}{x}\,dx + \frac{1}{x}\int_0^x u(s)\,ds\Big|_a^b - \int_a^b \frac{u(x)}{x}\,dx \\ &= \frac{1}{b}\int_0^b u(x)\,dx - \frac{1}{a}\int_0^a u(x)\,dx.\end{aligned}$$

Taking $b = t$ and $a \to 0$ in the last expression we get $\int_0^t \frac{v(x)}{x}\,dx = \frac{1}{t}\int_0^t u(x)\,dx$, and

$$u(x) = v(x) + \int_0^x \frac{v(t)}{t}\,dt. \tag{3.5}$$

We now prove that $x^{-s}v(x) \in L^G(\mathbb{R}_+)$. Indeed, by definition of v we have that

$$\int_0^\infty G\Big(\frac{|v(x)|}{x^s}\Big)\,dx = \int_0^\infty G\Big(\frac{|u(x) - \frac{1}{x}\int_0^x u(t)\,dt|}{x^s}\Big)\,dx.$$

Now, by using Jensen's inequality we get

$$\begin{aligned}\int_0^\infty G\Big(\frac{|u(x) - \frac{1}{x}\int_0^x u(t)\,dt|}{x^s}\Big)\,dx &\le \int_0^\infty G\Big(\frac{1}{x}\int_0^x \frac{|u(x)-u(t)|}{x^s}\,dt\Big)\,dx \\ &\le \int_0^\infty \frac{1}{x}\int_0^x G\Big(\frac{|u(x)-u(t)|}{x^s}\Big)\,dt\,dx,\end{aligned}$$

and the last term in the inequality above can be bounded as

$$\begin{aligned}\int_0^\infty \frac{1}{x}\int_0^x G\Big(\frac{|u(x)-u(t)|}{x^s}\Big)\,dt\,dx &\le \int_0^\infty\int_0^x G\Big(\frac{|u(x)-u(y)|}{|x-y|^s}\Big)\frac{dy\,dx}{|x-y|} \\ &\le \int_0^\infty\int_0^\infty G\Big(\frac{|u(x)-u(y)|}{|x-y|^s}\Big)\frac{dy\,dx}{|x-y|}.\end{aligned}$$

Since $x^{-s}v(x) \in L^G(\mathbb{R}_+)$, we are in a position to apply Lemma 3.4 to v, obtaining

$$\int_0^\infty G\Big(\Big|\frac{1}{x^s}\int_0^x \frac{v(t)}{t}\,dt\Big|\Big)\,dx \le \mathbf{C}_H \int_0^\infty G\Big(\frac{|v(x)|}{x^s}\Big)\,dx. \tag{3.6}$$

Therefore, from (3.5), property (G_2), and (3.6), we find that

$$\begin{aligned}\int_0^\infty G\left(\frac{|u(x)|}{x^s}\right)dx &\le \mathbf{C}\int_0^\infty G\left(\frac{|v(x)|}{x^s}\right)dx+\mathbf{C}\int_0^\infty G\left(\frac{1}{x^s}\left|\int_0^x \frac{v(t)}{t}\,dt\right|\right)dx\\ &\le \mathbf{C}(1+\mathbf{C}_H)\int_0^\infty G\left(\frac{|v(x)|}{x^s}\right)dx\\ &\le \mathbf{C}(1+\mathbf{C}_H)\Phi_{s,G}(u),\end{aligned}$$

giving the desired inequality. $\square$

As a corollary, we obtain the Hardy inequality for norms stated in Corollary 1.4.

Proof of Corollary 1.4: We assume without loss of generality that

$$\lim_{x\to 0}\frac{1}{x}\int_0^x u(x)\,dx = 0.$$

Then, the general case will follow by applying the inequality to $u-u_0$.

By using Theorem 1.1 together with (G_1) one gets that $\Phi_G(\frac{u}{x^s}) \le \Phi_{s,G}(\tilde{\mathbf{C}}_H u)$ for $u\in W^{s,G}(\mathbb{R}_+)$, where $\tilde{\mathbf{C}}_H=\max\{\mathbf{C}_H^{\frac{1}{p^-}},\mathbf{C}_H^{\frac{1}{p^+}}\}$. Then, given $u\in W^{s,G}(\mathbb{R}_+)$, from the last expression we find that $\Phi_G(u/\tilde{\mathbf{C}}_H[u]_{s,G}x^s)\le \Phi_{s,G}(u/[u]_{s,G})\le 1$. Hence, by definition of the Luxemburg norm we get

$$\left\|\frac{u}{x^s}\right\|_G = \inf\left\{\lambda : \Phi_G\left(\frac{u}{\lambda x^s}\right)\le 1\right\}\le \tilde{\mathbf{C}}_H[u]_{s,G}$$

and the proof concludes. $\square$

Now, we provide for the proof of the norm inequality given in Theorem 1.6.

Proof of Theorem 1.6: Given $u\in W^{s,G}(\mathbb{R}_+)$, by using the triangular inequality for the Luxemburg norm we obtain that

$$\left\|\frac{u}{x^s}\right\|_G \le \left\|\frac{u-\frac{1}{x}\int_0^x u(y)\,dy}{x^s}\right\|_G+\left\|\frac{1}{x^{1+s}}\int_0^x u(y)\,dy\right\|_G := (i)+(ii).$$

Let us find a bound for (i). By using Jensen's inequality we get

$$\begin{aligned}\int_0^\infty G\left(\frac{|u(x)-\frac{1}{x}\int_0^x u(t)\,dt|}{x^s}\right)dx &\le \int_0^\infty G\left(\frac{1}{x}\int_0^x\frac{|u(x)-u(t)|}{x^s}\,dt\right)dx\\ &\le \int_0^\infty\frac{1}{x}\int_0^x G\left(\frac{|u(x)-u(t)|}{x^s}\right)dt\,dx\\ &\le \int_0^\infty\int_0^x G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dy\,dx}{|x-y|}\\ &\le \Phi_{s,G}(u).\end{aligned}$$

The inequality above applied to $u/[u]_{s,G}$ gives that

$$\int_0^\infty G\left(\frac{|u(x) - \frac{1}{x}\int_0^x u(y)\,dy|}{x^s[u]_{s,G}}\right) dx \le \Phi_{s,G}\left(\frac{u}{[u]_{s,G}}\right) \le 1,$$

from where, from the definition of the Luxemburg norm, we obtain that (i) is bounded by $[u]_{s,G}$. Expression (3.2) from Corollary 3.3 gives that (ii) is less than $\frac{p^-}{(1+s)p^- - 1}\left\|\frac{u}{x^s}\right\|_G$. From these two relations we get that

$$\left\|\frac{u}{x^s}\right\|_G \le [u]_{s,G} + \frac{p^-}{(1+s)p^- - 1}\left\|\frac{u}{x^s}\right\|_G.$$

Finally, the condition $sp^- > 1$ leads to

$$\mathbf{C}_H = \left(1 - \frac{p^-}{(1+s)p^- - 1}\right)^{-1} = \frac{(1+s)p^- - 1}{sp^- - 1} > 0$$

and the desired inequality is obtained. □

4. Applications and examples

4.1. Lower bound of Dirichlet eigenvalues. Given a Young function G satisfying (L) and a fractional parameter $s \in (0,1)$ such that $sp^- > 1$, consider the eigenvalue problem for the fractional g-Laplacian operator in an open and bounded interval $\Omega \subset \mathbb{R}^+$ (see [**12, 29**]):

$$\begin{cases} (-\Delta_g)^s u = \lambda g(u)u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}\setminus\Omega. \end{cases} \tag{4.1}$$

This operator is well defined between $W^{s,G}(\mathbb{R})$ and its dual, and acts as

$$(-\Delta_g)^s u(x) := 2p.v.\int_{-\infty}^\infty g(|D_s u|)\frac{D_s u}{|D_s u|}\frac{dy}{|x-y|^{1+s}},$$

where $D_s u(x,y) = \frac{u(x)-u(y)}{|x-y|^s}$. The following representation formula holds:

$$\langle(-\Delta_g)^s u, v\rangle = \int_{-\infty}^\infty\int_{-\infty}^\infty g(|D_s u|)\frac{D_s u}{|D_s u|}D_s v\frac{dxdy}{|x-y|} \quad \forall v \in W^{s,G}(\mathbb{R}).$$

The natural space for solutions of (4.1) is $W_0^{s,G}(\Omega)$, i.e., functions in $W^{s,G}(\mathbb{R})$ such that $u = 0$ in $\mathbb{R}\setminus\Omega$.

Due to the possible lack of homogeneity of the problem, eigenpairs depend strongly on the normalization: for each $\alpha > 0$, $u_\alpha \in \mathcal{X}_\alpha := \{W_0^{s,G}(\Omega) : \Phi_G(u) = \alpha\}$ is an eigenfunction of (4.1) with eigenvalue λ_α if it holds that

$$\langle(-\Delta_g)^s u_\alpha, v\rangle = \lambda_\alpha\int_\Omega g(u_\alpha)u_\alpha\frac{u_\alpha}{|u_\alpha|}v\,dx \quad \forall v \in W_0^{s,G}(\Omega).$$

On the other hand, for each $\alpha > 0$ we define the minimizer (see [**11**, Theorem 2.12])

$$\Lambda_\alpha := \inf\{\Phi_{s,G}(u)/\Phi_G(u) : u \in \mathcal{X}_\alpha\}. \tag{4.2}$$

In [**29**] it is proved that for each $\alpha > 0$ the infimum in (4.2) is attained for some function $u_\alpha \in \mathcal{X}_\alpha$, and then, by a Lagrange multiplier argument, the existence of an eigenvalue λ_α of (4.1) with eigenfunction u_α is deduced. However, in contrast with the case of powers, in general λ_α does not admit a variational characterization and $\lambda_\alpha \neq \Lambda_\alpha$. Both constants are comparable with each other. Indeed, $\frac{p^-}{p^+}\Lambda_\alpha \leq \lambda_\alpha \leq \frac{p^+}{p^-}\Lambda_\alpha$.

Using Lemma 2.1 it is easy to see that for any function $u \in \mathcal{X}_\alpha$ it holds that

$$\int_\Omega G(|u|)\,dx \leq \int_\Omega G\left(\frac{|u(x)|}{x^s}\operatorname{diam}(\Omega)\right)dx \leq \psi_G(\operatorname{diam}(\Omega))\int_\Omega G\left(\frac{|u(x)|}{x^s}\right)dx,$$

where ψ_G is given in (1.3). Hence, from Theorem 1.1 we get that

$$(\psi_G(\operatorname{diam}(\Omega))\mathbf{C}_H)^{-1} \leq \Lambda_\alpha \leq \frac{p^+}{p^-}\lambda_\alpha.$$

4.2. Lower bound of weighted eigenvalues. With the same ideas of [**29**], we can consider eigenvalues λ_α and minimizers Λ_α of the weighted problem

$$\begin{cases} (-\Delta_g)^s u = \frac{\lambda}{|x|^s} g\left(\frac{|u|}{x^s}\right)\frac{u}{|u|} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}\setminus\Omega, \end{cases} \tag{4.3}$$

where $\Omega \subset \mathbb{R}^+$ is an open and bounded interval. Given $\alpha > 0$, a number λ_α is an eigenvalue of (4.3) with eigenfunction $u_\alpha \in \mathcal{X}_\alpha := \left\{W_0^{s,G}(\Omega) : \int_\Omega G\left(\frac{|u(x)|}{x^s}\right)dx = \alpha\right\}$ if it holds that

$$\langle(-\Delta_g)^s u_\alpha, v\rangle = \lambda_\alpha \int_\Omega g\left(\frac{|u_\alpha(x)|}{|x|^s}\right)\frac{u_\alpha(x)}{|u_\alpha(x)|}\frac{v(x)}{x^s}\,dx \quad \forall v \in W_0^{s,G}(\Omega).$$

We also define the number $\Lambda_\alpha := \inf\left\{\Phi_{s,G}(u)/\int_\Omega G\left(\frac{|u(x)|}{x^s}\right)dx : u \in \mathcal{X}_\alpha\right\}$.

We have again in this case that $\frac{p^-}{p^+}\Lambda_\alpha \leq \lambda_\alpha \leq \frac{p^+}{p^-}\Lambda_\alpha$. As a direct consequence of Theorem 1.1 we get $\frac{1}{\mathbf{C}_H} \leq \Lambda_\alpha \leq \frac{p^+}{p^-}\lambda_\alpha$.

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