

TOPOLOGICALLY SEMISIMPLE AND TOPOLOGICALLY PERFECT TOPOLOGICAL RINGS

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Abstract: Extending the Wedderburn–Artin theory of (classically) semisimple associative rings to the realm of topological rings with right linear topology, we show that the abelian category of left contra-modules over such a ring is split (equivalently, semisimple) if and only if the abelian category of discrete right modules over the same ring is split (equivalently, semisimple). Our results in this direction complement those of Iovanov–Mesyan–Reyes. An extension of Bass’ theory of left perfect rings to the topological realm is formulated as a list of conjecturally equivalent conditions, many equivalences and implications between which we prove. In particular, all conditions are equivalent for topological rings with a countable base of neighborhoods of zero and for topologically right coherent topological rings. Considering the rings of endomorphisms of modules as topological rings with the finite topology, we establish a close connection between the concept of a topologically perfect topological ring and the theory of modules with perfect decomposition. Our results also apply to endomorphism rings and direct sum decompositions of objects in certain additive categories more general than the categories of modules; we call them *topologically agreeable* categories. We show that any topologically agreeable split abelian category is Grothendieck and semisimple. We also prove that a module Σ -coperfect over its endomorphism ring has a perfect decomposition provided that either the endomorphism ring is commutative or the module is countably generated, partially answering a question of Angeleri Hügel and Saorín.

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Introduction

An abelian category \mathbf{A} is called *semisimple* if all its objects are (possibly infinite) coproducts of simple objects. For the category of modules over an associative ring $\mathbf{A} = S\text{-mod}$, this can be equivalently restated as the condition that all short exact sequences in \mathbf{A} split. This property is left-right symmetric: the category of left modules over an associative ring S is semisimple if and only if the category of right S -modules is. Such rings S are called *classically semisimple* (or “semisimple Artinian”). The classical Wedderburn–Artin theorem describes them as finite products of rings of matrices (of some finite size) over skew-fields.

An associative ring R is said to be *left perfect* if all flat left R -modules are projective, or equivalently, all descending chains of cyclic right R -modules terminate. The equivalence of these two and several other conditions describing left perfect rings was established in the classical paper of Bass [6, Theorem P]. In particular, one of these conditions characterizes perfect rings by their structural properties: a ring R is left perfect if and only if its Jacobson radical H is left T-nilpotent and the quotient ring R/H is classically semisimple.

In this paper we consider complete, separated topological associative rings with a right linear topology (which means that open right ideals form a base of neighborhoods of zero). Our aim is to extend the theorems of Wedderburn–Artin and Bass to such topological rings.

Right linear topological rings are ubiquitous, be it in commutative algebra, where one often considers the I -adic topology induced by an ideal I , the R -topology introduced by Matlis on a commutative domain (or even an arbitrary commutative ring) R , noncommutative generalizations of both the adic topologies and the R -topology in the form of Gabriel topologies, or the naturally arising pseudocompact topology on the vector space dual of a coalgebra over a field.

It is beyond our knowledge and beyond the scope of this introduction to give an adequate historical overview of topological structures in the theory of rings and modules, or in additive categories. The reader will find small elements of such historical discussion below in this introduction, as well as in Remarks 3.11 and 7.2 and elsewhere in the main body of the paper.

The key class of examples for us is provided by endomorphism rings of (possibly infinitely generated) modules. A classical construction equips any such endomorphism ring with the *finite topology*. In fact, a similar construction applies to endomorphism rings in any locally finitely generated Grothendieck category, providing us with even more examples. Conversely, as we explain in this paper, every complete, separated topological ring with right linear topology can be obtained as the ring of endomorphisms of a module equipped with the finite topology.

This directly relates the theory of topological rings to the theory of (decompositions of) modules over ordinary rings, objects in locally finitely generated Grothendieck categories or, more generally, objects in what we call *topologically agreeable* categories. Most remarkably, however, this interaction between the theory of topological rings with right linear topology and the theory of direct sum decompositions of modules brings applications in both directions.

One of the distinctive features of working with right linear topological rings, as opposed to ordinary rings, is that a symmetry between the categories of left and right modules is lost from the outset. With a complete, separated topological associative ring \mathfrak{S} with right linear topology, we associate the abelian category $\mathfrak{S}\text{-contra}$ of left \mathfrak{S} -*contra*modules and the abelian category $\text{discr-}\mathfrak{S}$ of *discrete* right \mathfrak{S} -modules. These are two abelian categories of a quite different nature: while $\text{discr-}\mathfrak{S}$ is a hereditary pretorsion class in $\text{mod-}\mathfrak{S}$ and a Grothendieck abelian category, $\mathfrak{S}\text{-contra}$ is a locally presentable abelian category with enough projective objects [47].

Then it turns out that $\mathfrak{S}\text{-contra}$ is a semisimple Grothendieck abelian category if and only if the abelian category $\text{discr-}\mathfrak{S}$ is semisimple. Moreover, the above two equivalent properties of a topological ring \mathfrak{S} are also equivalent to the seemingly weaker conditions that all short exact sequences are split in $\mathfrak{S}\text{-contra}$, or that all short exact sequences are split in $\text{discr-}\mathfrak{S}$. Topological rings \mathfrak{S} satisfying these equivalent conditions are called *topologically semisimple*. We describe them as the infinite topological products of the topological rings of infinite-sized, row-finite matrices over skew-fields.

An extension of the Wedderburn–Artin theory to topological rings was also studied in the paper of Iovanov, Mesyan, and Reyes [33], and the same class of topological rings (up to the passage to the opposite ring) was obtained as the result, characterized by a list of equivalent conditions different from ours. The authors of [33] discuss pseudocompact modules, while we prefer to consider discrete modules. There are no contra-modules in [33], so the topological semisimplicity is described in [33] in terms of modules on one side only, while we have both left and right modules of two different kinds. There are many equivalent characterizations of topologically semisimple topological rings in [33, Theorem 3.10], with the proof of the equivalence substantially based on the preceding results of the book of Warner [55]. So our results on topologically semisimple topological rings complement those of Iovanov, Mesyan, and Reyes by providing further conditions equivalent to the ones on their list.

Extending Bass’ theory of left perfect rings to the topological realm is a harder task, at which we only partially succeed. Given a complete, separated topological associative ring \mathfrak{R} with right linear topology, we show that projectivity of all flat left \mathfrak{R} -contra-modules implies the descending chain condition for cyclic discrete right \mathfrak{R} -modules. The converse implication is equivalent to a positive answer to a certain open question in

the theory of direct sum decompositions of modules (Question 0.1 below), as we explain, and we show that it holds under various additional assumptions including the cases where

- (1) the topological ring \mathfrak{R} has a countable base of neighborhoods of zero (proved in this paper),
- (2) the topological ring \mathfrak{R} is topologically right coherent (this is an interpretation of a result by Roos in [52], see below), or
- (3) the underlying ring of \mathfrak{R} is commutative (this follows from results in [45]).

Using a combination of contra-module-theoretic techniques developed in the paper [45] with results on direct sum decompositions of modules, we further show that all flat left contra-modules over a topological ring \mathfrak{R} are projective if and only if \mathfrak{R} has a certain set of structural properties. Namely, the topological Jacobson radical \mathfrak{J} of the topological ring \mathfrak{R} has to be *topologically left T -nilpotent* and *strongly closed in \mathfrak{R}* , and the topological quotient ring $\mathfrak{S} = \mathfrak{R}/\mathfrak{J}$ needs to be topologically semisimple. We call such (complete, separated, right linear) topological rings \mathfrak{R} *topologically left perfect*.

The classical Govorov–Lazard theorem tells us that flat modules over a ring are precisely the direct limits of (finitely generated) projective modules. In the contra-module context, it is easy to prove that direct limits of projective contra-modules are flat, but it is not known whether an analogue of the Govorov–Lazard theorem holds¹. Nevertheless, we prove that *if* the class of projective left \mathfrak{R} -contra-modules is closed under direct limits, *then* all flat left \mathfrak{R} -contra-modules are projective.

Further conditions on a topological ring \mathfrak{R} which, as we show, are equivalent to topological perfectness, are formulated in terms of *projective covers* in the abelian category $\mathfrak{R}\text{-contra}$. In fact, the main result of the paper [9] tells us that a direct limit of projective contra-modules is projective whenever it has a projective cover. On the other hand, one can show that all left contra-modules over a topologically left perfect topological ring have projective covers. Thus all flat left \mathfrak{R} -contra-modules are projective if and only if all flat left \mathfrak{R} -contra-modules have projective covers, if and only if all left \mathfrak{R} -contra-modules have projective covers, and if and only if the topological ring \mathfrak{R} is topologically left perfect.

It is worth mentioning that essentially we never consider topological modules in this paper, but only topological rings and topologies on additive categories. One reason for that is because topological modules rarely form abelian categories (pseudocompact modules are a notable exception, but we prefer to work with discrete modules). The following historical examples illustrate the point.

¹A counterexample is available now: Example 10.2 in [46].

Harrison ([32]) called an abelian group C “co-torsion” if $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, C) = 0 = \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, C)$ (in the present-day language, such abelian groups would be called “reduced cotorsion”). The category of abelian groups with these properties is abelian. Matlis ([35]) extended this definition to modules over a commutative domain R : an R -module C was called “cotorsion” in [35] if $\text{Hom}_R(Q, C) = 0 = \text{Ext}_R^1(Q, C)$, where Q is the field of fractions of R (in the modern language, such modules might be called “h-reduced Matlis cotorsion”). The category of R -modules with these properties is abelian whenever the projective dimension of the R -module Q does not exceed 1; such commutative domains are now known as *Matlis domains*.

The so-called “ R -topology” on the domain R and all R -modules was also defined and studied in the memoir [35]. One of the main results ([35, Theorem 6.10]) was that a bounded torsion R -module is cotorsion if and only if it is complete in the R -topology. This result was extended to arbitrary commutative rings R in the book [36, Corollary 2.3] and further to commutative rings R with a fixed multiplicative subset $S \subset R$ in the paper [42, Theorem 2.5], where the localization $S^{-1}R$ plays the role of the field/ring of fractions Q . The “ S -topology” on any R -module M (including the ring R itself) is defined by the rule that the submodules $sM \subset M$, where $s \in S$, form a base of neighborhoods of zero.

In the paper [42], the term “ S -contramodules” is used for what Matlis called cotorsion modules. These form a different but related category of contramodules as compared to the ones studied in the present paper. An R -module with bounded S -torsion is an S -contramodule if and only if it is complete in the S -topology. When the projective dimension of the R -module $S^{-1}R$ does not exceed 1, the category of S -contramodules is abelian [42, Theorem 3.4] (while the category of R -complete or S -complete modules is usually not abelian).

Thus, there is a strong connection between topological completeness of modules and certain homological properties of modules in this context, but the homological conditions are more convenient to work with, as they define better-behaved module categories. Contramodules over topological rings, which we study in this paper, can be viewed as one step further in this direction. Their categories in the above cases are very closely related, and often equivalent, to classes of modules defined by such homological conditions (see [40, Examples 2.4(3) and 5.4(2)] for a discussion of the comparison). Their advantage, however, is a broader generality and the fact that the concept is intrinsic to a complete, separated topological ring with right linear topology, without any reference to specific classes of modules.

Let us now look closer at the connection of our results to the existing literature and open problems in module theory and category theory. The theory of direct sum decompositions of modules goes back to the classical Krull–Schmidt–Remak–Azumaya uniqueness theorem [54, Section V.5], [19, Section 2]. There exists an extensive literature on this subject now, with lots of known results and open problems [16, 31, 37, 28, 26]. The fact that the finite topology on the endomorphism ring of a module is relevant in the study of its direct sum decompositions is well understood [13], but contra-modules over the topological rings of endomorphisms have not been used in such studies yet. This is a new technique which we bring to bear on the subject (using the approach originally developed in our previous paper [50]).

In particular, the above-mentioned open question was posed by Angeleri Hügel and Saorín as [3, Question 1 in Section 2]:

Question 0.1. *Let A be an associative ring and M be an A -module. Denote by R the ring of A -linear endomorphisms of M . Assume that M is Σ -coperfect over R . Does it follow that the A -module M has a perfect decomposition?*

Here an R -module is said to be *coperfect* if it satisfies the descending chain condition on cyclic (equivalently, finitely generated) R -submodules. An R -module M is Σ -coperfect if the countable direct sum $M^{(\omega)}$ of copies of M is a coperfect R -module. It follows from our results that the answer to Question 0.1 is positive whenever either the ring R is commutative or it is topologically right coherent in the finite topology, or the A -module M is countably generated.

It was shown in the same paper [3, Theorem 1.4] that an A -module M has perfect decomposition if and only if, for any direct system of A -modules $M_i \in \text{Add}(M)$ indexed by a linearly ordered set of indices i , the natural surjective A -module morphism $\bigoplus_i M_i \rightarrow \varinjlim_i M_i$ is split. In this paper we extend this result to objects of (the above-mentioned) topologically agreeable additive categories.

We stress again that interaction between the theory of topological rings with right linear topology and the theory of direct sum decompositions of modules brings applications in both directions. In particular, the above-mentioned results concerning flat contra-modules are applications of module theory to topological algebra. Another such application is the theorem that any split abelian category admitting a topologically agreeable structure is Grothendieck and semisimple. Our results concerning the question of Angeleri Hügel and Saorín, on the other hand, are applications of topological algebra to module theory.

Applications of topological rings to abelian categories, more specifically to certain classes of Grothendieck abelian categories, were initiated by Gabriel already in his dissertation [23]. This approach was further developed by Roos in [52]. In fact, Gabriel described locally finite Grothendieck categories in terms of pseudocompact topological rings [23, n^{os} IV.3-4]. Roos described locally Noetherian Grothendieck categories and their conjugate counterparts, the locally coperfect locally coherent Grothendieck categories, in terms of topologically coperfect and coherent topological rings [52, Theorem 6]. It follows from Roos' result that all flat left \mathfrak{R} -contramodules are projective whenever the category of discrete right \mathfrak{R} -modules is locally coherent and satisfies the descending chain condition for coherent (or finitely generated, or cyclic) objects/modules. This is what is behind one of the cases above where the descending chain condition on cyclic discrete right \mathfrak{R} -modules characterizes topological perfectness.

Finally, the theory of topological rings provides new cases where the following open problem due to Enochs ([26, Section 5.4]) can be answered affirmatively:

Question 0.2. *Let A be an associative ring and \mathcal{C} be a covering class of modules. Is \mathcal{C} closed under direct limits?*

The idea is again to study the problem for classes of the form $\mathcal{C} = \text{Add}(M)$, where M is a module, and translate the problem under suitable hypotheses to the question whether suitable contramodules over the endomorphism ring $\mathfrak{R} = \text{End}(M)^{\text{op}}$ have projective covers and, thus, whether \mathfrak{R} is topologically left perfect. This direction is already beyond the scope of this paper and we refer to [8, 9] for a detailed account.

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1. Preliminaries on topological rings

We mostly refer to [45, Section 2] (see also [43, Section 2]) and the references therein for the preliminary material, so the section below only contains a brief sketch of the key definitions and constructions.

All *topological abelian groups* in this paper are presumed to have a base of neighborhoods of zero consisting of open subgroups. Subgroups, quotient groups, and products of topological groups are endowed with

the induced/quotient/product topologies. The *completion* of a topological abelian group A is the abelian group $\mathfrak{A} = \varprojlim_U A/U$ (where U ranges over the open subgroups in A) endowed with the *projective limit topology*. A topological abelian group A is *complete* if the completion morphism $A \rightarrow \mathfrak{A}$ is surjective, and *separated* if this map is injective. The completion \mathfrak{A} of any topological abelian group A is complete and separated [45, Sections 2.1-2 and 8].

Unless otherwise stated, all *rings* are presumed to be associative and unital. The Jacobson radical of a ring R is denoted by $H(R)$. A topological ring R is said to have a *right linear topology* if open right ideals form a base of neighborhoods of zero in R . The completion \mathfrak{R} of a topological ring R with right linear topology is again a topological ring with right linear topology, and the completion map $R \rightarrow \mathfrak{R}$ is a continuous ring homomorphism [45, Section 2.3].

Let R be a topological ring with right linear topology and \mathfrak{R} be the completion of R . A right R -module \mathcal{N} is said to be *discrete* if the annihilator of every element of \mathcal{N} is an open right ideal in R . The full subcategory of discrete right R -modules $\text{discr-}R \subset \text{mod-}R$ is closed under submodules, quotients, and infinite direct sums in the category of right R -modules $\text{mod-}R$, so $\text{discr-}R$ is a Grothendieck abelian category. The categories of discrete right modules over a topological ring R and its completion \mathfrak{R} are naturally equivalent, $\text{discr-}R \cong \text{discr-}\mathfrak{R}$ [45, Section 2.4].

Given an abelian group A and a set X , we use the notation $A[X] = A^{(X)}$ for the direct sum of X copies of the group A . For a complete, separated topological abelian group \mathfrak{A} , we put $\mathfrak{A}[[X]] = \varprojlim_{\mathfrak{U} \subset \mathfrak{A}} (\mathfrak{A}/\mathfrak{U})[X]$, where the projective limit is taken over all the open subgroups $\mathfrak{U} \subset \mathfrak{A}$. The set $\mathfrak{A}[[X]]$ is interpreted as the set of all *infinite formal linear combinations* $\sum_{x \in X} a_x x$ of elements of X with the coefficients $a_x \in \mathfrak{A}$ forming an X -indexed family of elements $(a_x)_{x \in X}$ *converging to zero* in the topology of \mathfrak{A} . The latter condition means that, for every open subgroup $\mathfrak{U} \subset \mathfrak{A}$, the set of all $x \in X$ for which $a_x \notin \mathfrak{U}$ is finite. A closed subgroup \mathfrak{U} in a complete, separated topological abelian group \mathfrak{A} is said to be *strongly closed* if the quotient group $\mathfrak{A}/\mathfrak{U}$ is complete and the map $\mathfrak{A}[[X]] \rightarrow (\mathfrak{A}/\mathfrak{U})[[X]]$ induced by the morphism $\mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{U}$ is surjective for every set X [45, Sections 2.5 and 2.11].

The assignment of the set $\mathfrak{A}[[X]]$ to a set X is naturally extended to a covariant functor $X \mapsto \mathfrak{A}[[X]]: \mathbf{Sets} \rightarrow \mathbf{Sets}$ from the category of sets to itself (or, if one wishes, to the category \mathbf{Ab} of abelian groups). Given a complete, separated topological associative ring \mathfrak{R} with right linear topology, the functor $X \mapsto \mathfrak{R}[[X]]$ has a natural structure of a *monad*

on the category of sets [45, Section 2.6]. This means that, for every set X , one also considers a natural map $\epsilon_X : X \rightarrow \mathfrak{R}[[X]]$ (called the “point measure” map and defined in terms of the zero and unit elements in \mathfrak{R}) and a natural map $\phi_X : \mathfrak{R}[[\mathfrak{R}[[X]]]] \rightarrow \mathfrak{R}[[X]]$ (called the “opening of parentheses” map and defined in terms of the multiplication of pairs of elements and infinite sums of zero-converging families of elements in \mathfrak{R}), which satisfy certain associativity and unitality conditions.

A *left contra*module over a topological ring \mathfrak{R} is, by definition, a module (or, in the more conventional terminology, an “algebra”) over this monad on **Sets**. In other words, a left \mathfrak{R} -contra module \mathfrak{C} is a set endowed with a *left contra*action map $\pi_{\mathfrak{C}} : \mathfrak{R}[[\mathfrak{C}]] \rightarrow \mathfrak{C}$ satisfying the associativity and unitality equations with respect to the natural transformations ϕ and ϵ . The *free* left \mathfrak{R} -contra module $\mathfrak{R}[[X]]$ spanned by a set X is the free module/algebra over the monad $X \mapsto \mathfrak{R}[[X]]$ on **Sets**. The category of left \mathfrak{R} -contra modules $\mathfrak{R}\text{-contra}$ is a locally presentable abelian category with enough projective objects; the latter are precisely the direct summands of the free left \mathfrak{R} -contra modules $\mathfrak{R}[[X]]$. The free left \mathfrak{R} -contra module with one generator $\mathfrak{R} = \mathfrak{R}[[*]]$ is a projective generator of the abelian category $\mathfrak{R}\text{-contra}$. The underlying set of a left \mathfrak{R} -contra module carries a natural left \mathfrak{R} -module structure, which provides a faithful, exact, limit-preserving forgetful functor $\mathfrak{R}\text{-contra} \rightarrow \mathfrak{R}\text{-mod}$ [45, Section 2.7], [47, Sections 1.1-2 and 5].

A class of examples of left \mathfrak{R} -contra modules is constructed by dualizing discrete right \mathfrak{R} -modules. Let A be an associative ring and \mathcal{N} be an A - \mathfrak{R} -bimodule whose underlying right \mathfrak{R} -module is discrete. Let V be a left A -module. Then the abelian group $\mathfrak{D} = \text{Hom}_A(\mathcal{N}, V)$ has a natural structure of left \mathfrak{R} -contra module with the contraaction map given by the rule

$$\pi_{\mathfrak{D}} \left(\sum_{d \in \mathfrak{D}} r_d d \right) (b) = \sum_{d \in \mathfrak{D}} d(br_d) \quad \text{for all } b \in \mathcal{N}.$$

Here the sum on the right-hand side is finite because the annihilator of b is open in \mathfrak{R} and the family of elements $(r_d \in \mathfrak{R})_{d \in \mathfrak{D}}$ converges to zero in \mathfrak{R} [45, Section 2.8], [43, Section 2.8].

The *contratensor product* $\mathcal{N} \circ_{\mathfrak{R}} \mathfrak{C}$ of a discrete right \mathfrak{R} -module \mathcal{N} and a left \mathfrak{R} -contra module \mathfrak{C} is an abelian group constructed as the cokernel of (the difference of) the natural pair of abelian group homomorphisms $\mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{R}[[\mathfrak{C}]] \rightrightarrows \mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{C}$. Here one of the two maps $\mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{R}[[\mathfrak{C}]] \rightarrow \mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{C}$ is simply induced by the contraaction map $\pi_{\mathfrak{C}} : \mathfrak{R}[[\mathfrak{C}]] \rightarrow \mathfrak{C}$, while the other one is constructed in terms of the right action of \mathfrak{R} in \mathcal{N} (using the assumption that this right action is *discrete* in combination with the description of $\mathfrak{R}[[\mathfrak{C}]]$ as the set of all formal linear combinations of elements of \mathfrak{C} with *zero-convergent* families of coefficients in \mathfrak{R}). The

contratensor product $\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{C}$ is, generally speaking, a quotient group of the tensor product $\mathcal{N} \otimes_{\mathfrak{R}} \mathfrak{C}$.

For any A - \mathfrak{R} -bimodule \mathcal{N} whose underlying right \mathfrak{R} -module is discrete, any left A -module V , and any left \mathfrak{R} -contramodule \mathfrak{C} , there is a natural isomorphism of abelian groups [45, Section 2.8], [43, Section 2.8]

$$\text{Hom}^{\mathfrak{R}}(\mathfrak{C}, \text{Hom}_A(\mathcal{N}, V)) \cong \text{Hom}_A(\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{C}, V),$$

where $\text{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{D})$ is the notation for the group of all morphisms $\mathfrak{C} \rightarrow \mathfrak{D}$ in the abelian category $\mathfrak{R}\text{-contra}$.

For any left \mathfrak{R} -contramodule \mathfrak{C} and any set X , there is a natural isomorphism of abelian groups [45, Section 2.7]

$$\text{Hom}_{\mathfrak{R}}(\mathfrak{R}[[X]], \mathfrak{C}) \cong \text{Hom}_{\text{Sets}}(X, \mathfrak{C}).$$

For any discrete right \mathfrak{R} -module \mathcal{N} and any set X , there is a natural isomorphism of abelian groups [45, Section 2.8]

$$\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{R}[[X]] \cong \mathcal{N}[X] = \mathcal{N}^{(X)}.$$

2. Split and semisimple abelian categories

A nonzero object in an abelian category is *simple* if it has no nonzero proper subobjects. An object is *semisimple* if it is a coproduct of simple objects. An abelian category is called *semisimple* if all its objects are semisimple. We will say that an abelian category \mathbf{A} is *split* if all short exact sequences in \mathbf{A} split.

Lemma 2.1. *Every semisimple abelian category is split.*

Proof: The following proof was suggested to us by J. Rickard [51]. Let \mathbf{A} be a semisimple abelian category, and let $f: X \rightarrow Y$ be an epimorphism in \mathbf{A} . By assumption, the object Y is a coproduct of a family of simple objects, $Y = \coprod_i S_i$. Let $f_i: X_i \rightarrow S_i$ be the pullback of the morphism f along the split monomorphism $S_i \rightarrow Y$. In order to show that the epimorphism f has a section, it suffices to check that so does the epimorphism f_i for every index i (see [15, Proposition A.1]).

By assumption, the object X_i is a coproduct of a family of simple objects too: $X_i = \coprod_j T_{ij}$. The morphism $f_i: X_i \rightarrow S_i$ corresponds to a family of morphisms $f_{ij}: T_{ij} \rightarrow S_i$. By the Schur lemma, every morphism f_{ij} is either zero or an isomorphism. If $f_{ij} = 0$ for every j , then $f_i = 0$, which is impossible for an epimorphism f_i with a nonzero codomain S_i . Hence there exists an index j for which f_{ij} is an isomorphism. Now the composition of the inverse morphism $f_{ij}^{-1}: S_i \rightarrow T_{ij}$ with the split monomorphism $T_{ij} \rightarrow X_i$ provides a section of the epimorphism f_i . □

The material below in this section is essentially well known. We include it for the sake of completeness of the exposition.

An abelian category is *Ab5* if it is cocomplete and has exact functors of direct limits (= filtered colimits). A *Grothendieck category* is an abelian category which is *Ab5* and has a set of generators (or equivalently, a single generator).

A split Grothendieck abelian category is called *spectral* [24].

Remark 2.2. The theory of spectral categories is surprisingly complicated (as compared to the naïve expectation that all split abelian categories should be semisimple). The terminology “spectral category” refers to the spectral theory of operators in infinite-dimensional topological vector spaces (in functional analysis); in particular, a reference to the functional analysis concepts of “discrete” and “continuous” spectrum is presumed. A spectral category is called *discrete* if it is semisimple, and *continuous* if it has no simple objects. Every spectral category has a unique decomposition into the Cartesian product of a discrete and a continuous spectral category [54, Section V.6]. Furthermore, spectral categories \mathbf{A} with a chosen generator G correspond bijectively to left self-injective von Neumann regular rings R in the following way. To a pair (\mathbf{A}, G) the (opposite ring to) the ring of endomorphisms of the generator, $R = \text{Hom}_{\mathbf{A}}(G, G)^{\text{op}}$, is assigned (so R acts in G on the right). To a left self-injective von Neumann regular associative ring R , the full subcategory $\mathbf{A} = \text{Prod}(R) \subset R\text{-mod}$ of all the direct summands of products of copies of the (injective) free left R -module R is assigned, with the chosen generator $G = R$. The category \mathbf{A} can also be interpreted as a quotient category of $R\text{-mod}$ [54, Section XII.1]. A good exposition of some aspects of this theory can be found in [30, Chapter I]. A concrete class of examples of continuous spectral categories is described below in Example 2.9.

The following theorem characterizes and describes the semisimple Grothendieck (= discrete spectral) abelian categories (cf. [54, Proposition V.6.7]).

Theorem 2.3. *Let \mathbf{A} be an abelian category with set-indexed coproducts and a generator. Then the following conditions are equivalent:*

- (i) \mathbf{A} is *Ab5* and every object of \mathbf{A} is the sum of its simple subobjects;
- (ii) \mathbf{A} is *Ab5*, split, and every nonzero object of \mathbf{A} has a simple subquotient object;
- (iii) every object of \mathbf{A} is a coproduct of simple objects, and for every simple object $S \in \mathbf{A}$ the functor $\text{Hom}_{\mathbf{A}}(S, -): \mathbf{A} \rightarrow \mathbf{Ab}$ preserves coproducts;

- (iv) *there is a set X and an X -indexed family of division rings (= skew-fields) D_x , $x \in X$, such that the category \mathbf{A} is equivalent to the Cartesian product of the categories of vector spaces over D_x ,*

$$\mathbf{A} \cong \prod_{x \in X} D_x\text{-mod.}$$

Remark 2.4. The third condition in Theorem 2.3(ii), saying that every nonzero object of \mathbf{A} has a simple subquotient, is always satisfied for the abelian category of modules over an associative ring $\mathbf{A} = A\text{-mod}$ (because every nonzero A -module has a nonzero cyclic submodule, which in turn has a maximal proper submodule). Thus the category $A\text{-mod}$ is split/spectral if and only if it is semisimple. Moreover, for any topological ring R with right linear topology, the same applies to the abelian category $\mathbf{A} = \text{discr-}R$ of discrete right R -modules, which is also Grothendieck and has the property that every nonzero object has a simple subquotient.

The following lemma shows that in Ab5-categories, as in the categories of modules, the sum of a family of subobjects is direct whenever the sum of any finite subfamily of these objects is.

Lemma 2.5. *Let \mathbf{A} be an Ab5-category, $M \in \mathbf{A}$ be an object, and $(N_x \subset M)_{x \in X}$ be a family of subobjects in M . Suppose that for every finite subset $Z \subset X$ the induced morphism $\coprod_{z \in Z} N_z \rightarrow M$ is a monomorphism. Then the induced morphism $\coprod_{x \in X} N_x \rightarrow M$ is a monomorphism too.*

Proof: In any cocomplete category, one has $\coprod_{x \in X} N_x = \varinjlim_{Z \subset X} \coprod_{z \in Z} N_z$ (where the direct limit is taken over all the finite subsets $Z \subset X$). In an Ab5-category, the direct limit of a diagram of monomorphisms is a monomorphism. □

Corollary 2.6 ([54, Proposition V.6.2]). *Let \mathbf{A} be an Ab5-category, $M \in \mathbf{A}$ be an object, and $L \subset M$ be a subobject. Assume that M is the sum of a family $(S_x \subset M)_{x \in X}$ of simple subobjects of M (i.e., no proper subobject of M contains S_x for all $x \in X$). Then there exists a subset $Y \subset X$ such that $M = L \oplus \coprod_{y \in Y} S_y$.*

Proof: Provable by a standard Zorn lemma argument based on Lemma 2.5. □

The next lemma says that simple objects in Ab5-categories are *finitely generated* in the sense of [54, Section V.3] or [1, Section 1.E].

Lemma 2.7. *Let \mathbf{A} be an Ab5-category and $S \in \mathbf{A}$ be a simple object. Then the functor $\text{Hom}_{\mathbf{A}}(S, -) : \mathbf{A} \rightarrow \mathbf{Ab}$ preserves direct limits of diagrams of monomorphisms.*

Proof: Let $(m_{z,w}: M_w \rightarrow M_z)_{w < z}$ be a diagram of objects in \mathbf{A} and monomorphisms between them, indexed by some directed poset Z . Set $M = \varinjlim_{z \in Z} M_z$. Since \mathbf{A} is Ab5, the natural morphisms $M_z \rightarrow M$ are also monomorphisms, so one can consider the objects M_z as subobjects in M . Furthermore, the condition that \mathbf{A} is Ab5 can be equivalently expressed by saying that for any subobject $K \subset M$ one has $K = \varinjlim_{z \in Z} (K \cap M_z)$ [39, Section III.1].

Now let $f: S \rightarrow M$ be a morphism in \mathbf{A} and $f(S) \subset M$ be its image. Since S is simple, $f(S)$ is either simple or zero, and it follows that for every $z \in Z$ the intersection $f(S) \cap M_z$ is either the whole $f(S)$ or zero. We have $f(S) = \varinjlim_{z \in Z} f(S) \cap M_z$, so if $f(S) \cap M_z = 0$ for all $z \in Z$, then $f = 0$ and there is nothing to prove. Otherwise, there exists $z \in Z$ such that $f(S) \subset M_z$. Hence the morphism f factorizes through the monomorphism $M_z \rightarrow M$, as desired. \square

In particular, it follows from Lemma 2.7 that simple objects in Ab5-categories are *weakly finitely generated* in the sense of [50, Section 9.2].

Corollary 2.8. *Let \mathbf{A} be an Ab5-category and $S \in \mathbf{A}$ be a simple object. Then the functor $\text{Hom}_{\mathbf{A}}(S, -): \mathbf{A} \rightarrow \mathbf{Ab}$ preserves coproducts.* \square

Proof of Theorem 2.3: (i) \implies (ii) and (iii) follows from Corollaries 2.6 and 2.8.

(ii) \implies (i) Given an object $M \in \mathbf{A}$, consider its socle (= the sum of all simple subobjects) $N \subset M$. If $N \neq M$, then the quotient object M/N has a simple subquotient object. Since \mathbf{A} is split, this leads to a simple subobject in M not contained in N . The contradiction proves that $N = M$.

(iii) \implies (iv) It is important here that in any abelian category with a generator the isomorphism classes of simple objects form a set. Indeed, if $G \in \mathbf{A}$ is a generator and $S \in \mathbf{A}$ is simple, then there exists a nonzero morphism $G \rightarrow S$, so S is a quotient of G . Now the subobjects of G (hence also the quotient objects of G) form a set of the cardinality not exceeding that of the powerset of $\text{Hom}_{\mathbf{A}}(G, G)$. (If \mathbf{A} is split, all subobjects and quotient objects of G are direct summands, and their cardinality does not exceed that of the set of all idempotent endomorphisms of G .)

Now let $(S_x)_{x \in X}$ be a set of representatives of all the isomorphism classes of simple objects in \mathbf{A} . Put $D_x = \text{Hom}_{\mathbf{A}}(S_x, S_x)^{\text{op}}$ (by the Schur lemma, D_x are division rings). The desired equivalence of categories is provided by the functor $F: \mathbf{A} \rightarrow \times_{x \in X} D_x\text{-mod}$ taking an object $M \in \mathbf{A}$ to the collection of left D_x -vector spaces $(\text{Hom}_{\mathbf{A}}(S_x, M))_{x \in X}$.

The inverse functor G takes a collection of left D_x -vector spaces $(V_x \in D_x\text{-mod})_{x \in X}$ to the object $\coprod_{x \in X} (S_x \otimes_{D_x} V_x) \in \mathbf{A}$ (where $S_x \otimes_{D_x} -$ is a functor taking the coproduct $D_x^{(Y)}$ of Y copies of $D_x \in D_x\text{-mod}$ to the coproduct $S_x^{(Y)} \in \mathbf{A}$ of Y copies of the object S_x , for any set Y).

Essentially, the first condition in (iii) describes the objects of the category \mathbf{A} , and the second condition fully describes its morphisms. This allows us to prove that the functors F and G are mutually inverse equivalences.

The implication (iv) \implies (i) is obvious. \square

Example 2.9. Let us give an explicit example (or, rather, a class of examples) of *continuous* spectral categories \mathbf{A} with a chosen generator G . These examples have the additional advantage that the ring $R = \text{Hom}_{\mathbf{A}}(G, G)^{\text{op}}$ is *commutative*.

Following [54, Section XII.1], [30, Chapter I], and/or Remark 2.2 above, spectral categories \mathbf{A} with a chosen generator G are described by left self-injective von Neumann regular rings R . Isomorphism classes of simple objects in \mathbf{A} correspond to ring direct factors in R isomorphic to the (opposite ring of) the ring of endomorphisms of a vector space over a skew-field (the dimension of the vector space being equal to the multiplicity with which the simple object occurs in the chosen generator). So continuous spectral categories \mathbf{A} are described by left self-injective von Neumann regular rings R which have no such ring direct factors. In particular, if we want the ring R to be commutative, then such continuous spectral categories \mathbf{A} with a generator G are described by self-injective commutative von Neumann regular rings R such that no ring direct factor of R is a field.

Now let us restrict our attention to the following particular case. Let R be a Boolean ring, that is, an associative unital ring in which all the elements are idempotent. Then R is a commutative algebra over the field $\mathbb{Z}/2\mathbb{Z}$ and a von Neumann regular ring. Furthermore, there is a natural partial order on R which makes R a distributive lattice with complements (this structure is called a *Boolean algebra*) [54, Section III.4], [25]. According to [54, Section XII.3], a Boolean ring is self-injective if and only if its Boolean algebra is *complete* (that is, complete as a lattice). A Boolean ring has no field direct factors (i.e., ring direct factors isomorphic to $\mathbb{Z}/2\mathbb{Z}$) if and only if its Boolean algebra has no *atoms*.

A discussion of complete Boolean algebras can be found in [25, Chapter 38]; they are classified by *extremally disconnected* compact Hausdorff topological spaces (which means compact Hausdorff topological spaces in which the closure of every open subset is open; to such a space Z , the algebra of all its clopen subsets is assigned). Moving to a specific example

of a complete Boolean algebra, choose a Hausdorff topological space X without isolated points, and consider the Boolean algebra/ring R of all open or closed subsets in X considered *up to nowhere dense subsets*. The key observation is that, viewing nowhere dense subsets as negligible, there is *no difference* between open and closed subsets (since for any open subset $U \subset X$ with the closure $\bar{U} \subset X$, the complement $\bar{U} \setminus U$ is nowhere dense in X). Alternatively, the usual approach is to consider *regular open sets*, i.e., open subsets in X which coincide with the interior of their closure [25, Chapter 10]. These form the desired complete Boolean ring R without atoms, which is consequently self-injective commutative von Neumann regular without field direct factors.

3. Topologically agreeable additive categories

The following definitions and construction were suggested in the manuscript [14]. Let \mathbf{A} be an additive category with set-indexed coproducts. If set-indexed products exist in the category \mathbf{A} , one says that \mathbf{A} is *agreeable* if for every family of objects $N_x \in \mathbf{A}$ (indexed by elements x of some set X) the natural morphism from the coproduct to the product $\coprod_{x \in X} N_x \longrightarrow \prod_{x \in X} N_x$ is a monomorphism in \mathbf{A} .

This condition can be reformulated so as to avoid the assumption of the existence of products in \mathbf{A} . For every object M and a family of objects $N_x \in \mathbf{A}$, consider the natural map of abelian groups

$$\eta: \text{Hom}_{\mathbf{A}}(M, \coprod_{x \in X} N_x) \longrightarrow \prod_{x \in X} \text{Hom}_{\mathbf{A}}(M, N_x),$$

assigning to a morphism $f: M \longrightarrow \coprod_{x \in X} N_x$ the collection of its compositions $\eta(f) = (\pi_y \circ f)_{y \in X}$ with the projection morphisms $\pi_y: \prod_{x \in X} N_x \longrightarrow N_y$. An additive category \mathbf{A} with set-indexed coproducts is said to be *agreeable* if, for all objects M and $N_x \in \mathbf{A}$, the map η is injective. It is the latter, more general definition that was formulated in [14] and that we will use in the sequel.

Let \mathbf{A} be an agreeable category, $M \in \mathbf{A}$ be an object, and $(N_x \in \mathbf{A})_{x \in X}$ be a family of objects. A family of morphisms $(f_x: M \longrightarrow N_x)_{x \in X}$ is said to be *summable* if there exists a (necessarily unique, by assumption) morphism $f: M \longrightarrow \prod_{x \in X} N_x$ whose image under the map η is equal to the element $(f_x)_{x \in X} \in \prod_{x \in X} \text{Hom}_{\mathbf{A}}(M, N_x)$.

The particular case when all the objects N_x are one and the same, $N_x = N$, is important. Let M and N be two fixed objects in \mathbf{A} and $(f_x: M \longrightarrow N)_{x \in X}$ be a summable family of morphisms between them. The *sum* $\sum_{x \in X} f_x: M \longrightarrow N$ of the summable family of morphisms $(f_x)_{x \in X}$ is defined as the composition

$$M \xrightarrow{f} N^{(X)} \xrightarrow{\Sigma} N$$

of the related morphism $f: M \rightarrow N^{(X)} = \coprod_{x \in X} N$ with the natural summation morphism $\Sigma: N^{(X)} \rightarrow N$. The morphism Σ is defined by the condition that its composition $\Sigma \iota_x: N \rightarrow N^{(X)} \rightarrow N$ with the coproduct injection $\iota_x: N \rightarrow N^{(X)}$ is the identity morphism $N \rightarrow N$ for every $x \in X$.

Example 3.1. Any Grothendieck abelian category is agreeable. In fact, if \mathbf{A} is a complete, cocomplete abelian category with exact direct limits, then the natural morphism $\coprod_{x \in X} N_x \rightarrow \prod_{x \in X} N_x$ is a monomorphism for every family of objects $N_x \in \mathbf{A}$, since it is the direct limit of the split monomorphisms $\coprod_{z \in Z} N_z \rightarrow \prod_{x \in X} N_x$ taken over the directed poset of all finite subsets $Z \subset X$.

Remarks 3.2. (1) More generally, any abelian category satisfying Ab5 is agreeable. Indeed, let \mathbf{A} be an Ab5-category, $(N_x)_{x \in X}$ be a family of objects in \mathbf{A} , and $M \in \mathbf{A}$ be an object. Given a nonzero morphism $f: M \rightarrow \prod_{x \in X} N_x$, consider its image $f(M)$. The object $\prod_{x \in X} N_x$ is the direct limit of its subobjects $\prod_{z \in Z} N_z$, where Z ranges over all the finite subsets of X . Hence the subobject $f(M) \subset \prod_{x \in X} N_x$ is the direct limit of its subobjects $f(M) \cap \prod_{z \in Z} N_z$ (cf. the proof of Lemma 2.7). Since $f(M) \neq 0$, there exists a finite subset $Z \subset X$ such that the object $f(M) \cap \prod_{z \in Z} N_z$ is nonzero. It follows that the composition of the morphism f with the projection $\prod_{x \in X} N_x \rightarrow \prod_{z \in Z} N_z = \prod_{z \in Z} N_z$ is nonzero. Thus there exists $z \in Z$ for which the morphism $\pi_z \circ f: M \rightarrow N_z$ is nonzero, so $\eta(f) \neq 0$.

(2) Conversely, any agreeable abelian category \mathbf{A} satisfies Ab4, i.e., the functors of infinite coproducts in \mathbf{A} are exact (cf. [39, Section III.1]). Indeed, let $g_x: K_x \rightarrow L_x$ be a family of monomorphisms in \mathbf{A} ; we have to prove that the morphism $\coprod_{x \in X} g_x: \coprod_{x \in X} K_x \rightarrow \coprod_{x \in X} L_x$ is a monomorphism. Let $f: M \rightarrow \prod_{x \in X} K_x$ be a nonzero morphism. Then there exists $y \in X$ such that the composition of f with the projection map $\pi_y: \prod_{x \in X} K_x \rightarrow K_y$ is nonzero. Hence the composition $g_y \pi_y f: M \rightarrow K_y \rightarrow L_y$ is nonzero too. Denoting by ρ_y the projection map $\prod_{x \in X} L_x \rightarrow L_y$, we have $\rho_y \circ \prod_{x \in X} g_x = g_y \pi_y$. Hence $\rho_y \circ \prod_{x \in X} g_x \circ f = g_y \pi_y f \neq 0$, and it follows that the composition of f with the morphism $\prod_{x \in X} g_x$ is nonzero.

(3) Moreover, any complete agreeable abelian category with an injective cogenerator satisfies Ab5. This is the result of the paper [49].

In this paper, we will be interested mostly in a more special class of additive categories, which we call *topologically agreeable*. In fact, a topo-

logically agreeable category is an additive category with the following additional structure.

A *right topological* additive category \mathbf{A} is an additive category in which, for every pair of objects M and $N \in \mathbf{A}$, the abelian group $\text{Hom}_{\mathbf{A}}(M, N)$ is endowed with a topology in such a way that the following two conditions are satisfied:

- (i) the composition maps

$$\text{Hom}_{\mathbf{A}}(L, M) \times \text{Hom}_{\mathbf{A}}(M, N) \longrightarrow \text{Hom}_{\mathbf{A}}(L, N)$$

are continuous (as functions of two arguments) for all objects $L, M, N \in \mathbf{A}$;

- (ii) open $\text{Hom}_{\mathbf{A}}(N, N)$ -submodules form a base of neighborhoods of zero in $\text{Hom}_{\mathbf{A}}(M, N)$ for any two objects $M, N \in \mathbf{A}$.

A right topological additive category \mathbf{A} is said to be *complete* (resp., *separated*) if the topological abelian group $\text{Hom}_{\mathbf{A}}(M, N)$ is complete (resp., separated) for every pair of objects M and $N \in \mathbf{A}$.

For any object M in a right topological additive category \mathbf{A} , the topology on the group of endomorphisms $\text{Hom}_{\mathbf{A}}(M, M)$ makes it a topological ring with a *left* linear topology. Here the notation presumes that the ring $\text{Hom}_{\mathbf{A}}(M, M)$ acts on the object $M \in \mathbf{A}$ on the left. We will usually consider the opposite ring $\mathfrak{R} = \text{Hom}_{\mathbf{A}}(M, M)^{\text{op}}$, which acts on M on the right. Hence \mathfrak{R} is a topological ring with a right linear topology. When the topological additive category \mathbf{A} is complete (resp., separated), so is the ring \mathfrak{R} .

Lemma 3.3. *Let \mathbf{A} be a right topological additive category, let M and $N \in \mathbf{A}$ be two objects, and let X be a set such that a coproduct $N^{(X)}$ of X copies of N exists in \mathbf{A} . Let $(f_x : M \rightarrow N)_{x \in X}$ be a family of morphisms converging to zero in the topology of the abelian group $\text{Hom}_{\mathbf{A}}(M, N)$. Then the family of morphisms $(\iota_x f_x)_{x \in X}$ converges to zero in the group $\text{Hom}_{\mathbf{A}}(M, N^{(X)})$.*

Proof: For any two elements x and $y \in X$, denote by $\sigma_{x,y} : N^{(X)} \rightarrow N^{(X)}$ the automorphism permuting the coordinates x and y . In particular, $\sigma_{x,x} = \text{id}_{N^{(X)}}$; and for any $x, y \in X$ we have $\sigma_{x,y} \iota_y = \iota_x : N \rightarrow N^{(X)}$. Choose a fixed element $x_0 \in X$. Since the family of morphisms $f_x : M \rightarrow N$ converges to zero in the topology of $\text{Hom}_{\mathbf{A}}(M, N)$, it follows from the continuity axiom (i) that the family of morphisms $(\iota_{x_0} f_x : M \rightarrow N^{(X)})_{x \in X}$ converges to zero in the topology of the group $\text{Hom}_{\mathbf{A}}(M, N^{(X)})$. By axiom (ii) (applied to the objects M and $N^{(X)} \in \mathbf{A}$), we can conclude that the family of morphisms $\iota_x f_x = \sigma_{x,x_0} \iota_{x_0} f_x : M \rightarrow N^{(X)}$ also converges to zero in the topology of $\text{Hom}_{\mathbf{A}}(M, N^{(X)})$. □

Lemma 3.4. *Let \mathbf{A} be an additive category that is simultaneously agreeable and complete, separated right topological. Let M and $N \in \mathbf{A}$ be two objects. Then any family of morphisms $f_x \in \text{Hom}_{\mathbf{A}}(M, N)$ converging to zero in the topology of the abelian group $\text{Hom}_{\mathbf{A}}(M, N)$ is summable in the agreeable category \mathbf{A} . Moreover, the sum $\sum_{x \in X}^{\text{top}} f_x \in \text{Hom}_{\mathbf{A}}(M, N)$ defined as the limit of finite partial sums in the topology of the group $\text{Hom}_{\mathbf{A}}(M, N)$ coincides with the sum $\sum_{x \in X} f_x = \sum_{x \in X}^{\text{agr}} f_x$ computed in the agreeable category \mathbf{A} (so our notation is unambiguous).*

Proof: By Lemma 3.3, the family of elements $(\iota_x f_x)_{x \in X}$ converges to zero in the topological abelian group $\text{Hom}_{\mathbf{A}}(M, N^{(X)})$. Since this group is complete and separated by assumption, the sum

$$f = \sum_{x \in X}^{\text{top}} \iota_x f_x : M \longrightarrow N^{(X)},$$

understood as the limit of finite partial sums, is well defined. Using the continuity of composition again, one can see that

$$\pi_y \circ f = \pi_y \circ \sum_{x \in X}^{\text{top}} (\iota_x \circ f_x) = \sum_{x \in X}^{\text{top}} (\pi_y \circ \iota_x \circ f_x) = f_y, \quad y \in X,$$

so $\eta(f) = (f_x)_{x \in X}$. This proves that the family of morphisms (f_x) is summable in the agreeable category \mathbf{A} . Finally, the same continuity axiom (i) implies that

$$\sum_{x \in X}^{\text{agr}} f_x = \Sigma \circ f = \sum_{x \in X}^{\text{top}} (\Sigma \circ \iota_x \circ f_x) = \sum_{x \in X}^{\text{top}} f_x,$$

so the two notions of infinite summation agree. \square

Notice that the converse assertion to Lemma 3.4 certainly does *not* hold in general. In fact, endowing all the abelian groups $\text{Hom}_{\mathbf{A}}(M, N)$ with the discrete topology defines a complete, separated right topological additive category structure on any additive category \mathbf{A} in such a way that no infinite family of nonzero morphisms converges to zero in $\text{Hom}_{\mathbf{A}}(M, N)$.

A *topologically agreeable* category \mathbf{A} is an agreeable additive category endowed with a complete, separated right topological additive category structure in such a way that, for any two objects $M, N \in \mathbf{A}$, every summable family of morphisms $f_x : M \longrightarrow N$ converges to zero in the topology of $\text{Hom}_{\mathbf{A}}(M, N)$.

Examples 3.5. (1) Any full subcategory closed under coproducts in an agreeable category is agreeable.

(2) Any full subcategory closed under coproducts in a topologically agreeable category is topologically agreeable.

An additive category \mathbf{A} is said to be *idempotent-complete* if all the idempotent endomorphisms of objects in \mathbf{A} have their images in \mathbf{A} . Given an additive category \mathbf{A} , the additive category obtained by adjoining to \mathbf{A} the images of all the idempotent endomorphisms of its objects is called the *idempotent completion* of \mathbf{A} (see, e.g., [4, Exposé IV, Exercice 7.5(b)], [5, Section 1], or [20, Section 4.4]).

Examples 3.6. (1) The idempotent completion of any agreeable additive category is agreeable.

(2) Any structure of a right topological category on an additive category \mathbf{A} can be extended in a unique way to a structure of right topological category on the idempotent completion of \mathbf{A} . Indeed, given a topological abelian group \mathfrak{A} and its continuous idempotent endomorphism $e: \mathfrak{A} \rightarrow \mathfrak{A}$, there exists a unique topology on the abelian group $e\mathfrak{A}$ for which both the inclusion $e\mathfrak{A} \rightarrow \mathfrak{A}$ and the projection $e: \mathfrak{A} \rightarrow e\mathfrak{A}$ are continuous. Hence both the existence and the uniqueness follow from the continuity axiom (i).

(3) In the context of (2), if \mathbf{A} is a topologically agreeable category, then so is the idempotent completion of \mathbf{A} .

Example 3.7. (1) For any associative ring A , the category of left A -modules $\mathbf{A} = A\text{-mod}$ is topologically agreeable. Indeed, \mathbf{A} is agreeable by Example 3.1, and the right topological structure on \mathbf{A} is defined by the classical construction of the *finite topology* on the group of morphisms $\text{Hom}_A(M, N)$ between two left A -modules. Specifically, a base of neighborhoods of zero in $\text{Hom}_A(M, N)$ is provided by the annihilators of finitely generated A -submodules $E \subset M$ (see the references in [45, Example 2.13], and [50, Section 7.1] for further discussion).

One easily observes that a family of morphisms $(f_x: M \rightarrow N_x)_{x \in X}$ in $A\text{-mod}$ corresponds to a morphism $M \rightarrow \bigoplus_{x \in X} N_x$ if and only if the family is *locally finite*, that is, for every finitely generated submodule $E \subset M$ the set of all $x \in X$ for which $f_x|_E \neq 0$ is finite. When $N_x = N$ is one and the same module for all $x \in X$, this is equivalent to convergence of the family of elements $(f_x)_{x \in X}$ to zero in the finite topology on the group $\text{Hom}_A(M, N)$.

(2) More generally, the same construction as in (1) provides a topologically agreeable category structure on any locally finitely generated (Grothendieck) abelian category (in the sense of [1, Section 1.E]). This includes all locally finitely presentable abelian categories, and in particular, all locally Noetherian and locally coherent Grothendieck categories (cf. the discussion in Section 13 below).

Examples 3.8. As is essentially shown in the paper [50, Sections 9-10], further examples of topologically agreeable additive categories include:

(1) all locally weakly finitely generated abelian categories [50, Section 9.2], and

(2) all the additive categories admitting a *closed additive functor* into a locally weakly finitely generated abelian category, as discussed in [50, Section 9.3] (in particular, the categories of comodules over coalgebras and corings [50, Proposition 10.4] and semimodules over semialgebras [50, Proposition 10.8]).

Examples 3.9. (1) More generally, let \mathbf{A} be an additive category, \mathbf{C} be an agreeable additive category, and $F: \mathbf{A} \rightarrow \mathbf{C}$ be a faithful functor preserving coproducts. Then the category \mathbf{A} is agreeable.

(2) Let \mathbf{A} be an additive category, \mathbf{C} be a right topological additive category, and $F: \mathbf{A} \rightarrow \mathbf{C}$ be a faithful additive functor. Then for any two objects M and $N \in \mathbf{C}$ one can endow the group $\text{Hom}_{\mathbf{A}}(M, N)$ with the induced topology of a subgroup in the topological abelian group $\text{Hom}_{\mathbf{C}}(F(M), F(N))$. This construction makes \mathbf{A} a right topological additive category.

(3) Let \mathbf{A} be an additive category and \mathbf{C} be a topologically agreeable additive category. An additive functor $F: \mathbf{A} \rightarrow \mathbf{C}$ is said to be *closed* if it is faithful, preserves coproducts, and for every pair of objects M and $N \in \mathbf{A}$, the image of the injective map $F: \text{Hom}_{\mathbf{A}}(M, N) \rightarrow \text{Hom}_{\mathbf{C}}(F(M), F(N))$ is a closed subgroup in $\text{Hom}_{\mathbf{C}}(F(M), F(N))$.

Given a closed functor $F: \mathbf{A} \rightarrow \mathbf{C}$, the induced right topological category structure on \mathbf{A} (as in (2)) is clearly complete and separated. Moreover, this topology makes \mathbf{A} a topologically agreeable category.

Indeed, let $(f_x: M \rightarrow N)_{x \in X}$ be a summable family of morphisms in the agreeable category \mathbf{A} , and let $f: M \rightarrow N^{(X)}$ be the related morphism to the coproduct. Then the family of morphisms $(F(f_x): F(M) \rightarrow F(N))_{x \in X}$ is summable in \mathbf{C} , since there is the morphism $F(f): F(M) \rightarrow F(N^{(X)}) = F(N)^{(X)}$. Since \mathbf{C} is topologically agreeable by assumption, it follows that the family of morphisms $(F(f_x))_{x \in X}$ converges to zero in the topological group $\text{Hom}_{\mathbf{C}}(F(M), F(N))$. Therefore, the family of morphisms $(f_x)_{x \in X}$ converges to zero in the induced topology of the subgroup $\text{Hom}_{\mathbf{A}}(M, N) \subset \text{Hom}_{\mathbf{C}}(F(M), F(N))$.

Examples 3.10. A topologically agreeable category structure on a given agreeable category does *not* need to be unique. It is instructive to start with the following example of a bijective continuous homomorphism of

complete, separated topological rings with right linear topologies $f: \mathfrak{R}' \rightarrow \mathfrak{R}''$ such that the induced map $f[[X]]: \mathfrak{R}'[[X]] \rightarrow \mathfrak{R}''[[X]]$ is bijective for every set X , yet the inverse homomorphism $f^{-1}: \mathfrak{R}'' \rightarrow \mathfrak{R}'$ is not continuous. This will show that a complete, separated topology on an abelian group is in no way determined by the related zero-convergent families of elements and their sums.

(1) Let R be the ring of (commutative) polynomials in an uncountable set of variables x_i over a field k , and let $S \subset R$ be the multiplicative subset generated by the elements $x_i \in R$. Let \mathfrak{R}' denote the ring R endowed with the discrete topology, and let \mathfrak{R}'' be the ring R endowed with the S -topology (in which the ideals sR , $s \in S$, form a base of neighborhoods of zero). By [26, Proposition 1.16], \mathfrak{R}'' is a complete, separated topological ring. One can easily see that *no infinite family of nonzero elements converges to zero* in \mathfrak{R}'' . The identity map $f: \mathfrak{R}' \rightarrow \mathfrak{R}''$ is a continuous ring homomorphism, and one has $\mathfrak{R}'[[X]] = R[X] = \mathfrak{R}''[[X]]$ for any set X ; still the map $f^{-1}: \mathfrak{R}'' \rightarrow \mathfrak{R}'$ is not continuous. (Cf. [43, Remark 6.3].)

(2) Now let us present an example of an agreeable category with two different topologically agreeable structures. For this purpose, one does not have to look any further than the categories of modules $\mathbf{A} = A\text{-mod}$ over associative rings A . The constructions of Examples 3.7(1) and 3.8(1) provide two different topologically agreeable structures on $A\text{-mod}$, generally speaking.

A left A -module D is said to be *weakly finitely generated* [50, Section 9.2] if, for any family of left A -modules $(N_x)_{x \in X}$, the natural map of abelian groups $\bigoplus_{x \in X} \text{Hom}_A(D, N_x) \rightarrow \text{Hom}_A(D, \bigoplus_{x \in X} N_x)$ is an isomorphism. Such modules D are known as *dually slender* or “small” in the literature [18, 57] (cf. [50, Remark 9.4]); and an associative ring A is said to be *left steady* if all such modules are finitely generated. In the *weakly finite topology* of Example 3.8(1), for any left A -modules M and N , annihilators of weakly finitely generated submodules $D \subset M$ form a base of neighborhoods of zero in the topological group $\text{Hom}_A(M, N)$.

Let \mathbf{A}'' denote the topologically agreeable category of left A -modules with the finite topology on the groups of homomorphisms, and let \mathbf{A}' stand for the topologically agreeable category of left A -modules with the weakly finite topology on the Hom groups. Then the identity functor $F: \mathbf{A}' \rightarrow \mathbf{A}''$ induces continuous bijective maps $\text{Hom}_{\mathbf{A}'}(M, N) \rightarrow \text{Hom}_{\mathbf{A}''}(M, N)$ for all left A -modules M and N , but the inverse map $\text{Hom}_{\mathbf{A}''}(M, N) \rightarrow \text{Hom}_{\mathbf{A}'}(M, N)$ does not need to be continuous.

(3) To give a specific example, let $A = k\{x, y\}$ be the free associative algebra with two generators. Then any injective A -module is weakly finitely generated [57, Lemma 3.2]. Let M be an (infinitely generated) injective cogenerator of $A\text{-mod}$. Then the ring $\mathfrak{R}' = \text{Hom}_{A'}(M, M)^{\text{op}}$ is discrete, while the ring $\mathfrak{R}'' = \text{Hom}_{A''}(M, M)^{\text{op}}$ is not. Indeed, let $\mathfrak{U} \subset \mathfrak{R}''$ be an open right ideal. Then \mathfrak{U} contains the annihilator of some finitely generated submodule $E \subset M$. This annihilator is nonzero, as $\text{Hom}_A(M/E, M) \neq 0$. Thus the zero ideal is not open in \mathfrak{R}'' . But it is open in \mathfrak{R}' , since M is weakly finitely generated (so one can take $D = M$).

As in (1), we have a continuous bijective map of complete, separated topological rings with right linear topologies $f: \mathfrak{R}' \rightarrow \mathfrak{R}''$, where \mathfrak{R}' is discrete but \mathfrak{R}'' is not. Still, no infinite family of nonzero elements converges to zero in \mathfrak{R}'' .

(4) In fact, the functor $F: A' \rightarrow A''$ is an equivalence of topologically agreeable categories *if and only if* the ring A is left steady. Indeed, let M be a weakly finitely generated left A -module that is not finitely generated. Put $N = \bigoplus_{E \subset M} M/E$, where E ranges over all the finitely generated A -submodules of M . Then the topological abelian group $\mathfrak{R}' = \text{Hom}_{A'}(M, N)$ is discrete, since M is weakly finitely generated. Let us show that the complete, separated topological abelian group $\mathfrak{R}'' = \text{Hom}_{A''}(M, N)$ is *not* discrete. Let $\mathfrak{U} \subset \mathfrak{R}''$ be an open subgroup. Then \mathfrak{U} contains the annihilator of some finitely generated submodule $E \subset M$. By construction, this annihilator is nonzero, as $\text{Hom}_A(M/E, N) \neq 0$. So the zero subgroup is not open in \mathfrak{R}'' .

Set $L = M \oplus N$, and consider the complete, separated right linear topological rings $\mathfrak{R}' = \text{Hom}_{A'}(L, L)^{\text{op}}$ and $\mathfrak{R}'' = \text{Hom}_{A''}(L, L)^{\text{op}}$. Then the identity map $f: \mathfrak{R}' \rightarrow \mathfrak{R}''$ is a continuous ring homomorphism such that the induced map $f[[X]]: \mathfrak{R}'[[X]] \rightarrow \mathfrak{R}''[[X]]$ is bijective for every set X . Yet it is clear from the previous paragraph and the argument in Example 3.6(2) that the map $f^{-1}: \mathfrak{R}'' \rightarrow \mathfrak{R}'$ is not continuous.

(5) Slightly more generally, let A be an associative ring and M be a *self-small* left A -module, i.e., a module for which the natural map $\bigoplus_{x \in X} \text{Hom}_A(M, M) \rightarrow \text{Hom}_A(M, \bigoplus_{x \in X} M)$ is an isomorphism for any set X . Then the finite topology on the ring $\mathfrak{R}'' = \text{Hom}_{A''}(M, M)^{\text{op}}$ is a complete, separated right linear topology in which no infinite family of nonzero elements converges to zero. The weakly finite topology on the ring $\mathfrak{R}' = \text{Hom}_{A'}(M, M)^{\text{op}}$ has the same properties.

Remark 3.11. Beyond the historical discussion in [45, Example 2.13] and [50, Section 7.1], let us mention one class of examples of topological rings of endomorphisms which appears in the literature. Let R be a commutative domain and Q be its ring of fractions. Consider the R -module $K = Q/R$. Then the R -algebra $\text{Hom}_R(K, K)$ is commutative [35, Proposition 7.1] and isomorphic to the completion of R in the R -topology [35, Proposition 6.4]. The completion (projective limit) topology on $\text{Hom}_R(K, K)$ as the completion of R coincides with the R -topology [35, Theorems 6.8 and 6.10] and with the finite topology on $\text{Hom}_R(K, K)$ as the endomorphism ring.

The similar results hold for an arbitrary commutative ring R and its full ring of quotients Q [36, Section 2], or even more generally, for a commutative ring R with a multiplicative subset of regular elements $S \subset R$ and the R -module $K = (S^{-1}R)/R$. In the latter case, the commutative R -algebra $\text{Hom}_R(K, K)$ is isomorphic to the completion of R in the S -topology; the completion topology on $\text{Hom}_R(K, K)$ coincides with the S -topology and with the finite topology of the endomorphism ring [42, Proposition 3.2, and Theorems 2.3 and 2.5]. A construction of a two-sided linear topology on the endomorphism ring of a similar module K for a noncommutative ring R was suggested in the paper [21, Proposition 3.5].

In the rest of this section, we discuss the question of the *existence* of a topologically agreeable structure and the consequences of such existence.

Remark 3.12. The reader should be warned that the abelian category $\mathfrak{R}\text{-contra}$ of left contra-modules over a topological ring \mathfrak{R} is rarely agreeable, generally speaking (see, e.g., the discussion in [44, Section 1.5]). However, the additive category of *projective* left \mathfrak{R} -contra-modules $\mathfrak{R}\text{-contra}_{\text{proj}}$ is agreeable, and in fact has a natural topologically agreeable category structure, which can be explicitly constructed as follows. The construction of the matrix topology in Section 5 (extended from the square row-zero-convergent matrices with entries in \mathfrak{R} to the rectangular ones in the obvious way) provides an agreeable topologization on the full subcategory of *free* left \mathfrak{R} -contra-modules in $\mathfrak{R}\text{-contra}$. This topologization can be extended to the whole category $\mathfrak{R}\text{-contra}_{\text{proj}}$ as explained in Examples 3.6.

More generally, the following lemma holds.

Lemma 3.13. *Let \mathbf{B} be a cocomplete abelian category with a projective generator. Assume that the full subcategory of projective objects $\mathbf{B}_{\text{proj}} \subset \mathbf{B}$ is agreeable. Then the category \mathbf{B} is locally presentable.*

Proof: This is explained in the discussion in [47, Section 1.2]. In fact, let P be a projective generator of \mathbf{B} . Then we claim that the category \mathbf{B} is locally κ -presentable, where κ is the successor cardinal of the cardinality of the set $\text{Hom}_{\mathbf{B}}(P, P)$.

Indeed, the category \mathbf{B} is equivalent to the category of modules over the monad $\mathbb{T}_P: X \mapsto \text{Hom}_{\mathbf{B}}(P, P^{(X)})$ on the category of sets, so it suffices to show that this monad is κ -accessible (see [47, Section 1.1] or [40, Section 1]). For this purpose, one simply observes that a summable (in the terminology of [47], “admissible”) family of morphisms $(f_x: P \rightarrow P)_{x \in X}$ cannot have any given nonzero morphism $P \rightarrow P$ repeated in it more than a finite number of times, or otherwise the cancellation trick leads to a contradiction. Thus the cardinality of any summable family of nonzero morphisms $(f_x: P \rightarrow P)$ is smaller than κ . \square

Given an additive category \mathbf{A} with set-indexed coproducts and an object $M \in \mathbf{A}$, we denote by $\text{Add}(M) \subset \mathbf{A}$ the full subcategory consisting of all the direct summands of coproducts of copies of M in \mathbf{A} . The following theorem is the main result of what we call “generalized tilting theory” (see [50, 48]).

Theorem 3.14. (i) *Let \mathbf{A} be an idempotent-complete additive category with set-indexed coproducts and $M \in \mathbf{A}$ be an object. Then there exists a unique abelian category \mathbf{B} with enough projective objects such that the full subcategory $\mathbf{B}_{\text{proj}} \subset \mathbf{B}$ of projective objects in \mathbf{B} is equivalent to the full subcategory $\text{Add}(M) \subset \mathbf{A}$, that is,*

$$\mathbf{A} \supset \text{Add}(M) \cong \mathbf{B}_{\text{proj}} \subset \mathbf{B}.$$

- (ii) *In the context of (i), assume additionally that the additive category \mathbf{A} is agreeable. Then the abelian category \mathbf{B} is locally presentable and, for any family of projective objects $(P_x \in \mathbf{B})_{x \in X}$, the natural morphism $\coprod_{x \in X} P_x \rightarrow \prod_{x \in X} P_x$ is a monomorphism in \mathbf{B} .*
- (iii) *In the context of (ii), assume additionally that the additive category \mathbf{A} is topologically agreeable. Then the category \mathbf{B} is equivalent to the category of left contra-modules over the topological ring $\mathfrak{R} = \text{Hom}_{\mathbf{A}}(M, M)^{\text{op}}$, that is, $\mathbf{B} = \mathfrak{R}\text{-contra}$. The equivalence of additive categories $\text{Add}(M) \cong \mathfrak{R}\text{-contra}_{\text{proj}}$ takes the object $M \in \text{Add}(M)$ to the free left \mathfrak{R} -contra-module with one generator $\mathfrak{R} \in \mathfrak{R}\text{-contra}_{\text{proj}}$.*

Proof: Part (i) is [48, Theorem 1.1(a)] (see also [50, Section 6.3]). In part (ii), we observe that the category $\text{Add}(M)$ is agreeable by Example 3.5(1), and it follows that the category \mathbf{B}_{proj} is agreeable too. Let $P \in \mathbf{B}_{\text{proj}}$ be the object corresponding to the object $M \in \text{Add}(M)$ under

the equivalence $\text{Add}(M) \cong \mathbf{B}_{\text{proj}}$; then P is a projective generator of \mathbf{B} . By Lemma 3.13, the category \mathbf{B} is locally presentable. Finally, part (iii) essentially follows from Lemma 3.4 (cf. the proofs of [50, Theorems 7.1, 9.9, and 9.11]). \square

If \mathbf{A} is abelian, the equivalence of categories $\text{Add}(M) \cong \mathbf{B}_{\text{proj}}$ in Theorem 3.14 can be extended to a pair of adjoint functors between the whole categories \mathbf{A} and \mathbf{B} [48, Section 1]. The fully faithful functor $\text{Add}(M) \cong \mathbf{B}_{\text{proj}} \hookrightarrow \mathbf{B}$ extends to a left exact functor $\Psi: \mathbf{A} \rightarrow \mathbf{B}$, while the fully faithful functor $\mathbf{B}_{\text{proj}} \cong \text{Add}(M) \hookrightarrow \mathbf{A}$ extends to a right exact functor $\Phi: \mathbf{B} \rightarrow \mathbf{A}$, with the functor Φ left adjoint to Ψ .

We will need the following more explicit description of the functor Ψ [50, proof of Proposition 6.2], [48, Section 1]. The abelian category \mathbf{B} can be constructed as the category of modules over the additive monad $\mathbb{T}_M: X \mapsto \text{Hom}_{\mathbf{A}}(M, M^{(X)})$ on the category of sets. Hence the category \mathbf{B} is endowed with a faithful, exact, limit-preserving forgetful functor $\mathbf{B} \rightarrow \mathbf{Ab}$ to the category of abelian groups. The composition of the right exact functor $\Psi: \mathbf{A} \rightarrow \mathbf{B}$ with the exact forgetful functor $\mathbf{B} \rightarrow \mathbf{Ab}$ is computed as the functor $\text{Hom}_{\mathbf{A}}(M, -): \mathbf{A} \rightarrow \mathbf{Ab}$.

In particular, in the context of Theorem 3.14(iii), the monad \mathbb{T}_M is isomorphic to the monad $X \mapsto \mathfrak{R}[[X]]$ on the category of sets. The forgetful functor $\mathbf{B} \cong \mathfrak{R}\text{-contra} \rightarrow \mathbf{Ab}$ assigns to a left \mathfrak{R} -contramodule its underlying abelian group.

We say that an agreeable category is *topologizable* if it admits a topologically agreeable category structure. Notice that any topologizable category is (agreeable, hence) additive with set-indexed coproducts by definition.

The following theorem is one of the main results of this paper (cf. the closely related Theorem 6.6). Its proof will be given below in Section 11.

Theorem 3.15. *Any topologizable split abelian category is Ab5 and semisimple. So any topologizable spectral category is discrete spectral.*

4. Topological rings as endomorphism rings

The aim of this section is to show that the categories of modules over associative rings are, in a rather strong sense, representative among all the topologically agreeable categories (so some results about direct sum decompositions in topologically agreeable categories follow from the same results for modules, as we will see below in Section 10). More specifically, in this section we prove that all complete, separated topological

rings with right linear topology can be realized as the rings of endomorphisms of modules over associative rings (endowed with the finite topology).

Let \mathbf{C} be a small category. Then the category $\text{Funct}(\mathbf{C}, \mathbf{Ab})$ of all functors from \mathbf{C} to the category of abelian groups is a locally finitely presentable Grothendieck abelian category. According to Example 3.7(2), the category $\text{Funct}(\mathbf{C}, \mathbf{Ab})$ is topologically agreeable, so for any two functors $F, G: \mathbf{C} \rightarrow \mathbf{Ab}$ there is a natural complete, separated topology on the abelian group $\text{Hom}_{\text{Funct}}(F, G)$. Moreover, the ring $\text{Hom}_{\text{Funct}}(F, F)^{\text{op}}$ is a topological ring with right linear topology.

This topology can be explicitly described as follows. By definition, a base of neighborhoods of zero in $\text{Hom}_{\text{Funct}}(F, G)$ consists of the annihilators of finitely generated subfunctors $E \subset F$. Simply put, this means that one has to choose a finite sequence of objects $C_1, \dots, C_m \in \mathbf{C}$ and some element $e_j \in F(C_j)$ for every $j = 1, \dots, m$, and consider the subgroup in $\text{Hom}_{\text{Funct}}(F, G)$ consisting of all the natural transformations $F \rightarrow G$ annihilating the chosen elements e_j . Subgroups of this form constitute a base of neighborhoods of zero in $\text{Hom}_{\text{Funct}}(F, G)$.

More generally, let \mathbf{D} be a (not necessarily small) category and $\mathbf{C} \subset \mathbf{D}$ be a small subcategory. For every functor $F: \mathbf{D} \rightarrow \mathbf{Ab}$ and object $D \in \mathbf{D}$ consider the induced homomorphism of abelian groups

$$F_{\mathbf{C}/\mathbf{D}}: \bigoplus_{C \rightarrow D} F(C) \longrightarrow F(D),$$

where the direct sum on the left-hand side is taken over all pairs (an object $C \in \mathbf{C}$, a morphism $C \rightarrow D$ in \mathbf{D}). We will say that the full subcategory $\mathbf{C} \subset \mathbf{D}$ is *weakly dense for F* if the map $F_{\mathbf{C}/\mathbf{D}}$ is surjective for every object $D \in \mathbf{D}$. The full subcategory in the category of functors $\text{Funct}(\mathbf{D}, \mathbf{Ab})$ consisting of all the functors $F: \mathbf{D} \rightarrow \mathbf{Ab}$ for which the full subcategory $\mathbf{C} \subset \mathbf{D}$ is weakly dense will be denoted by $\text{Funct}(\mathbf{D}/\mathbf{C}, \mathbf{Ab}) \subset \text{Funct}(\mathbf{D}, \mathbf{Ab})$.

Clearly, if the full subcategory $\mathbf{C} \subset \mathbf{D}$ is weakly dense for a functor $F: \mathbf{D} \rightarrow \mathbf{Ab}$, then any morphism $F \rightarrow G$ of functors $\mathbf{D} \rightarrow \mathbf{Ab}$ is determined by its restriction to the full subcategory $\mathbf{C} \subset \mathbf{D}$. In particular, the restriction functor

$$\rho = \rho_{\mathbf{D}/\mathbf{C}}: \text{Funct}(\mathbf{D}/\mathbf{C}, \mathbf{Ab}) \longrightarrow \text{Funct}(\mathbf{C}, \mathbf{Ab})$$

assigning to a functor $F: \mathbf{D} \rightarrow \mathbf{Ab}$ its restriction $\rho(F) = F|_{\mathbf{C}}$ to the full subcategory $\mathbf{C} \subset \mathbf{D}$ is faithful. It follows that morphisms $F \rightarrow G$ between any two fixed functors $F \in \text{Funct}(\mathbf{D}/\mathbf{C}, \mathbf{Ab})$ and $G \in \text{Funct}(\mathbf{D}, \mathbf{Ab})$ form a set rather than a proper class. Moreover, the following assertion holds.

Lemma 4.1. *The restriction functor $\rho_{D/C}$ is a closed additive functor in the sense of Example 3.8(2) or 3.9(3), and [50, Section 9.3].*

Proof: Essentially, we have to check that, given two functors $F \in \text{Funct}(D/C, \text{Ab})$ and $G \in \text{Funct}(D, \text{Ab})$, a morphism of functors $g: F|_C \rightarrow G|_C$ can be extended to a morphism of functors $f: F \rightarrow G$ provided that, for every finitely generated subfunctor $E \subset F|_C$ there exists a morphism of functors $h: F \rightarrow G$ such that the restriction of $h|_C$ to E coincides with the restriction of g to E .

The latter condition means that, for every finite sequence of objects $C_1, \dots, C_m \in C$ and any elements $e_j \in F(C_j)$, $j = 1, \dots, m$, there exists a morphism of functors $h: F \rightarrow G$ such that $h(e_j) = g(e_j)$ for all $1 \leq j \leq m$. We have to show that there exists a morphism of functors $f: F \rightarrow G$ such that $f|_C = g$; in other words, this means that for every object $D \in D$ the map of abelian groups

$$\bigoplus_{C \rightarrow D} g_C: \bigoplus_{C \rightarrow D} F(C) \longrightarrow \bigoplus_{C \rightarrow D} G(C)$$

descends to a map $f_D: F(D) \rightarrow G(D)$ forming a commutative square with the maps $F_{C/D}$ and $G_{C/D}$. Let $e \in \bigoplus_{C \rightarrow D} F(C)$ be an element annihilated by the surjective map $F_{C/D}$; then the element e has a finite number of nonzero components $e_j \in F(C_j)$, $C_j \in C$, $1 \leq j \leq m$. Now the existence of a morphism of functors $h: F \rightarrow G$ agreeing with the morphism of functors $g: F|_C \rightarrow G|_C$ on the elements $e_j \in F(C_j)$ guarantees that the element e is also annihilated by the composition of maps $\bigoplus_{C \rightarrow D} F(C) \rightarrow \bigoplus_{C \rightarrow D} G(C) \rightarrow G(D)$, implying the existence of the desired map f_D . □

In view of Lemma 4.1, for any two functors $F \in \text{Funct}(D/C, \text{Ab})$ and $G \in \text{Funct}(D, \text{Ab})$, the group $\text{Hom}_{\text{Funct}}(F, G)$ is a closed subgroup of the topological abelian group $\text{Hom}_{\text{Funct}}(F|_C, G|_C)$. We endow the group $\text{Hom}_{\text{Funct}}(F, G)$ with the induced topology, making it a complete, separated topological abelian group. In particular, the ring $\text{Hom}_{\text{Funct}}(F, F)^{\text{op}}$ becomes a complete, separated topological ring with a right linear topology and a closed subring in $\text{Hom}_{\text{Funct}}(F|_C, F|_C)^{\text{op}}$.

Explicitly, a base of neighborhoods of zero in $\text{Hom}_{\text{Funct}}(F, G)$ is provided by the annihilators of finite sets of elements $e_j \in F(C_j)$, $C_j \in C$. Now we observe that, due to the condition of surjectivity of the maps $F_{C/D}$ imposed on the functor F , the collection of the annihilator subgroups of all finite sets of elements $e_j \in F(D_j)$, $D_j \in D$, $1 \leq j \leq m$, is another base of neighborhoods of zero for the same topology on $\text{Hom}_{\text{Funct}}(F, G)$. Thus the complete, separated topology on the group $\text{Hom}_{\text{Funct}}(F, G)$ that we have constructed depends only on the (possibly

large) category \mathcal{D} and the functors F and G , and does *not* depend on the choice of a weakly dense small subcategory $\mathcal{C} \subset \mathcal{D}$ for the functor F .

In the rest of this section we apply the above considerations to one specific large category \mathcal{D} with a small subcategory \mathcal{C} and a functor $F: \mathcal{D} \rightarrow \mathbf{Ab}$. Namely, let R be a topological ring with a right linear topology, and let $\mathcal{D} = \text{discr-}R$ be the abelian category of discrete right R -modules. Furthermore, let $\mathcal{C} \subset \mathcal{D}$ be the full subcategory consisting of all the cyclic discrete right R -modules R/I , where $I \subset R$ ranges over the open right ideals in R . Finally, let $F: \text{discr-}R \rightarrow \mathbf{Ab}$ be the forgetful functor, assigning to a discrete right R -module \mathcal{N} its underlying abelian group \mathcal{N} .

Proposition 4.2. *The restriction map*

$$\text{Hom}_{\text{Funct}}(F, F) \longrightarrow \text{Hom}_{\text{Funct}}(F|_{\mathcal{C}}, F|_{\mathcal{C}})$$

is bijective (hence a topological ring isomorphism). The complete, separated topological ring $\text{Hom}_{\text{Funct}}(F, F)^{\text{op}}$ is naturally isomorphic, as a topological ring, to the completion \mathfrak{R} of the topological ring R (with the projective limit topology on \mathfrak{R}).

Proof: Since $\text{discr-}R = \text{discr-}\mathfrak{R}$, we can replace the topological ring R by its completion \mathfrak{R} from the outset and assume that we are dealing with the category $\mathcal{D} = \text{discr-}\mathfrak{R}$, its full subcategory \mathcal{C} spanned by the cyclic discrete right modules $\mathfrak{R}/\mathfrak{I}$, where \mathfrak{I} runs over the open right ideals in \mathfrak{R} , and the forgetful functor $F: \text{discr-}\mathfrak{R} \rightarrow \mathbf{Ab}$. Then the right action of \mathfrak{R} on the discrete right modules over it provides a natural ring homomorphism $\mathfrak{R} \rightarrow \text{Hom}_{\text{Funct}}(F, F)^{\text{op}}$. Since we already know that $\text{Hom}_{\text{Funct}}(F, F)$ is a closed subring in $\text{Hom}_{\text{Funct}}(F|_{\mathcal{C}}, F|_{\mathcal{C}})$ with the induced topology, it suffices to show that the composition $\mathfrak{R} \rightarrow \text{Hom}_{\text{Funct}}(F, F)^{\text{op}} \rightarrow \text{Hom}_{\text{Funct}}(F|_{\mathcal{C}}, F|_{\mathcal{C}})^{\text{op}}$ is an isomorphism of topological rings.

Since the topological ring \mathfrak{R} is separated, for any element $t \in \mathfrak{R}$ there exists an open right ideal $\mathfrak{J} \subset \mathfrak{R}$ not containing t . Then the element t acts nontrivially on the coset $1 + \mathfrak{J} \in F(\mathfrak{R}/\mathfrak{J})$, taking it to the coset $t + \mathfrak{J} \neq 0$. This proves injectivity of the map $\mathfrak{R} \rightarrow \text{Hom}_{\text{Funct}}(F|_{\mathcal{C}}, F|_{\mathcal{C}})^{\text{op}}$. To prove surjectivity, consider a natural transformation $\tau: F|_{\mathcal{C}} \rightarrow F|_{\mathcal{C}}$. For every open right ideal $\mathfrak{J} \in \mathfrak{R}$, we can apply τ to the coset $1 + \mathfrak{J} \in F(\mathfrak{R}/\mathfrak{J})$, obtaining an element $t_{\mathfrak{J}} + \mathfrak{J} = \tau_{\mathfrak{R}/\mathfrak{J}}(1 + \mathfrak{J}) \in F(\mathfrak{R}/\mathfrak{J}) = \mathfrak{R}/\mathfrak{J}$. For every open right ideal $\mathfrak{K} \subset \mathfrak{J}$, there is a morphism $p_{\mathfrak{J}, \mathfrak{K}}: \mathfrak{R}/\mathfrak{K} \rightarrow \mathfrak{R}/\mathfrak{J}$ in the category \mathcal{C} taking the coset $r + \mathfrak{K}$ to the coset $r + \mathfrak{J}$ for every $r \in \mathfrak{R}$. The commutativity equation $\tau_{\mathfrak{R}/\mathfrak{J}}F|_{\mathcal{C}}(p_{\mathfrak{J}, \mathfrak{K}}) = F|_{\mathcal{C}}(p_{\mathfrak{J}, \mathfrak{K}})\tau_{\mathfrak{R}/\mathfrak{K}}$ on the natural transformation τ applied to the coset $1 + \mathfrak{K} \in F(\mathfrak{R}/\mathfrak{K})$ implies that $p_{\mathfrak{J}, \mathfrak{K}}(t_{\mathfrak{K}}) = t_{\mathfrak{J}}$. Thus the family of cosets $(t_{\mathfrak{J}} + \mathfrak{J} \in \mathfrak{R}/\mathfrak{J})$ represents a well-defined element t of the projective limit $\varprojlim_{\mathfrak{J} \subset \mathfrak{R}} \mathfrak{R}/\mathfrak{J} = \mathfrak{R}$.

Let us show that our natural transformation τ is the image of the element t under the map $\mathfrak{R} \rightarrow \text{Hom}_{\text{Funct}}(F|_{\mathbb{C}}, F|_{\mathbb{C}})^{\text{op}}$. We have to check that, for every open right ideal $\mathfrak{J} \subset \mathfrak{R}$ and every coset $s + \mathfrak{J} \in \mathfrak{R}/\mathfrak{J}$, the image of $s + \mathfrak{J}$ under τ is given by the rule $\tau_{\mathfrak{R}/\mathfrak{J}}(s + \mathfrak{J}) = st + \mathfrak{J}$. Choose an open right ideal $\mathfrak{K} \subset \mathfrak{R}$ such that $s\mathfrak{K} \subset \mathfrak{J}$. Then there is a morphism $s_{\mathfrak{J},\mathfrak{K}}: \mathfrak{R}/\mathfrak{K} \rightarrow \mathfrak{R}/\mathfrak{J}$ in the category \mathbb{C} taking the coset $r + \mathfrak{K}$ to the coset $sr + \mathfrak{J}$ for every $r \in \mathfrak{R}$. Now the commutativity equation $\tau_{\mathfrak{R}/\mathfrak{J}}F|_{\mathbb{C}}(s_{\mathfrak{J},\mathfrak{K}}) = F|_{\mathbb{C}}(s_{\mathfrak{J},\mathfrak{K}})\tau_{\mathfrak{R}/\mathfrak{K}}$ on the natural transformation τ applied to the coset $1 + \mathfrak{K} \in F(\mathfrak{R}/\mathfrak{K})$ implies the desired equality $\tau_{\mathfrak{R}/\mathfrak{J}}(s + \mathfrak{J}) = st + \mathfrak{J}$.

It remains to show that the topologies on the rings \mathfrak{R} and $\text{Hom}_{\text{Funct}}(F, F)$ agree. Indeed, for any finite sequence of discrete right \mathfrak{R} -modules $\mathcal{N}_1, \dots, \mathcal{N}_m$ and any chosen elements $e_j \in \mathcal{N}_j$, the intersection of the annihilators of the elements e_j is an open right ideal in \mathfrak{R} . Conversely, any open right ideal $\mathfrak{J} \subset \mathfrak{R}$ is the annihilator of the single coset $1 + \mathfrak{J} \in \mathfrak{R}/\mathfrak{J}$. □

Proposition 4.3. *Let \mathbb{C} be a small category and $F: \mathbb{C} \rightarrow \text{Ab}$ be a functor. Then there exists an associative ring A and a left A -module M such that the topological ring $\text{Hom}_{\text{Funct}}(F, F)^{\text{op}}$ is isomorphic to the topological ring $\text{Hom}_A(M, M)^{\text{op}}$ (with the finite topology on the latter; see Example 3.7(1)).*

Proof: Set $M = \bigoplus_{C \in \mathbb{C}} F(C)$ to be the direct sum of the abelian groups $F(C)$ over all the objects $C \in \mathbb{C}$. Then every natural transformation $\tau: F \rightarrow F$ acts naturally by an abelian group homomorphism $\tau_M: M \rightarrow M$ by the direct sum of the maps $\tau_C: F(C) \rightarrow F(C)$.

For every object $C \in \mathbb{C}$, let $p_C: M \rightarrow M$ denote the projector onto the direct summand $F(C)$ in M . For every morphism $s: C \rightarrow D$ in \mathbb{C} , let $s_{C,D}: M \rightarrow M$ denote the map whose restriction to $F(C) \subset M$ is equal to the composition of the map $F(s): F(C) \rightarrow F(D)$ with the inclusion $F(D) \rightarrow M$, while the restriction of $s_{C,D}$ to $F(C')$ is zero for all $C' \in \mathbb{C}$, $C' \neq C$. So, in particular, $p_C = (\text{id}_C)_{C,C}$.

Let A be the subring (with unit) in $\text{Hom}_{\mathbb{Z}}(M, M)$ generated by the maps p_C and $s_{C,D}$, $C, D \in \mathbb{C}$. Then the maps τ_M are exactly all the abelian group endomorphisms of M commuting with all the maps p_C and $s_{C,D}$. Indeed, any map $t: M \rightarrow M$ commuting with the projectors p_C for all $C \in \mathbb{C}$ has the form $t = \bigoplus_{C \in \mathbb{C}} t_C$ for some maps $t_C: F(C) \rightarrow F(C)$. If the map t also commutes with the maps $s_{C,D}$ for all morphisms $s: C \rightarrow D$ in \mathbb{C} , then the collection of maps t_C is an endomorphism of the functor F . Thus $\text{Hom}_{\text{Funct}}(F, F) \cong \text{Hom}_A(M, M)$.

Furthermore, the annihilators of finite subsets in M form the same collection of subgroups in $\text{Hom}_A(M, M)$ as the annihilators of finite sequences of elements $e_j \in F(C_j)$, $C_j \in \mathbf{C}$, $j = 1, \dots, m$. So the topologies on the rings $\text{Hom}_{\text{Funct}}(F, F)$ and $\text{Hom}_A(M, M)$ also agree. \square

Corollary 4.4. *For every complete, separated topological ring \mathfrak{R} with right linear topology, there exists an associative ring A and a left A -module M such that \mathfrak{R} is isomorphic, as a topological ring, to the ring $\text{Hom}_A(M, M)^{\text{op}}$ (endowed with the finite topology).*

Proof: The assertion follows from Propositions 4.2 and 4.3. For convenience, let us spell out the specific construction of the ring A and the module M that we obtain. The underlying abelian group of M is the direct sum $\bigoplus_{\mathfrak{J} \subset \mathfrak{R}} \mathfrak{R}/\mathfrak{J}$ of the quotient groups of \mathfrak{R} by its open right ideals. So M is naturally a discrete right \mathfrak{R} -module (as it should be). The ring A acting on M on the left is simplest constructed as the ring of endomorphisms of the right \mathfrak{R} -module M . The above proof of Proposition 4.3 provides a smaller ring A which works just as well. Namely, it is the subring (with unit) in $\text{Hom}_{\mathfrak{R}^{\text{op}}}(M, M)$ or $\text{Hom}_{\mathbb{Z}}(M, M)$ generated by the compositions

$$M \longrightarrow \mathfrak{R}/\mathfrak{J} \xrightarrow{s_{\mathfrak{J}, \mathfrak{J}}} \mathfrak{R}/\mathfrak{J} \longrightarrow M$$

of the direct summand projections $M \longrightarrow \mathfrak{R}/\mathfrak{J}$, the direct summand inclusions $\mathfrak{R}/\mathfrak{J} \longrightarrow M$, and the maps $s_{\mathfrak{J}, \mathfrak{J}}$ mentioned in the proof of Proposition 4.2. \square

Remark 4.5. Let B be a set of open right ideals forming a base of neighborhoods of zero in a topological ring R with right linear topology. Consider the category $\mathbf{D} = \text{discr-}R$ and the full subcategory $\mathbf{C} \subset \mathbf{D}$ consisting of all the cyclic discrete right R -modules R/I with the ideal $I \in B$. Let $F: \text{discr-}R \longrightarrow \mathbf{Ab}$ be the forgetful functor. The assertion of Proposition 4.2 remains valid in this context, and the proof is essentially the same. In the particular case of a complete, separated topological ring \mathfrak{R} with a right linear topology, the construction of Corollary 4.4 gets modified accordingly, providing a “smaller” ring A and left A -module $M = \bigoplus_{\mathfrak{J} \in B} \mathfrak{R}/\mathfrak{J}$ such that \mathfrak{R} is isomorphic, as a topological ring, to $\text{Hom}_A(M, M)^{\text{op}}$.

The following question was raised at the end of [33, Section 2]. Let k be a field and V be an infinite-dimensional k -vector space. Can one characterize complete, separated (associative and unital) topological k -algebras \mathfrak{R} with right linear topology that can be realized as closed subalgebras of $\text{Hom}_k(V, V)^{\text{op}}$?

The results of this section together with the first paragraph of this remark allow us to give a complete answer to this question. Let $\lambda = \dim_k V$

denote the cardinality of a basis of V . Then a complete, separated topological k -algebra \mathfrak{R} with right linear topology can be realized as a closed subalgebra in $\text{Hom}_k(V, V)^{\text{op}}$ (with the finite topology on $\text{Hom}_k(V, V)^{\text{op}}$ and the induced topology on \mathfrak{R}) if and only if the following two conditions hold:

- (i) \mathfrak{R} has a base of neighborhoods of zero of the cardinality not exceeding λ ; and
- (ii) for every open right ideal $\mathfrak{J} \subset \mathfrak{R}$, one has $\dim_k \mathfrak{R}/\mathfrak{J} \leq \lambda$.

Indeed, if (i) and (ii) hold and B is a base of neighborhoods of zero in \mathfrak{R} consisting of at most λ open right ideals, then one can consider the right \mathfrak{R} -module $N = \bigoplus_{\mathfrak{J} \in B} \mathfrak{R}/\mathfrak{J}$. By assumption, $\dim_k N \leq \lambda$. If the inequality is strict, one can replace N by the direct sum $M = N^{(\lambda)}$ of λ copies of N ; otherwise, put $M = N$. Denoting by A the ring of endomorphisms of the right \mathfrak{R} -module M , one has $\mathfrak{R} = \text{End}_A(M, M)^{\text{op}}$, essentially by the first paragraph of this remark. By [33, Lemma 2.2(2)], \mathfrak{R} is a closed subring in $\text{Hom}_k(M, M)^{\text{op}}$.

Conversely, let $\mathfrak{R} \subset \text{End}_k(V, V)^{\text{op}}$ be a closed subring with the induced topology. Choose a basis $(v_i)_{i \in \lambda}$ of the vector space V . Then the annihilators of finite subsets of $\{v_i\}$ in \mathfrak{R} provide a base of neighborhoods of zero in \mathfrak{R} of the cardinality at most λ . This proves (i). Furthermore, for any open right ideal $\mathfrak{J} \subset \mathfrak{R}$, there exists a finite-dimensional subspace $W \subset V$ such that \mathfrak{J} contains the annihilator of W in \mathfrak{R} . Then the action of \mathfrak{R} in V provides a natural injective map $\mathfrak{R}/\mathfrak{J} \rightarrow \text{Hom}_k(W, V)$. Since $\dim_k \text{Hom}_k(W, V) = \lambda$, this proves (ii).

Thus, in particular, any complete, separated topological k -algebra \mathfrak{R} with a right linear topology can be realized as a closed subalgebra in the algebra $\text{Hom}_k(V, V)^{\text{op}}$ of endomorphisms of a k -vector space V . Moreover, one can realize \mathfrak{R} as a “bicommutant” or “bicentralizer”, i.e., the algebra of all k -linear operators commuting with a certain ring A of such operators acting on V .

The above construction is certainly not new. A far more advanced result concerning realization of algebras over commutative rings as endomorphism algebras of modules is known as Corner’s realization theorem [26, Theorem 20.1].

5. Matrix topologies

Let \mathfrak{R} be a complete, separated topological ring with a right linear topology and Y be a set. The aim of this section is to construct a complete, separated right linear topology on a certain ring $\mathfrak{S} = \mathfrak{Mat}_Y(\mathfrak{R})$ of Y -sized matrices with the entries in \mathfrak{R} , in such a way that the cate-

gory of left \mathfrak{S} -contramodules is equivalent to the category of left \mathfrak{R} -contramodules and the category of discrete right \mathfrak{S} -modules equivalent to the category of discrete right \mathfrak{R} -modules. Furthermore, the free \mathfrak{R} -contramodule $\mathfrak{R}[[Y]]$ with the set of generators Y corresponds to the free \mathfrak{S} -contramodule with one generator \mathfrak{S} under the equivalence of categories $\mathfrak{R}\text{-contra} \cong \mathfrak{S}\text{-contra}$.

Specifically, we denote by $\mathfrak{Mat}_Y(\mathfrak{R})$ the set of all *row-zero-convergent* matrices with entries in \mathfrak{R} , meaning matrices $(a_{x,y} \in \mathfrak{R})_{x,y \in Y}$ such that for every $x \in Y$ the family of elements $(a_{x,y})_{y \in Y}$ converges to zero in the topology of \mathfrak{R} . The abelian group $\mathfrak{Mat}_Y(\mathfrak{R})$, with the obvious entrywise additive structure, is in fact an associative ring with the unit element $1 = (\delta_{x,y})_{x,y \in Y}$ and the matrix multiplication

$$(ab)_{x,z} = \sum_{y \in Y} a_{x,y} b_{y,z}, \quad a, b \in \mathfrak{Mat}_Y(\mathfrak{R}),$$

defined using the multiplication in \mathfrak{R} and the infinite summation of zero-converging families of elements, understood as the limit of finite partial sums in the topology of \mathfrak{R} . It is important for this construction that the topology in \mathfrak{R} is right linear, so whenever a family of elements $a_y \in \mathfrak{R}$ converges to zero, so does the family of elements $a_y b_y$, for *any* family of elements $b_y \in \mathfrak{R}$. For a similar reason, the matrix ab is row-zero-convergent whenever the matrices a and b are. The multiplication in $\mathfrak{Mat}_Y(\mathfrak{R})$ is associative, because for any three matrices $a, b, c \in \mathfrak{Mat}_Y(\mathfrak{R})$ and any two fixed indices x and $w \in Y$, the whole $(Y \times Y)$ -indexed family of triple products $(a_{x,y} b_{y,z} c_{z,w})_{y,z \in Y}$ converges to zero in the topology of \mathfrak{R} .

Let us define a topology on $\mathfrak{Mat}_Y(\mathfrak{R})$. For any finite subset $X \subset Y$ and any open right ideal $\mathfrak{J} \subset \mathfrak{R}$, denote by $\mathfrak{K}_{X,\mathfrak{J}} \subset \mathfrak{Mat}_Y(\mathfrak{R})$ the subgroup consisting of all matrices $a = (a_{x,y})_{x,y \in Y}$ such that $a_{x,y} \in \mathfrak{J}$ for all $x \in X \subset Y$ and $y \in Y$.

Lemma 5.1. *The collection of all the subgroups $\mathfrak{K}_{X,\mathfrak{J}}$, where X ranges over the finite subsets in Y and \mathfrak{J} ranges over the open right ideals in \mathfrak{R} , is a base of neighborhoods of zero in a complete, separated right linear topology on the ring $\mathfrak{Mat}_Y(\mathfrak{R})$.*

Proof: One easily observes that the collection of all subgroups $\mathfrak{K}_{X,\mathfrak{J}} \subset \mathfrak{Mat}_Y(\mathfrak{R})$ is a topology base. The quotient group $\mathfrak{Mat}_Y(\mathfrak{R})/\mathfrak{K}_{X,\mathfrak{J}}$ is the group of all row-finite rectangular $(X \times Y)$ -matrices with the entries in $\mathfrak{R}/\mathfrak{J}$. For a fixed finite subset $X \subset Y$, the projective limit of the groups $\mathfrak{Mat}_Y(\mathfrak{R})/\mathfrak{K}_{X,\mathfrak{J}}$ over all the open right ideals $\mathfrak{J} \subset \mathfrak{R}$ is the group of all row-zero-convergent rectangular $(X \times Y)$ -matrices (with a finite number of rows indexed by the set X) with entries in \mathfrak{R} . Passing to the projective limit over the finite subsets $X \subset Y$, one obtains the whole

group $\text{Mat}_Y(\mathfrak{R})$. So our topology on this group is complete and separated.

It is also easy to observe that $\mathfrak{K}_{X,\mathfrak{J}}$ is a right ideal in the ring $\text{Mat}_Y(\mathfrak{R})$. It remains to check that, for any matrix $a \in \text{Mat}_Y(\mathfrak{R})$, any finite subset $X \subset Y$ and any open right ideal $\mathfrak{J} \subset \mathfrak{R}$, there exists a finite subset $W \subset Y$ and an open right ideal $\mathfrak{J} \subset \mathfrak{R}$ such that $a\mathfrak{K}_{W,\mathfrak{J}} \subset \mathfrak{K}_{X,\mathfrak{J}}$ in $\text{Mat}_Y(\mathfrak{R})$. Indeed, since a is a row-zero-convergent matrix, there is a finite subset $W \subset Y$ such that one has $a_{x,y} \in \mathfrak{J}$ for all $x \in X$ and $y \in Y \setminus W$. Now $(a_{x,w})_{x \in X, w \in W}$ is just a finite matrix, and it remains to choose an open right ideal $\mathfrak{J} \subset \mathfrak{R}$ in such a way that $a_{x,w}\mathfrak{J} \subset \mathfrak{J}$ for all $x \in X$ and $w \in W$. \square

Proposition 5.2. *For any complete, separated topological ring \mathfrak{R} with a right linear topology and any nonempty set Y , the abelian category of discrete right modules over the topological ring $\text{Mat}_Y(\mathfrak{R})$ is equivalent to the abelian category of discrete right \mathfrak{R} -modules.*

Proof: The functor $\mathcal{V}_Y: \text{discr-}\mathfrak{R} \rightarrow \text{discr-}\text{Mat}_Y(\mathfrak{R})$ assigns to a discrete right \mathfrak{R} -module \mathcal{N} the direct sum $\mathcal{V}_Y(\mathcal{N}) = \mathcal{N}^{(Y)}$ of Y copies of \mathcal{N} , viewed as the group of finitely supported Y -sized rows of elements of \mathcal{N} . In other words, an element $m \in \mathcal{V}_Y(\mathcal{N})$ is a Y -indexed family of elements $(m_y \in \mathcal{N})_{y \in Y}$ such that $m_y = 0$ for all but a finite subset of indices $y \in Y$. The ring $\text{Mat}_Y(\mathfrak{R})$ acts on $\mathcal{V}_Y(\mathcal{N})$ by the usual formula for the right action of matrices in rows,

$$(ma)_y = \sum_{x \in Y} m_x a_{xy}, \quad m \in \mathcal{V}_Y(\mathcal{N}), \quad a \in \text{Mat}_Y(\mathfrak{R}).$$

One can easily check that the row ma is finitely supported, using the assumptions that the row m is finitely supported, the matrix a is row-zero-convergent, and the right action of \mathfrak{R} on \mathcal{N} is discrete. The action of \mathcal{V}_Y on morphisms between discrete right \mathfrak{R} -modules is defined in the obvious way. The resulting functor \mathcal{V}_Y is clearly exact and faithful, so it remains to check that it is surjective on morphisms and on the isomorphism classes of objects.

The closed subring of diagonal matrices in $\text{Mat}_Y(\mathfrak{R})$ is isomorphic, as a topological ring, to the product \mathfrak{R}^Y of Y copies of the ring \mathfrak{R} (endowed with the product topology). Let \mathcal{M} be a discrete right $\text{Mat}_Y(\mathfrak{R})$ -module; then \mathcal{M} can also be considered as a discrete right module over the subring $\mathfrak{R}^Y \subset \text{Mat}_Y(\mathfrak{R})$. Then the description of discrete modules over a product of topological rings [45, Lemma 8.1(a)] shows that \mathcal{M} decomposes naturally into a direct sum of discrete right \mathfrak{R} -modules $\mathcal{M} = \bigoplus_{y \in Y} \mathcal{N}_y$, with the componentwise action of the diagonal matrices from \mathfrak{R}^Y in $\bigoplus_{y \in Y} \mathcal{N}_y$.

Let $e_{x,y} \in \mathfrak{Mat}_Y(\mathfrak{R})$ denote the elementary matrix with the entry 1 in the position $(x, y) \in Y \times Y$ and the entry 0 in all other positions. Then the action of the elements $e_{x,y}$ and $e_{y,x}$ on the discrete right $\mathfrak{Mat}_Y(\mathfrak{R})$ -module \mathcal{M} provides an isomorphism of discrete right \mathfrak{R} -modules $\mathcal{N}_x \cong \mathcal{N}_y$. This makes all the discrete right \mathfrak{R} -modules \mathcal{N}_y , $y \in Y$, naturally isomorphic to each other, so we can set $\mathcal{N} = \mathcal{N}_y$. This provides an inverse functor $\text{discr-}\mathfrak{Mat}_Y(\mathfrak{R}) \longrightarrow \text{discr-}\mathfrak{R}$.

Now we have a natural isomorphism of abelian groups $\mathcal{M} \cong \mathcal{V}_Y(\mathcal{N})$ which agrees with the action of both the diagonal matrices from \mathfrak{R}^Y and the elementary matrices $e_{x,y}$ on these two discrete right $\mathfrak{Mat}_Y(\mathfrak{R})$ -modules. Since the subring generated by these two kinds of matrices in $\mathfrak{Mat}_Y(\mathfrak{R})$ contains all the finitely supported matrices and is, therefore, dense in the topology of $\mathfrak{Mat}_Y(\mathfrak{R})$, it follows that we have a natural isomorphism of discrete right $\mathfrak{Mat}_Y(\mathfrak{R})$ -modules $\mathcal{M} \cong \mathcal{V}_Y(\mathcal{N})$. This proves surjectivity of the functor \mathcal{V}_Y on objects; and its surjectivity on morphisms is also clear from the above arguments. \square

Proposition 5.3. *For any complete, separated topological ring \mathfrak{R} with a right linear topology and any nonempty set Y , the abelian category of left contra-modules over the topological ring $\mathfrak{Mat}_Y(\mathfrak{R})$ is equivalent to the abelian category of left \mathfrak{R} -contra-modules.*

Proof: The functor $\mathfrak{V}_Y: \mathfrak{R}\text{-contra} \longrightarrow \mathfrak{Mat}_Y(\mathfrak{R})\text{-contra}$ assigns to a left \mathfrak{R} -contra-module \mathfrak{C} the product $\mathfrak{V}_Y(\mathfrak{C}) = \mathfrak{C}^Y$ of Y copies of \mathfrak{C} , viewed as the group of all Y -sized columns of elements of \mathfrak{C} . So the elements $d \in \mathfrak{V}_Y(\mathfrak{C})$ are described as Y -indexed families of elements $(d_y \in \mathfrak{C})_{y \in Y}$. The left contraaction of the ring $\mathfrak{Mat}_Y(\mathfrak{R})$ on the set $\mathfrak{V}_Y(\mathfrak{C})$ is defined by a “contra” (infinite summation) version of the usual formula for the left action of matrices on columns,

$$\pi_{\mathfrak{V}_Y(\mathfrak{C})} \left(\sum_{d \in \mathfrak{V}_Y(\mathfrak{C})} a_d d \right)_x = \pi_{\mathfrak{C}} \left(\sum_{d \in \mathfrak{V}_Y(\mathfrak{C}), y \in Y} a_{d,x,y} d_y \right).$$

Here $a_d = (a_{d,x,y} \in \mathfrak{R})_{x,y \in Y}$ is an element of the ring $\mathfrak{Mat}_Y(\mathfrak{R})$, defined for every $d \in \mathfrak{V}_Y(\mathfrak{C})$; the family of elements $(a_d)_{d \in \mathfrak{V}_Y(\mathfrak{C})}$ converges to zero in the topology of $\mathfrak{Mat}_Y(\mathfrak{R})$. The expression in parentheses on the left-hand side is an element of the set $\mathfrak{Mat}_Y(\mathfrak{R})[[\mathfrak{V}_Y(\mathfrak{C})]]$ of infinite formal linear combinations of elements of $\mathfrak{V}_Y(\mathfrak{C})$ with zero-convergent families of coefficients in $\mathfrak{Mat}_Y(\mathfrak{R})$. The whole left-hand side is the x -indexed component of the element of $\mathfrak{V}_Y(\mathfrak{C})$ that we want to obtain by applying the left $\mathfrak{Mat}_Y(\mathfrak{R})$ -contraaction map to a given element of $\mathfrak{Mat}_Y(\mathfrak{R})[[\mathfrak{V}_Y(\mathfrak{C})]]$.

The expression in parentheses on the right-hand side is an element of the set $\mathfrak{R}[[\mathfrak{C}]]$. The sum on the right-hand side is understood as

the limit of finite partial sums in the projective limit topology of the group $\mathfrak{R}[[\mathfrak{C}]] = \varprojlim_{\mathfrak{J} \subset \mathfrak{R}} (\mathfrak{R}/\mathfrak{J})[\mathfrak{C}]$ (where \mathfrak{J} ranges over the open right ideals in \mathfrak{R}). To check that this sum converges in $\mathfrak{R}[[\mathfrak{C}]]$, it suffices to observe that, for any fixed index x , the double-indexed family of elements $(a_{d,x,y} \in \mathfrak{R})_{d \in \mathfrak{V}_Y(\mathfrak{C}), y \in Y}$ converges to zero in \mathfrak{R} . The latter observation follows from the definition of the topology on $\mathfrak{Mat}_Y(\mathfrak{R})$.

Checking the contraunitality of this left contraaction of $\mathfrak{Mat}_Y(\mathfrak{R})$ in $\mathfrak{V}_Y(\mathfrak{C})$ is easy; and the contraassociativity follows from the contraassociativity of the left \mathfrak{R} -contraaction in \mathfrak{C} , essentially, for the same reason as the matrix multiplication is generally associative. To simplify the task of checking the details, one can use the notation of [41, Section 1.2] and [44, Section 2.1] for the contraaction operation and the contraassociativity axiom.

As in the previous proof, the action of \mathfrak{V}_Y on morphisms of left \mathfrak{R} -contramodules is defined in the obvious way, and the resulting functor $\mathfrak{V}_Y: \mathfrak{R}\text{-contra} \rightarrow \mathfrak{Mat}_Y(\mathfrak{R})\text{-contra}$ is clearly exact and faithful. So it remains to check that it is surjective on morphisms and on the isomorphism classes of objects.

Let \mathfrak{D} be a left $\mathfrak{Mat}_Y(\mathfrak{R})$ -contramodule. Restricting the $\mathfrak{Mat}_Y(\mathfrak{R})$ -contraaction in \mathfrak{D} to the closed subring of diagonal matrices $\mathfrak{R}^Y \subset \mathfrak{Mat}_Y(\mathfrak{R})$ and using the description of contramodules over a product of topological rings given in [45, Lemma 8.1(b)], we obtain a functorial decomposition of \mathfrak{D} into a direct product of left \mathfrak{R} -contramodules $\mathfrak{D} = \prod_{y \in Y} \mathfrak{C}_y$, with the componentwise contraaction of the diagonal matrices from \mathfrak{R}^Y in $\prod_{y \in Y} \mathfrak{C}_y$. As in the previous proof, the action of the elementary matrices $e_{x,y} \in \mathfrak{Mat}_Y(\mathfrak{R})$ provides natural isomorphisms of left \mathfrak{R} -contramodules $\mathfrak{C}_x \cong \mathfrak{C}_y$. So we can set $\mathfrak{C} = \mathfrak{C}_y$; this defines an exact inverse functor $\mathfrak{Mat}_Y(\mathfrak{R})\text{-contra} \rightarrow \mathfrak{R}\text{-contra}$.

Now we have a natural isomorphism of abelian groups $\mathfrak{D} = \mathfrak{V}_Y(\mathfrak{C})$, and it essentially remains to show that this is an isomorphism of $\mathfrak{Mat}_Y(\mathfrak{R})$ -contramodules. For this purpose, we will demonstrate that a left $\mathfrak{Mat}_Y(\mathfrak{R})$ -contramodule structure on any contramodule \mathfrak{D} can be expressed in terms of the contraaction of the diagonal subring \mathfrak{R}^Y and the action of the elementary matrices $e_{x,y}$.

Indeed, let $a_d = (a_{d,x,y} \in \mathfrak{R})_{x,y \in Y} \in \mathfrak{Mat}_Y(\mathfrak{R})$, $d \in \mathfrak{D}$ be a \mathfrak{D} -indexed family of elements converging to zero in $\mathfrak{Mat}_Y(\mathfrak{R})$. Put $\mathfrak{S} = \mathfrak{Mat}_Y(\mathfrak{R})$ for brevity. For every element $d \in \mathfrak{D}$ and a pair of indices $x, y \in Y$, consider the element $e_{x,y}d \in \mathfrak{S}[[\mathfrak{D}]]$. This is a finite formal linear combination of elements of the set \mathfrak{D} with exactly one nonzero coefficient in \mathfrak{S} .

Furthermore, for every fixed index $x \in Y$, consider the infinite formal linear combination of finite formal linear combinations $\sum_{d,y} a_{d,x,y}(e_{x,y}d)$. Here the coefficients are the matrix entries $a_{d,x,y} \in \mathfrak{R}$ viewed as the scalar (hence diagonal) matrices $a_{d,x,y} \in \mathfrak{R} \subset \mathfrak{Mat}_Y(\mathfrak{R})$. The $(\mathfrak{D} \times Y)$ -indexed family of elements $(a_{d,x,y})_{d \in \mathfrak{D}, y \in Y}$ converges to zero in \mathfrak{R} , because the \mathfrak{D} -indexed family of matrices $(a_d)_{d \in \mathfrak{D}}$ consists of row-zero-convergent matrices and converges to zero in $\mathfrak{Mat}_Y(\mathfrak{R})$. As the embedding of the subring of scalar matrices $\mathfrak{R} \rightarrow \mathfrak{Mat}_Y(\mathfrak{R})$ is continuous, the $(\mathfrak{D} \times Y)$ -indexed family of elements $(a_{d,x,y})_{d,y}$ converges to zero in $\mathfrak{S} = \mathfrak{Mat}_Y(\mathfrak{R})$ as well. So we have $\sum_{d,y} a_{d,x,y}(e_{x,y}d) \in \mathfrak{S}[[\mathfrak{S}[[\mathfrak{D}}]]]$ for every $x \in Y$.

Finally, we consider the element

$$(1) \quad \sigma = \sum_{x \in Y} e_{x,x} \left(\sum_{d \in \mathfrak{D}, y \in Y} a_{d,x,y}(e_{x,y}d) \right) \in \mathfrak{S}[[\mathfrak{S}[[\mathfrak{S}[[\mathfrak{D}}]]]]].$$

This is an infinite formal linear combination of elements of the set $\mathfrak{S}[[\mathfrak{S}[[\mathfrak{D}}]]]$ with the coefficients $(e_{x,x} \in \mathfrak{S})_{x \in Y}$, which form a Y -indexed family of elements converging to zero in the topology of $\mathfrak{S} = \mathfrak{Mat}_Y(\mathfrak{R})$. In fact, the elements $e_{x,x}$ belong to the closed subring of diagonal matrices $\mathfrak{R}^Y \subset \mathfrak{Mat}_Y(\mathfrak{R})$.

Now the (iterated) contraassociativity axiom tells us that all the compositions of “opening of parentheses” (monad multiplication) and contraaction maps acting from $\mathfrak{S}[[\mathfrak{S}[[\mathfrak{S}[[\mathfrak{D}}]]]]]$ into \mathfrak{D} are equal to each other. In particular, for any set Z there is the squared monad multiplication map

$$\phi_Z^{(2)} = \phi_Z \circ \mathfrak{S}[[\phi_Z]] = \phi_Z \circ \phi_{\mathfrak{S}[[Z]]} : \mathfrak{S}[[\mathfrak{S}[[\mathfrak{S}[[Z]]]]]] \longrightarrow \mathfrak{S}[[Z]].$$

Setting $Z = \mathfrak{D}$ and applying this map to the element σ , we obtain

$$(2) \quad \phi_{\mathfrak{D}}^{(2)}(\sigma) = \sum_{d \in \mathfrak{D}} \sum_{x,y \in Y} (e_{x,x} a_{d,x,y} e_{x,y}) d = \sum_{d \in \mathfrak{D}} a_d d,$$

since $a_d = \sum_{x,y \in Y} a_{d,x,y} e_{x,y} = \sum_{x,y \in Y} e_{x,x} a_{d,x,y} e_{x,y}$ as the limit of finite partial sums converging in the topology of $\mathfrak{Mat}_Y(\mathfrak{R})$.

On the other hand, for any set Z endowed with an (arbitrary) map of sets $\pi_Z : \mathfrak{S}[[Z]] \rightarrow Z$, there is the iterated map

$$\pi_Z^{(3)} = \pi_Z \circ \mathfrak{S}[[\pi_Z]] \circ \mathfrak{S}[[\mathfrak{S}[[\pi_Z]]]] : \mathfrak{S}[[\mathfrak{S}[[\mathfrak{S}[[Z]]]]]] \longrightarrow Z.$$

In the situation at hand with $Z = \mathfrak{D}$, we see from (1) that the value of $\pi_{\mathfrak{D}}^{(3)}(\sigma) \in \mathfrak{D}$ is uniquely determined by the action of the elements $e_{x,y}$ and the contraaction of the diagonal subring $\mathfrak{R}^Y \subset \mathfrak{Mat}_Y(\mathfrak{R})$ in \mathfrak{D} . The contraassociativity equation

$$\pi_{\mathfrak{D}}(\phi_{\mathfrak{D}}^{(2)}(\sigma)) = \pi_{\mathfrak{D}}^{(3)}(\sigma)$$

together with the equality (2) tells us that the whole left contraaction of $\mathfrak{Mat}_Y(\mathfrak{R})$ in \mathfrak{D} is determined by (and expressed explicitly by the above formulas in terms of) these data, concluding the proof. \square

Lemma 5.4. *In the category equivalence $\mathfrak{Mat}_Y(\mathfrak{R})\text{-contra} \cong \mathfrak{R}\text{-contra}$ of Proposition 5.3, the free left contra module with one generator $\mathfrak{Mat}_Y(\mathfrak{R})$ over the topological ring $\mathfrak{Mat}_Y(\mathfrak{R})$ corresponds to the free left contra module $\mathfrak{R}[[Y]]$ with the set of generators Y over the topological ring \mathfrak{R} .*

Proof: Following the proof of Proposition 5.3 and [45, Lemma 8.1(b)], the left \mathfrak{R} -contra module \mathfrak{C} corresponding to a left $\mathfrak{Mat}_Y(\mathfrak{R})$ -contra module \mathfrak{D} can be computed as the \mathfrak{R} -subcontra module $\mathfrak{C} = e_{x,x}\mathfrak{D}$ in \mathfrak{D} , where x is any chosen element of the set Y . Hence, in the particular case of the free left $\mathfrak{Mat}_Y(\mathfrak{R})$ -contra module $\mathfrak{D} = \mathfrak{Mat}_Y(\mathfrak{R})$, the left \mathfrak{R} -contra module \mathfrak{C} can be described as the set of all zero-convergent Y -sized rows (or rectangular $(\{x\} \times Y)$ -matrices with one row) with entries in \mathfrak{R} , which is isomorphic to $\mathfrak{R}[[Y]]$ as an \mathfrak{R} -contra module. \square

Remark 5.5. The above proof of Proposition 5.3 has the advantage of being direct and explicit, but it is quite involved. There is an alternative indirect argument based on the result of Corollary 4.4.

Let A be an associative ring and M be a left A -module such that the topological ring \mathfrak{R} is isomorphic to the topological ring of endomorphism $\text{Hom}_A(M, M)^{\text{op}}$ of the A -module M . Then the topological ring $\mathfrak{Mat}_Y(\mathfrak{R})$ (with the above-defined topology on it) is isomorphic to the topological ring of endomorphisms $\text{Hom}_A(M^{(Y)}, M^{(Y)})^{\text{op}}$ of the direct sum $M^{(Y)}$ of Y copies of the A -module M . By Theorem 3.14(iii), we have

$$\mathfrak{R}\text{-contra}_{\text{proj}} \cong \text{Add}(M) = \text{Add}(M^{(Y)}) \cong \mathfrak{Mat}_Y(\mathfrak{R})\text{-contra}_{\text{proj}}.$$

An abelian category with enough projective objects is determined by its full subcategory of projective objects, so the equivalence of additive categories $\mathfrak{R}\text{-contra}_{\text{proj}} \cong \mathfrak{Mat}_Y(\mathfrak{R})\text{-contra}_{\text{proj}}$ extends uniquely to an equivalence of abelian categories $\mathfrak{R}\text{-contra} \cong \mathfrak{Mat}_Y(\mathfrak{R})\text{-contra}$. This is, of course, the same equivalence of categories $\mathfrak{R}\text{-contra} \cong \mathfrak{Mat}_Y(\mathfrak{R})\text{-contra}$ as the one provided by the constructions in the proof of Proposition 5.3.

Yet another proof of Proposition 5.3 can be found in [50, Theorem 7.9 and Example 7.10]. The direct approach worked out above in this section also has another advantage, though: on par with the equivalence of the contra module categories, it allows us to obtain an equivalence of the categories of discrete modules in Proposition 5.2.

6. Topologically semisimple topological rings

The concept of a topologically semisimple right linear topological ring is based on Theorem 6.2, which we prove in this section. An important related result is Theorem 6.6, whose proof we postpone to Section 11.

Given an additive category \mathbf{A} with set-indexed coproducts, we denote by $\text{Add}_\infty(\mathbf{A}, \mathbf{Ab})$ the full subcategory in $\text{Funct}(\mathbf{A}, \mathbf{Ab})$ consisting of all the functors $\mathbf{A} \rightarrow \mathbf{Ab}$ preserving all coproducts. If $M \in \mathbf{A}$ is an object, we also denote by $\{M\} = \{M\}_{\mathbf{A}}$ the full subcategory in \mathbf{A} spanned by the single object M . When $\mathbf{A} = \text{Add}(M)$, the restriction functor $\text{Add}_\infty(\mathbf{A}, \mathbf{Ab}) \rightarrow \text{Funct}(\{M\}, \mathbf{Ab})$ taking a functor $F: \mathbf{A} \rightarrow \mathbf{Ab}$ to the functor $F|_{\{M\}}: \{M\} \rightarrow \mathbf{Ab}$ is faithful. Hence morphisms between any two fixed objects in $\text{Add}_\infty(\mathbf{A}, \mathbf{Ab})$ form a set.

Similarly, given a cocomplete additive category \mathbf{A} , we denote by $\text{Rex}_\infty(\mathbf{A}, \mathbf{Ab})$ the full subcategory in $\text{Funct}(\mathbf{A}, \mathbf{Ab})$ consisting of all the functors $\mathbf{A} \rightarrow \mathbf{Ab}$ preserving all colimits (or equivalently, all coproducts and cokernels). If an object $G \in \mathbf{A}$ is a generator, then the restriction functor $\text{Rex}_\infty(\mathbf{A}, \mathbf{Ab}) \rightarrow \text{Funct}(\{G\}, \mathbf{Ab})$ is faithful. So morphisms between any two fixed objects in $\text{Rex}_\infty(\mathbf{A}, \mathbf{Ab})$ form a set.

Lemma 6.1. *Let \mathfrak{R} be a complete, separated topological ring with a right linear topology. Then the functor*

$$\Theta: \text{discr-}\mathfrak{R} \longrightarrow \text{Rex}_\infty(\mathfrak{R}\text{-contra}, \mathbf{Ab})$$

induced by the pairing functor of contratensor product

$$\odot_{\mathfrak{R}}: \text{discr-}\mathfrak{R} \times \mathfrak{R}\text{-contra} \longrightarrow \mathbf{Ab}$$

is fully faithful.

Proof: The free left \mathfrak{R} -contramodule \mathfrak{R} with one generator is a generator of $\mathfrak{R}\text{-contra}$, so morphisms between any two fixed objects in $\text{Rex}_\infty(\mathfrak{R}\text{-contra}, \mathbf{Ab})$ form a set. The functor $\odot_{\mathfrak{R}}$ preserves colimits (in both its arguments), so the functor Θ indeed takes values in $\text{Rex}_\infty(\mathfrak{R}\text{-contra}, \mathbf{Ab})$. Furthermore, let us consider the composition of Θ with the restriction functor $\rho: \text{Rex}_\infty(\mathfrak{R}\text{-contra}, \mathbf{Ab}) \rightarrow \text{Funct}(\{\mathfrak{R}\}, \mathbf{Ab})$

$$\text{discr-}\mathfrak{R} \xrightarrow{\Theta} \text{Rex}_\infty(\mathfrak{R}\text{-contra}, \mathbf{Ab}) \xrightarrow{\rho} \text{Funct}(\{\mathfrak{R}\}, \mathbf{Ab}).$$

The category $\text{Funct}(\{\mathfrak{R}\}, \mathbf{Ab}) \cong (\text{Hom}_{\mathfrak{R}\text{-contra}}(\mathfrak{R}, \mathfrak{R}))\text{-mod} \cong \text{mod-}\mathfrak{R}$ is equivalent to the category of right \mathfrak{R} -modules, and the functor $\rho \circ \Theta$ is isomorphic to the fully faithful inclusion functor $\text{discr-}\mathfrak{R} \rightarrow \text{mod-}\mathfrak{R}$ (due to the natural isomorphism of abelian groups $\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{R} \cong \mathcal{N}$ for any discrete right \mathfrak{R} -module \mathcal{N}). According to the discussion preceding the lemma, the functor ρ is faithful. Since ρ is faithful and $\rho \circ \Theta$ is fully faithful, it follows that Θ is fully faithful. □

Theorem 6.2. *Let \mathfrak{S} be a complete, separated topological ring with a right linear topology. Then the following conditions are equivalent:*

- (i) *the abelian category \mathfrak{S} -contra is Ab5 and semisimple;*
- (ii) *the abelian category $\text{discr-}\mathfrak{S}$ is split (or equivalently, semisimple);*
- (iii) *there exists an associative ring A and a semisimple left A -module M such that \mathfrak{S} is isomorphic, as a topological ring, to the endomorphism ring of M endowed with the finite topology, $\mathfrak{S} \cong \text{End}_A(M)^{\text{op}} = \text{Hom}_A(M, M)^{\text{op}}$;*
- (iv) *there is a set X , an X -indexed family of nonempty sets Y_x , and an X -indexed family of division rings D_x , $x \in X$, such that the topological ring \mathfrak{S} is isomorphic to the product of the endomorphism rings of Y_x -dimensional vector spaces over D_x ,*

$$\mathfrak{S} \cong \prod_{x \in X} \text{End}_{D_x}(D_x^{(Y_x)})^{\text{op}}.$$

Here $\text{End}_{D_x}(D_x^{(Y_x)})^{\text{op}} = \mathfrak{Mat}_{Y_x}(D_x)$ is, generally speaking, the ring of row-finite infinite matrices of size Y_x with entries in D_x . It is endowed with the finite topology of the endomorphism ring of the D_x -module $D_x^{(Y_x)}$ (see Example 3.7(1)), which coincides with the topology of the ring of matrices with entries in the discrete ring D_x (as defined in Section 5). The ring \mathfrak{S} is isomorphic, as a topological ring, to a (generally speaking) infinite product of such rings of infinite matrices, endowed with the product topology. Topological rings satisfying the equivalent conditions of Theorem 6.2 are called *topologically semisimple*.

Remark 6.3. Associative rings of the form described above appear in the theory of direct sum decompositions of modules [31, 2, 3], where the authors usually seem to say that such a ring \mathfrak{S} is von Neumann regular (which it is – but it is a very special kind of von Neumann regular ring). Certainly, \mathfrak{S} is not classically semisimple as an abstract ring, generally speaking; it is not Artinian, and the categories $\mathfrak{S}\text{-mod}$ and $\text{mod-}\mathfrak{S}$ of left and right modules over it are not semisimple. But as a topological ring, \mathfrak{S} is topologically semisimple in the sense of the above theorem. (See Remark 10.5 below for further discussion.)

Remark 6.4. It is instructive to consider the simple objects of the semisimple abelian categories $\text{discr-}\mathfrak{S}$ and $\mathfrak{S}\text{-contra}$. There is only one simple discrete right module over the topological ring $\mathfrak{S}_x = \text{End}_{D_x}(D_x^{(Y_x)})^{\text{op}}$, namely, the Y_x -dimensional vector space $D_x^{(Y_x)}$. There is also only one simple left \mathfrak{S}_x -contramodule, namely, the product $D_x^{Y_x}$ of Y_x copies of D_x . The discrete module and contramodule structures on these objects were explicitly described in the proofs of Propositions 5.2 and 5.3.

For the ring $\mathfrak{S} = \prod_{x \in X} \mathfrak{S}_x$, both the simple discrete right modules and the simple left contra-modules are indexed by the set X (see [45, Lemma 8.1]).

Remark 6.5. The same class of topological rings (up to switching the roles of the left and right sides) as in Theorem 6.2 was characterized by a list of many equivalent conditions in the paper [33, Theorem 3.10], with the proof of the equivalence based on a preceding result in the book [55, Theorem 29.7]. In particular, our condition (iv) of Theorem 6.2 is the same as condition (d) of [33, Theorem 3.10].

Proof of Theorem 6.2: By Remark 2.4, the abelian category $\text{discr-}\mathfrak{S}$ is split if and only if it is semisimple. We will prove the implications

$$(i) \implies (ii) \implies (iv) \implies (iii) \implies (i).$$

(i) \implies (ii) The argument is based on Lemma 6.1. By Theorem 2.3(iv), the category $\mathfrak{S}\text{-contra}$ being Ab5 and semisimple means a category equivalence $\mathfrak{S}\text{-contra} \cong \times_{x \in X} D_x\text{-mod}$ for some set of indices X and a family of skew-fields $(D_x)_{x \in X}$. The category $\mathfrak{S}\text{-contra}$ is split abelian, so the two full subcategories $\text{Rex}_\infty(\mathfrak{S}\text{-contra}, \text{Ab})$ and $\text{Add}_\infty(\mathfrak{S}\text{-contra}, \text{Ab})$ in $\text{Funct}(\mathfrak{S}\text{-contra}, \text{Ab})$ coincide.

Furthermore, for any functor $N: \times_{x \in X} D_x\text{-mod} \longrightarrow \text{Ab}$ which is co-product-preserving, the image of the one-dimensional left vector space D_x over D_x is naturally a left module over the ring $\text{Hom}_{D_x\text{-mod}}(D_x, D_x) = D_x^{\text{op}}$, i.e., a right D_x -vector space. The functor N is uniquely determined by the collection of right D_x -vector spaces $(N(D_x) \in \text{mod-}D_x)_{x \in X}$, which can be arbitrary. So the assignment $N \mapsto (N(D_x))_{x \in X}$ establishes a category equivalence

$$\text{Add}_\infty\left(\times_{x \in X} D_x\text{-mod}, \text{Ab}\right) \cong \times_{x \in X} \text{mod-}D_x.$$

By Lemma 6.1, it follows that $\text{discr-}\mathfrak{S}$ is a full subcategory in a semisimple abelian category $\times_{x \in X} \text{mod-}D_x$. It remains to observe that any abelian category which can be embedded as a full subcategory into a split abelian category is split.

(ii) \implies (iv) The argument is based on Proposition 4.2. By Theorem 2.3(iv), we have $\text{discr-}\mathfrak{S} \cong \times_{x \in X} \text{mod-}D_x$ for some set of indices X and a family of skew-fields $(D_x)_{x \in X}$. The forgetful functor $F: \text{discr-}\mathfrak{S} \longrightarrow \text{Ab}$ (assigning to every discrete right \mathfrak{S} -module \mathcal{N} its underlying abelian group \mathcal{N}) can thus be interpreted as a functor

$$F: \times_{x \in X} \text{mod-}D_x \longrightarrow \text{Ab}.$$

We know that the functor F is faithful and preserves colimits/coproducts.

As above, the image of the one-dimensional right vector space D_x over D_x under the functor F is naturally a left module over the ring $\text{Hom}_{\text{mod-}D_x}(D_x, D_x) = D_x$. Denote this left D_x -module by $V_x = F(D_x)$, and let Y_x be a set such that $V_x \cong D_x^{(Y_x)}$ in $D_x\text{-mod}$. Since the functor F is faithful, the set Y_x is nonempty.

Now an endomorphism $t: F \rightarrow F$ of the functor F is uniquely determined by the collection of left D_x -module morphisms $t_{D_x}: V_x \rightarrow V_x$, which can be arbitrary. In view of Proposition 4.2, this provides the desired ring isomorphism $\mathfrak{S} \cong \prod_{x \in X} \text{End}_{D_x}(D_x^{(Y_x)})^{\text{op}}$. Finally, the topology on the ring \mathfrak{S} is also described by Proposition 4.2, which allows us to identify it with the product of the finite topologies on the rings of row-finite matrices.

(iv) \implies (iii) Set $A = \prod_{x \in X} D_x$ and $M = \bigoplus_{x \in X} D_x^{(Y_x)}$.

(iii) \implies (i) The argument is based on the generalized tilting theory. By Theorem 3.14(i)(iii), we have a category equivalence $\text{Add}(M) \cong \mathfrak{S}\text{-contra}_{\text{proj}}$. Since M is a semisimple A -module, the category $\text{Add}(M)$ is abelian, Grothendieck, and semisimple.

An abelian category \mathcal{B} with enough projective objects is uniquely determined by its full subcategory of projective objects $\mathcal{B}_{\text{proj}}$. In particular, if $\mathcal{B}_{\text{proj}}$ happens to be split abelian, then all objects of \mathcal{B} are projective.

In the situation at hand, we conclude that the category $\mathfrak{S}\text{-contra} = \mathfrak{S}\text{-contra}_{\text{proj}} \cong \text{Add}(M)$ is Ab5 and semisimple.

Alternatively, the implication (iii) \implies (iv) is easy to prove, and (iv) \implies (i) holds by Proposition 5.3 and [45, Lemma 8.1(b)] (while (iv) \implies (ii) follows directly from Proposition 5.2 and [45, Lemma 8.1(a)]).

Notice also that it is clear from condition (iv) that the functor $\Theta: \text{discr-}\mathfrak{S} \rightarrow \text{Rex}_{\infty}(\mathfrak{S}\text{-contra}, \text{Ab})$ from Lemma 6.1 (which was used in the proof of the implication (i) \implies (ii)) is actually an equivalence of categories for a topologically semisimple topological ring \mathfrak{S} . \square

Our next theorem shows that, similarly to Theorem 6.2(ii), the semi-simplicity condition in Theorem 6.2(i) can be relaxed to the splitness condition.

Theorem 6.6. *Let \mathfrak{S} be a complete, separated topological ring with a right linear topology. Then the abelian category $\mathfrak{S}\text{-contra}$ is split if and only if it is Grothendieck and semisimple.*

Remarks 6.7. Notice that the abelian category of \mathfrak{S} -contramodules is rarely Ab5 or Grothendieck, generally speaking. Theorem 6.6 says that if it is split, then it is both Grothendieck and semisimple.

It would be interesting to know an example of a cocomplete split abelian category with a generator which is not Grothendieck (i.e., does not satisfy Ab5). We are not aware of any such examples.

Theorem 6.6 is essentially the same result as the above Theorem 3.15. Its proof is based on known results in the theory of direct sum decompositions of modules [3]. We present it below in Section 11.

7. Topologically left T-nilpotent subsets

This section contains a technical lemma which will be useful in Section 10.

Let R be a separated topological ring with a right linear topology. A subset $E \subset R$ is said to be *topologically nil* if for every element $a \in E$ the sequence of elements a^n , $n = 1, 2, \dots$, converges to zero in the topology of R as $n \rightarrow \infty$. A subset $E \subset R$ is *topologically left T-nilpotent* if for every sequence of elements $a_1, a_2, a_3, \dots \in E$ the sequence of elements $a_1, a_1a_2, a_1a_2a_3, \dots, a_1a_2 \cdots a_n, \dots \in R$ converges to zero as $n \rightarrow \infty$. In other words, this means that for every open right ideal $I \subset R$ there exists $n \geq 1$ such that $a_1a_2 \cdots a_n \in I$ (cf. [45, Section 6]).

Lemma 7.1. *Let \mathfrak{R} be a complete, separated topological ring with a right linear topology and $E \subset \mathfrak{R}$ be a topologically left T-nilpotent subset. Denote by \mathfrak{E} the topological closure of the subring without unit generated by E in \mathfrak{R} . Then \mathfrak{E} is also a topologically left T-nilpotent subset in \mathfrak{R} .*

Proof: Let E' denote the multiplicative subsemigroup (without unit) generated by $E \cup -E$ in \mathfrak{R} . Clearly, if E is topologically left T-nilpotent, then so is E' .

Let E'' denote the additive subgroup generated by E' in \mathfrak{R} . Our next aim is to show that E'' is topologically left T-nilpotent in \mathfrak{R} .

Indeed, let $(b_n)_{n \geq 1}$ be a sequence of elements in E'' . Then $b_n = a_{n,1} + \cdots + a_{n,m_n}$, where $a_{n,j} \in E'$ and m_n are some nonnegative integers. We need to show that the sequence of products $b_1b_2 \cdots b_n$ converges to zero in \mathfrak{R} .

Consider the following rooted tree A . The root vertex (that is, the only vertex of depth 0) has m_1 children, marked by the elements $a_{1,1}, \dots, a_{1,m_1} \in \mathfrak{R}$.

These are the vertices of depth 1. Each of them has m_2 children. The children of the vertex of depth 1 marked by the element a_{1,j_1} are marked by the elements $a_{1,j_1}a_{2,1}, \dots, a_{1,j_1}a_{2,m_2} \in \mathfrak{R}$.

Generally, every vertex of depth $n - 1$ has m_n children. If a vertex of depth $n - 1$ is marked by an element $r \in \mathfrak{R}$, then its children are marked by the elements $ra_{n,1}, \dots, ra_{n,m_n} \in \mathfrak{R}$. So every vertex of depth $n \geq 1$ is marked by a product of the form $a_{1,j_1}a_{2,j_2} \cdots a_{n,j_n} \in \mathfrak{R}$, where $1 \leq j_i \leq m_i$.

Now let $\mathfrak{J} \subset \mathfrak{R}$ be an open right ideal. Notice that if a vertex v in our tree A is marked by an element a_v belonging to \mathfrak{J} , then all the vertices below v in A are also marked by elements belonging to \mathfrak{J} . So we can consider the reduced tree $A_{\mathfrak{J}}$ obtained by deleting from A all the vertices v for which $a_v \in \mathfrak{J}$.

Since the subset $E' \subset \mathfrak{R}$ is topologically left T-nilpotent, every branch of the tree A eventually encounters a vertex marked by an element from \mathfrak{J} . Hence the reduced tree $A_{\mathfrak{J}}$ has no infinite branches. It is also locally finite by construction (i.e., every vertex has only a finite number of children). By the Kőnig lemma, it follows that the whole reduced tree $A_{\mathfrak{J}}$ is finite.

Thus there exists an integer $n \geq 1$ such that $A_{\mathfrak{J}}$ has no vertices of depth greater than $n - 1$. Then the product $b_1 b_2 \cdots b_n$ belongs to the ideal \mathfrak{J} .

The subset $E'' \subset \mathfrak{R}$ is exactly the subring without unit generated by E in \mathfrak{R} . We have shown that E'' is topologically left T-nilpotent. It remains to check that the topological closure \mathfrak{E} of E'' in \mathfrak{R} is also topologically left T-nilpotent.

Let $(c_n \in \mathfrak{E})_{n \geq 1}$ be a sequence of elements and $\mathfrak{J} \subset \mathfrak{R}$ be an open right ideal. Since E'' is dense in \mathfrak{E} , there exists an element $b_1 \in E''$ such that $c_1 - b_1 \in \mathfrak{J}$.

Furthermore, there exists an open right ideal $\mathfrak{J}_1 \subset \mathfrak{R}$ such that $b_1 \mathfrak{J}_1 \subset \mathfrak{J}$. Let $b_2 \in E''$ be an element such that $c_2 - b_2 \in \mathfrak{J}_1$. Then we have $b_1(c_2 - b_2) \in \mathfrak{J}$.

Proceeding by induction, for every $i \geq 2$ we choose an open right ideal $\mathfrak{J}_{i-1} \subset \mathfrak{R}$ such that $b_1 \cdots b_{i-1} \mathfrak{J}_{i-1} \subset \mathfrak{J}$, and an element $b_i \in E''$ such that $c_i - b_i \in \mathfrak{J}_{i-1}$. Then we have $b_1 \cdots b_{i-1}(c_i - b_i) \in \mathfrak{J}$. Now

$$c_1 \cdots c_n - b_1 \cdots b_n = (c_1 - b_1)c_2 \cdots c_n + b_1(c_2 - b_2)c_3 \cdots c_n + \cdots + b_1 \cdots b_{n-2}(c_{n-1} - b_{n-1})c_n + b_1 \cdots b_{n-1}(c_n - b_n) \in \mathfrak{J}$$

for every $n \geq 1$. It remains to choose $n \geq 1$ such that $b_1 b_2 \cdots b_n \in \mathfrak{J}$, and conclude that $c_1 c_2 \cdots c_n \in \mathfrak{J}$. □

Remark 7.2. Topologically T-nilpotent ideals have been considered in the literature in the following context. Let R be a commutative domain. Endow R with the R -topology; so nonzero ideals form a base of neighborhoods of zero in R . An ideal in R was called “topologically T-nilpotent” in [53] if it is topologically T-nilpotent (in the sense of our definition above) in the R -topology. A commutative local domain R was called “topologically T-nilpotent” (TTN) in [53] if its maximal ideal is topologically T-nilpotent. A commutative local domain is topologically

T-nilpotent in this sense if and only if it is *almost perfect* in the sense of the papers [10, 11].

More generally, let R be a prime ring (i.e., an associative ring in which the product of any two nonzero two-sided ideals is nonzero). Then nonzero two-sided ideals form a base of a topology on R , making R a topological ring. For a prime local ring R , the Jacobson radical of R is topologically left T-nilpotent in the described topology if and only if the ring R is *left almost perfect* in the sense of the paper [22].

To give another generalization, let R be a commutative local ring and $S \subset R$ be a multiplicative subset. Endow R with the S -topology. Then the maximal ideal of R is topologically T-nil in the sense of our definition above if and only if the ring R is *S - h -nil* in the sense of [7, Section 6].

8. Lifting orthogonal idempotents

In this section we show that an (infinite, zero-convergent) complete family of orthogonal idempotents can be lifted modulo a topologically nil strongly closed two-sided ideal. In order to do so, we have to fill (what we think is) a gap in the proof of [38, Lemma 8]. The results of this section will be used in Section 10.

First, we recall a lemma from [45] concerning the lifting of a single idempotent. Given a subgroup K in a topological abelian group \mathfrak{A} , we denote by $\overline{K} \subset \mathfrak{A}$ the topological closure of K in \mathfrak{A} . We also recall the notation $H(R)$ for the Jacobson radical of an associative ring R (when \mathfrak{R} is a topological ring, $H(\mathfrak{R})$ denotes the Jacobson radical of the abstract ring \mathfrak{R} , which ignores the topology).

Lemma 8.1. *Let \mathfrak{R} be a complete, separated topological ring with a right linear topology, and let $\mathfrak{H} \subset \mathfrak{R}$ be a topologically nil closed two-sided ideal. Let $f \in \mathfrak{R}$ be an element such that $f^2 - f \in \mathfrak{H}$. Then there exists an element $e \in f + \mathfrak{H} \subset \mathfrak{R}$ such that $e^2 = e$ in \mathfrak{R} and $e \in \overline{f\mathfrak{R}f} \subset \mathfrak{R}$.*

Proof: A particular natural choice of the element e is provided by the construction in the proof of [45, Lemma 10.3]. The first two desired properties of the element e are established in the argument in [45], and the last one is clear from the construction. \square

As pointed out in [13], once every individual idempotent in an orthogonal family has been lifted, orthogonalizing the lifted idempotents becomes a task solvable under weaker assumptions. The next proposition is a restatement of [38, Lemma 8] with the left and right sides switched (as the authors of [38] work with left linear topologies on rings and we prefer the right linear ones).

Proposition 8.2. *Let \mathfrak{R} be a complete, separated topological ring with a right linear topology, Z be a linearly ordered set of indices, and $(e_z \in \mathfrak{R})_{z \in Z}$ be a family of idempotent elements such that $e_w e_z \in H(\mathfrak{R})$ for every pair of indices $z < w$ in Z , the family of elements e_z converges to zero in the topology of \mathfrak{R} , and the element $u = \sum_{z \in Z} e_z$ is invertible in \mathfrak{R} . Then $(u^{-1}e_z)_{z \in Z}$ and $(e_z u^{-1})_{z \in Z}$ are two families of orthogonal idempotents, converging to zero in \mathfrak{R} with $\sum_{z \in Z} u^{-1}e_z = 1 = \sum_{z \in Z} e_z u^{-1}$.*

Note that if in the context of the proposition one has $u = 1$, then it follows that $e_w e_z = 0$ for all $w \neq z$. (Cf. the discussion in [38, Corollary 4].)

The proof of Proposition 8.2 is based on two lemmas. The following one is just [38, Lemma 1] with the left and right sides switched.

Lemma 8.3. *Let R be an associative ring, $e = e^2 \in R$ be an idempotent element, and $a \in R$ be an element such that $ea e \equiv e \pmod{H(R)}$. Then there exists an element $f = f^2 \in aRe$ such that $Re = Rf$. \square*

The next lemma is our (expanded) version of [38, Lemma 2].

Lemma 8.4. *Let R be an associative ring, Z be a finite, linearly ordered set of indices, and $(e_z \in R)_{z \in Z}$ be idempotent elements such that $e_w e_z \in H(R)$ for all pairs of indices $z < w$ in Z . Then*

- (i) *the sum of left ideals $\sum_{z \in Z} Re_z$ is direct, and a direct summand in the left R -module R ;*
- (ii) *for any right R -module N , the sum of subgroups $\sum_{z \in Z} Ne_z$ in N is direct.*

Proof: The assertion of [38, Lemma 2] is essentially our part (i); the difference is that we have added part (ii).

For $|Z| = 2$, the argument in [38] uses the previous lemma in order to find a pair of orthogonal idempotents f_1 and $f_2 \in R$ such that $Re_1 = Rf_1$ and $Re_2 = Rf_2$. It follows that $Ne_1 = Nf_1$ and $Ne_2 = Nf_2$. Then it is clear that $R = Rf_1 \oplus Rf_2 \oplus R(1 - f_1 - f_2)$, and similarly $N = Nf_1 \oplus Nf_2 \oplus N(1 - f_1 - f_2)$.

For $|Z| \geq 3$, the argument in [38] proceeds by induction, using the case $|Z| = 2$ in order to make the induction step. This works for part (ii) in exactly the same way as for part (i).

The nature of the induction step is such that assuming the original idempotents $e_z \in R$ to be only “half-orthogonal modulo $H(R)$ ” (for $z < w$) is more convenient than requiring two-sided orthogonality modulo the Jacobson radical. \square

Proof of Proposition 8.2: By continuity of multiplication in \mathfrak{R} , both the families of elements $(u^{-1}e_z)_{z \in Z}$ and $(e_z u^{-1})_{z \in Z}$ converge to zero, and $\sum_{z \in Z} u^{-1}e_z = 1 = \sum_{z \in Z} e_z u^{-1}$. We have to prove that these are two families of orthogonal idempotents. Here it suffices to check that $e_w u^{-1}e_z = \delta_{z,w}e_z$ for all $z, w \in Z$.

For every fixed $w \in Z$, we have

$$(3) \quad e_w = e_w \left(\sum_{z \in Z} u^{-1}e_z \right) = e_w u^{-1}e_w + \sum_{z \in Z, z \neq w} e_w u^{-1}e_z,$$

where the infinite sum is understood as the limit of finite partial sums in the topology of \mathfrak{R} . Let $\mathfrak{J} \subset \mathfrak{R}$ be an open right ideal. Then there exists an open right ideal $\mathfrak{I} \subset \mathfrak{R}$ such that $e_w u^{-1}\mathfrak{I} \subset \mathfrak{I}$. Since the family $(e_z)_{z \in Z}$ converges to zero in \mathfrak{R} by assumption, for all but a finite subset of indices $z \in Z$ we have $e_z \in \mathfrak{I}$, hence $e_w u^{-1}e_z \in \mathfrak{I}$.

Considering the equation (3) modulo \mathfrak{I} (that is, as an equation in $\mathfrak{R}/\mathfrak{I}$), the converging infinite sum on the right-hand side reduces to a finite one. Let $Z' \subset Z$ denote a finite subset of indices such that $w \in Z'$ and $e_w u^{-1}e_z \in \mathfrak{I}$ for $z \in Z \setminus Z'$. Applying Lemma 8.4(ii) to the ring $R = \mathfrak{R}$, the finite set of idempotents $(e_z \in \mathfrak{R})_{z \in Z'}$, and the right R -module $N = \mathfrak{R}/\mathfrak{I}$, we obtain that the sum $\sum_{z \in Z'} N e_z$ is direct. Hence it follows from the equation (3) taken modulo \mathfrak{I} that

$$e_w u^{-1}e_w \equiv e_w \pmod{\mathfrak{I}} \quad \text{and} \quad e_w u^{-1}e_z \equiv 0 \pmod{\mathfrak{I}} \text{ for } z \neq w.$$

Since this holds modulo every open right ideal $\mathfrak{I} \subset \mathfrak{R}$, and the topological ring \mathfrak{R} is assumed to be separated, we can conclude that $e_w u^{-1}e_w = e_w$ and $e_w u^{-1}e_z = 0$ in \mathfrak{R} for all $z \neq w$, as desired. \square

Combining Lemma 8.1 with Proposition 8.2, we obtain the following theorem.

Theorem 8.5. *Let \mathfrak{R} be a complete, separated topological ring with a right linear topology and $\mathfrak{H} \subset \mathfrak{R}$ be a topologically nil closed two-sided ideal. Let $(f_z \in \mathfrak{R})_{z \in Z}$ be a family of elements such that $f_z^2 - f_z \in \mathfrak{H}$ for all $z \in Z$ and $f_w f_z \in \mathfrak{H}$ for all $z \neq w, z, w \in Z$. Assume further that the family of elements $(f_z \in \mathfrak{R})_{z \in Z}$ converges to zero in the topology of \mathfrak{R} and $\sum_{z \in Z} f_z \in 1 + \mathfrak{H}$. Then there exist elements $e_z \in f_z + \mathfrak{H} \subset \mathfrak{R}$ such that $e_z^2 = e_z$ for all $z \in Z$ and $e_w e_z = 0$ for all $z \neq w, z, w \in Z$. Moreover, the family of elements $(e_z \in \mathfrak{R})_{z \in Z}$ converges to zero in \mathfrak{R} and $\sum_{z \in Z} e_z = 1$. In addition, one can choose the elements e_z in such a way that $e_z \in \overline{\mathfrak{R} f_z}$ for all $z \in Z$, or in such a way that $e_z \in \overline{f_z \mathfrak{R}}$ for all $z \in Z$, as one wishes.*

Proof: According to Lemma 8.1, there exist idempotent elements $e'_z \in f_z + \mathfrak{H}$ for which one also has $e'_z \in \overline{f_z \mathfrak{R} f_z}$. Hence for every open right ideal $\mathfrak{J} \subset \mathfrak{R}$ one has $e'_z \in \mathfrak{J}$ whenever $f_z \in \mathfrak{J}$, so the family of elements e'_z converges to zero in \mathfrak{R} since the family of elements f_z does. Furthermore, one has $\sum_{z \in Z} e'_z \in \sum_{z \in Z} f_z + \mathfrak{H} = 1 + \mathfrak{H}$. Besides, $e'_w e'_z \in f_w f_z + \mathfrak{H} = \mathfrak{H}$ for all $z \neq w$.

By [45, Lemma 7.6(a)], any topologically nil left or right ideal in \mathfrak{R} is contained in $H(\mathfrak{R})$; hence $\mathfrak{H} \subset H(\mathfrak{R})$. We also observe that the element $u = \sum_{z \in Z} e'_z$ is invertible in \mathfrak{R} , since $u \in 1 + \mathfrak{H}$. Thus Proposition 8.2 is applicable to the family of elements $e'_z \in \mathfrak{R}$, and one can set, at one's choice, either $e_z = u^{-1} e'_z$ for all $z \in Z$ or $e_z = e'_z u^{-1}$ for all $z \in Z$. Finally, in both cases $e_z \in e'_z + \mathfrak{H} = f_z + \mathfrak{H}$, since $u \in 1 + \mathfrak{H}$. \square

Corollary 8.6. *Let \mathfrak{R} be a complete, separated topological ring with a right linear topology, let $\mathfrak{H} \subset \mathfrak{R}$ be a topologically nil, strongly closed two-sided ideal, and let $\mathfrak{S} = \mathfrak{R}/\mathfrak{H}$ be the quotient ring. Let $(\bar{e}_z \in \mathfrak{S})_{z \in Z}$ be a set of orthogonal idempotents in \mathfrak{S} . Assume further that the family of elements \bar{e}_z converges to zero in \mathfrak{S} and $\sum_{z \in Z} \bar{e}_z = 1$ in \mathfrak{S} . Then there exists a set of orthogonal idempotents $(e_z \in \mathfrak{R})_{z \in Z}$ such that $\bar{e}_z = e_z + \mathfrak{H}$ for every $z \in Z$. Moreover, the family of elements e_z converges to zero in \mathfrak{R} and $\sum_{z \in Z} e_z = 1$ in \mathfrak{R} .*

Proof: As the ideal $\mathfrak{H} \subset \mathfrak{R}$ is strongly closed by assumption, the zero-convergent family of elements $\bar{e}_z \in \mathfrak{S}$ can be lifted to *some* zero-convergent family of elements $f_z \in \mathfrak{R}$. This is enough to satisfy the assumptions of Theorem 8.5. \square

9. Split direct limits

The aim of this section and the next one is to discuss the contramodule-theoretic implications and categorical generalizations of the characterization of modules with perfect decompositions in [3, Theorem 1.4]. In particular, in this section we extend the result of [3, Theorem 1.4(2) \iff (3)] to the context of additive categories.

One difference from the approach in [3] is that, given an object M in an additive or abelian category \mathcal{B} , we want to formulate the condition of split direct limits in $\text{Add}(M) \subset \mathcal{B}$ as *intrinsic* to the category $\mathcal{A} = \text{Add}(M)$, so that it makes sense independently of the ambient category \mathcal{B} .

Let X be a directed poset and \mathcal{A} be an additive category. We will say that \mathcal{A} has *X -sized coproducts* if coproducts of families of objects of cardinality not exceeding the cardinality of X exist in \mathcal{A} . An additive category \mathcal{A} is said to have *X -shaped direct limits* if the direct limit of any X -shaped diagram $X \longrightarrow \mathcal{A}$ exists in it.

Recall that whenever X -shaped direct limits exist, they always preserve epimorphisms and cokernels. Assuming in addition that X -sized coproducts exist, for any X -shaped diagram $(g_{yx}: N_x \rightarrow N_y)_{x < y \in X}$ in \mathbf{A} there is a natural right exact sequence

$$(4) \quad \coprod_{x < y} N_x \longrightarrow \coprod_{x \in X} N_x \longrightarrow \varinjlim_{x \in X} N_x \longrightarrow 0.$$

In other words, the rightmost nontrivial morphism in this sequence is the cokernel of the leftmost one. In particular, it is an epimorphism. (In fact, all these properties hold for colimits of X -shaped diagrams for an arbitrary small category X .)

Lemma 9.1. *Let X be a directed poset and \mathbf{A} be an idempotent-complete additive category with X -shaped direct limits and X -sized coproducts. Consider the following properties:*

- (i) *the direct limit of any X -shaped diagram of split monomorphisms is a split monomorphism in \mathbf{A} ;*
- (ii) *the direct limit of any X -shaped diagram of split epimorphisms is a split epimorphism in \mathbf{A} ;*
- (iii) *for any X -shaped diagram $(N_x \rightarrow N_y)_{x < y \in X}$ in \mathbf{A} , the natural epimorphism $\coprod_{x \in X} N_x \rightarrow \varinjlim_{x \in X} N_x$ is split.*

Then the implications (i) \implies (ii) \iff (iii) hold.

Proof: (i) \implies (ii) Let $(M_x \rightarrow M_y)_{x < y} \rightarrow (N_x \rightarrow N_y)_{x < y}$ be an X -indexed diagram of split epimorphisms $M_x \rightarrow N_x$ in \mathbf{A} . Let L_x be the kernel of the split epimorphism $M_x \rightarrow N_x$; then $(L_x \rightarrow L_y)_{x < y} \rightarrow (M_x \rightarrow M_y)_{x < y}$ is an X -indexed diagram of split monomorphisms. By assumption, $f: \varinjlim_{x \in X} L_x \rightarrow \varinjlim_{x \in X} M_x$ is a split monomorphism in \mathbf{A} . Since direct limits preserve cokernels, the morphism $\varinjlim_{x \in X} M_x \rightarrow \varinjlim_{x \in X} N_x$ is the cokernel of f , hence a split epimorphism.

(ii) \implies (iii) Given an X -shaped diagram $(g_{yx}: N_x \rightarrow N_y)_{x < y \in X}$, we consider the following X -shaped diagram $(M_x \rightarrow M_y)_{x < y}$ in the category \mathbf{A} . For every index $x \in X$, the object M_x is the coproduct of N_z over all the indices $z \leq x$ in X . For every pair of indices $x < y$, the morphism $M_x \rightarrow M_y$ is the subcoproduct inclusion of the coproduct indexed by $\{z \mid z \leq x\}$ into the coproduct indexed by $\{z \mid z \leq y\}$. Then one has $\varinjlim_{x \in X} M_x = \coprod_{x \in X} N_x$. Furthermore, for every $x \in X$ there is a split epimorphism $M_x \rightarrow N_x$ in \mathbf{A} with the components $g_{x,z}: N_z \rightarrow N_x$, $z \leq x$. Taken together, these morphisms form a natural morphism of diagrams $(M_x \rightarrow M_y)_{x < y} \rightarrow (N_x \rightarrow N_y)_{x < y}$, and the induced morphism of direct limits $\varinjlim_{x \in X} M_x \rightarrow \varinjlim_{x \in X} N_x$

is the epimorphism $\coprod_{x \in X} N_x \longrightarrow \varinjlim_{x \in X} N_x$ we are interested in. Thus the latter epimorphism is split whenever (ii) holds.

(iii) \implies (ii) Let $(M_x \longrightarrow M_y)_{x < y} \longrightarrow (N_x \longrightarrow N_y)_{x < y}$ be an X -indexed diagram of split epimorphisms. Then the induced morphism $\coprod_x M_x \longrightarrow \coprod_x N_x$ is a split epimorphism too. By assumption, so is the natural morphism $\coprod_x N_x \longrightarrow \varinjlim_x N_x$. Now it follows from commutativity of the square diagram $\coprod_x M_x \longrightarrow \coprod_x N_x \longrightarrow \varinjlim_x N_x$, $\coprod_x M_x \longrightarrow \varinjlim_x M_x \longrightarrow \varinjlim_x N_x$ that the morphism $\varinjlim_x M_x \longrightarrow \varinjlim_x N_x$ is also a split epimorphism (cf. [3, proof of Theorem 1.4(2) \implies (3)]). \square

We will say that an additive category \mathbf{A} with X -sized coproducts and X -shaped direct limits has *split X -direct limits* if the condition of Lemma 9.1(i) is satisfied, that is, the direct limit of X -shaped diagrams in \mathbf{A} preserves split monomorphisms.

Lemma 9.2. *Let \mathbf{B} be an idempotent-complete additive category with X -sized coproducts and $\mathbf{A} \subset \mathbf{B}$ be a full subcategory closed under direct summands and X -sized coproducts. In this setting,*

- (i) *if \mathbf{B} has split X -direct limits, then so does \mathbf{A} ;*
- (ii) *if \mathbf{A} has split X -direct limits, then \mathbf{A} is closed under X -shaped direct limits in \mathbf{B} .*

Proof: (i) Let $N = (N_x \longrightarrow N_y)_{x < y}$ be an X -shaped diagram in \mathbf{A} . By assumption, the direct limit of the diagram N exists in \mathbf{B} and is isomorphic to a direct summand of the coproduct $\coprod_{x \in X} N_x$. Since \mathbf{A} is closed under direct summands and X -sized coproducts in \mathbf{B} , the direct limit of the diagram N computed in \mathbf{B} belongs to \mathbf{A} . It follows that the direct limit of the diagram N exists in \mathbf{A} and coincides with the direct limit of the same diagram in \mathbf{B} .

(ii) Once again, let $N = (N_x \longrightarrow N_y)_{x < y}$ be an X -shaped diagram in \mathbf{A} . Consider the diagram $M = (M_x \longrightarrow M_y)_{x < y}$ constructed in the proof of Lemma 9.1(ii) \implies (iii) and denote by $K = (K_x \longrightarrow K_y)_{x < y}$ the kernel of the natural termwise split epimorphism of diagrams $M \longrightarrow N$. It follows from the construction that $K_z \cong \coprod_{x < z} N_x$ for each $z \in X$. Applying the same construction to the diagram K , we obtain a termwise split epimorphism of diagrams $L \longrightarrow K$, where

$$L_y = \coprod_{z \leq y} K_z \cong \coprod_{z \leq y} \coprod_{x < z} N_x = \coprod_{x < z \leq y} N_x.$$

Now in both the categories \mathbf{A} and \mathbf{B} , the diagram N is the cokernel of the composition of morphisms of diagrams $L \longrightarrow K \longrightarrow M$. Furthermore,

in both the categories \mathbf{A} and \mathbf{B} we have $\varinjlim_x M_x = \coprod_x N_x$ and $\varinjlim_y L_y = \coprod_y K_y \cong \coprod_{x < y} N_x$. The morphism of diagrams $L \rightarrow M$ induces a morphism of their direct limits

$$f: \coprod_{x < y} N_x \cong \varinjlim_x L_x \rightarrow \varinjlim_x M_x \cong \coprod_x N_x,$$

which coincides with the leftmost morphism in the sequence (4).

Now, for any of our two categories $\mathbf{C} = \mathbf{A}$ or $\mathbf{C} = \mathbf{B}$, the direct limit of the diagram N in \mathbf{C} exists if and only if the cokernel of the morphism f exists in \mathbf{C} , and if they do exist then they coincide, $\varinjlim_x^{\mathbf{C}} N_x = \text{coker}^{\mathbf{C}}(f)$. In particular, X -shaped direct limits exist in \mathbf{A} by assumption, so we have an object $\text{coker}^{\mathbf{A}}(f) = \varinjlim_x^{\mathbf{A}} N_x$. Moreover, since we are assuming that \mathbf{A} has split X -direct limits, in view of Lemma 9.1(i) \implies (ii) the morphism f is the composition of a split epimorphism $\varinjlim_x L_x \rightarrow \varinjlim_x K_x$ and a split monomorphism $\varinjlim_x K_x \rightarrow \varinjlim_x M_x$ in \mathbf{A} . Therefore, the cokernel of f also exists in \mathbf{B} and coincides with the cokernel of f in \mathbf{A} , that is, $\text{coker}^{\mathbf{B}}(f) = \text{coker}^{\mathbf{A}}(f)$. Finally, we have $\varinjlim_x^{\mathbf{B}} N_x = \text{coker}^{\mathbf{B}}(f)$. \square

As in the preceding proof, we will use the notation $\varinjlim_{x \in X}^{\mathbf{A}}$ for the direct limit in a category \mathbf{A} (when the category is not clear from the context). Now we can formulate our categorical version of [3, Theorem 1.4(2) \iff (3)] in a way which resembles the original result.

Corollary 9.3. *Let X be a directed poset, \mathbf{B} be an abelian category with X -sized coproducts and exact functors of X -shaped direct limits, and $\mathbf{A} \subset \mathbf{B}$ be a full subcategory closed under direct summands and X -sized coproducts. Then the following conditions are equivalent:*

- (i) *the additive category \mathbf{A} has X -split direct limits;*
- (ii) *the direct limit in \mathbf{B} of any X -shaped diagram of split monomorphisms in \mathbf{A} is a split monomorphism;*
- (iii) *the direct limit in \mathbf{B} of any X -shaped diagram of split epimorphisms in \mathbf{A} is a split epimorphism;*
- (iv) *for any X -shaped diagram $(N_x \rightarrow N_y)_{x < y \in X}$ in \mathbf{A} , the natural epimorphism $\coprod_{x \in X} N_x \rightarrow \varinjlim_{x \in X}^{\mathbf{B}} N_x$ in the category \mathbf{B} is split.*

If any one of these conditions holds, then \mathbf{A} is closed under X -shaped direct limits in \mathbf{B} (so the X -shaped direct limits in \mathbf{A} and \mathbf{B} agree).

Proof: (i) \implies (ii) Assume that \mathbf{A} has X -split direct limits. Then, by Lemma 9.2(ii), \mathbf{A} is closed under X -shaped direct limits in \mathbf{B} . Hence (ii) follows by the definition of \mathbf{A} having X -split direct limits.

(ii) \iff (iii) holds since the direct limit in \mathbf{B} of any X -shaped diagram of split short exact sequences in \mathbf{A} is a short exact sequence. Only here in the implication (iii) \implies (ii) we are using the assumption that X -shaped direct limits in \mathbf{B} are exact.

(iii) \iff (iv) is provable in the same way as Lemma 9.1(ii) \iff (iii) (with the direct limits of X -shaped diagrams in \mathbf{A} computed in \mathbf{A} replaced by the direct limits of X -shaped diagrams in \mathbf{A} computed in \mathbf{B}).

(ii)+(iv) \implies (i) In the context of (iv), since \mathbf{A} is closed under X -sized coproducts in \mathbf{B} by assumption, the coproduct $\coprod_{x \in X} N_x$ is the same in \mathbf{A} and in \mathbf{B} . Being a direct summand of this coproduct, the object $\varinjlim_{x \in X}^{\mathbf{B}} N_x$ consequently also belongs to \mathbf{A} . Hence \mathbf{A} is closed under X -shaped direct limits in \mathbf{B} . Now (ii) tells us that \mathbf{A} has X -split direct limits. \square

Given an additive category \mathbf{A} with set-indexed coproducts, we will say that \mathbf{A} has *split direct limits* if it has X -split direct limits for every directed poset X . It is clear from [1, Lemma 1.6] that a category \mathbf{A} has split direct limits whenever it has X -split direct limits for every *linearly ordered* (or even well-ordered) index set X .

Example 9.4. Let \mathbf{A} be a locally Noetherian Grothendieck abelian category and $\mathbf{A}_{\text{inj}} \subset \mathbf{A}$ be the full subcategory of injective objects in \mathbf{A} . Then the category \mathbf{A}_{inj} has split direct limits. Indeed, the full subcategory \mathbf{A}_{inj} is closed under direct limits in \mathbf{A} , hence all (coproducts and) direct limits exist in \mathbf{A}_{inj} . Furthermore, the direct limit of any directed diagram of split monomorphisms in \mathbf{A}_{inj} is a monomorphism in \mathbf{A} between two objects of \mathbf{A}_{inj} . Clearly, any such monomorphism is split.

In view of Lemma 9.2(i), it follows that, for any injective object $J \in \mathbf{A}$, the full subcategory $\text{Add}(J) \subset \mathbf{A}$ has split direct limits.

In the rest of this section, we discuss split direct limits in the full subcategory $\text{Add}(M)$ of direct summands of coproducts of copies of a fixed object M in a topologically agreeable abelian category \mathbf{A} . The special case in which $\mathbf{A} = A\text{-mod}$ is the category of modules over an associative ring plays an important role. The aim is to interpret the condition of split direct limits in $\text{Add}(M)$ as a “naturally sounding” property of the related contramodule category $\text{Hom}_{\mathbf{A}}(M, M)^{\text{op-contra}}$.

Lemma 9.5. *Let \mathbf{A} be a cocomplete abelian category, $M \in \mathbf{A}$ be an object, and \mathbf{B} be the abelian category with enough projective objects for which $\mathbf{B}_{\text{proj}} \cong \text{Add}(M) \subset \mathbf{A}$, as in Theorem 3.14(i). Let X be a small category.*

Assume that the full subcategory \mathbf{B}_{proj} is closed under X -shaped colimits in \mathbf{B} . Then the full subcategory $\text{Add}(M)$ is closed under X -shaped colimits in \mathbf{A} , and for any X -shaped diagram $(N_x \rightarrow N_y)_{x \rightarrow y \in X}$ in $\text{Add}(M)$ the natural epimorphism $\coprod_{x \in X} N_x \rightarrow \varinjlim_{x \in X}^{\mathbf{A}} N_x$ in the category \mathbf{A} is split.

Proof: According to the discussion near the end of Section 3, the equivalence of categories $\mathbf{B}_{\text{proj}} \cong \text{Add}(M)$ can be extended to a pair of adjoint functors $\Psi: \mathbf{A} \rightarrow \mathbf{B}$ and $\Phi: \mathbf{B} \rightarrow \mathbf{A}$. Given an X -shaped diagram $N \in \text{Add}(M)^X$, consider the diagram $P = \Psi(N) \in (\mathbf{B}_{\text{proj}})^X$. Then we have $N = \Phi(P)$. Since the functor Φ is a left adjoint, it preserves colimits, and it follows that

$$\varinjlim_{x \in X} N_x = \Phi \left(\varinjlim_{x \in X} P_x \right) \quad \text{and} \quad \coprod_{x \in X} N_x = \Phi \left(\coprod_{x \in X} P_x \right).$$

Now we have $\varinjlim_{x \in X} P_x \in \mathbf{B}_{\text{proj}}$ by assumption, hence $\varinjlim_{x \in X} N_x \in \text{Add}(M)$. Moreover, the natural morphism $f: \coprod_{x \in X} P_x \rightarrow \varinjlim_{x \in X} P_x$ is an epimorphism between projective objects in \mathbf{B} , hence a split epimorphism. Thus the morphism $\Phi(f): \coprod_{x \in X} N_x \rightarrow \varinjlim_{x \in X} N_x$ is a split epimorphism in \mathbf{A} (and in $\text{Add}(M)$). \square

Proposition 9.6. *Let \mathfrak{R} be a complete, separated topological ring with a right linear topology, and let X be a directed poset. Then the category of projective left \mathfrak{R} -contramodules $\mathfrak{R}\text{-contra}_{\text{proj}}$ has X -split direct limits if and only if the full subcategory $\mathfrak{R}\text{-contra}_{\text{proj}}$ is closed under X -shaped direct limits in $\mathfrak{R}\text{-contra}$.*

Proof: “Only if” holds by Lemma 9.2(ii). To prove “if”, we use Corollary 4.4. Let A be an associative ring and M a left A -module such that \mathfrak{R} is isomorphic to $\text{Hom}_A(M, M)^{\text{op}}$ as a topological ring. By Theorem 3.14(i)(iii), we have $\text{Add}(M) \cong \mathfrak{R}\text{-contra}_{\text{proj}}$. By Lemma 9.5, it follows that for any X -shaped diagram N in $\text{Add}(M)$ the natural epimorphism of left A -modules $\coprod_{x \in X} N_x \rightarrow \varinjlim_{x \in X}^{A\text{-mod}} N_x$ is split. Applying Corollary 9.3(iv) \implies (i) to the full subcategory $\text{Add}(M)$ in the Ab5 -category $A\text{-mod}$, we conclude that the category $\text{Add}(M)$ has split direct limits. Hence so does the category $\mathfrak{R}\text{-contra}_{\text{proj}} \cong \text{Add}(M)$. \square

Remark 9.7. The above proof is quite indirect. The problem is that direct limits are not exact in the abelian category $\mathfrak{R}\text{-contra}$, so Corollary 9.3 is not applicable to $\mathbf{A} = \mathfrak{R}\text{-contra}_{\text{proj}}$ and $\mathbf{B} = \mathfrak{R}\text{-contra}$. It would be interesting to know whether the assertion of Proposition 9.6 holds for abelian categories of a more general nature than the categories of contramodules over topological rings – such as, e.g., cocomplete abelian categories \mathbf{B} with a projective generator in which the additive category of projective objects \mathbf{B}_{proj} is agreeable. (Cf. Example 11.4 below.)

Example 9.8. The following simple counterexample purports to show why Proposition 9.6 is nontrivial. Let k be a field and $\mathbf{A} = \mathbf{B} = k\text{-vect}^{\text{op}}$ be the opposite category to the category of k -vector spaces. This category is abelian, all its objects are projective, and all short exact sequences split. So \mathbf{A} satisfies the conditions of Lemma 9.1(ii)(iii) and $\mathbf{A} \subset \mathbf{B}$ satisfies the conditions of Corollary 9.3(iii)(iv). But direct limits are not exact in $\mathbf{A} = \mathbf{B}$, so the conditions in Lemma 9.1(i) and Corollary 9.3(i)(ii) are *not* satisfied and \mathbf{A} does not have split direct limits.

Corollary 9.9. *Let \mathbf{A} be an idempotent-complete topologically agreeable additive category, $M \in \mathbf{A}$ be an object, and $\mathfrak{R} = \text{Hom}_{\mathbf{A}}(M, M)^{\text{op}}$ be the ring of endomorphisms of M , endowed with its complete, separated, right linear topology. Let X be a directed poset. Then the full subcategory $\text{Add}(M) \subset \mathbf{A}$ has X -split direct limits if and only if the full subcategory $\mathfrak{R}\text{-contra}_{\text{proj}}$ is closed under X -shaped direct limits in $\mathfrak{R}\text{-contra}$.*

Proof: By Theorem 3.14(i)(iii), we have an equivalence of additive categories $\text{Add}(M) \cong \mathfrak{R}\text{-contra}_{\text{proj}}$. So the assertion follows from Proposition 9.6. \square

10. Objects with perfect decompositions

Let \mathbf{A} be an agreeable category. A family of nonzero objects $(M_z \in \mathbf{A})_{z \in Z}$ is said to be *locally T-nilpotent* if for any sequence of indices $z_1, z_2, z_3, \dots \in Z$ and any sequence of nonisomorphisms $f_i \in \text{Hom}_{\mathbf{A}}(M_{z_i}, M_{z_{i+1}})$ the family of morphisms $f_n f_{n-1} \cdots f_1 : M_{z_1} \rightarrow M_{z_{n+1}}$, $n = 1, 2, 3, \dots$, is summable (i.e., corresponds to a morphism $f : M_{z_1} \rightarrow \prod_{n=1}^{\infty} M_{z_{n+1}}$; see Section 3).

Notice that the ring of endomorphisms of every object in a locally T-nilpotent family is necessarily local, as for any noninvertible endomorphism $h \in \text{Hom}_{\mathbf{A}}(M_z, M_z)$ a morphism $(1-h)^{-1} = \sum_{n=0}^{\infty} h^n : M_z \rightarrow M_z$ exists. So nonautomorphisms form a two-sided ideal in $\text{Hom}_{\mathbf{A}}(M_z, M_z)$. It follows that the object M_z is indecomposable.

An object $M \in \mathbf{A}$ is said to have a *perfect decomposition* if there exists a locally T-nilpotent family of objects $(M_z)_{z \in Z}$ in \mathbf{A} such that $M \cong \prod_{z \in Z} M_z$.

In view of the discussion in the first half of Section 9, the result of [3, Theorem 1.4] can be formulated in our language as follows.

Theorem 10.1 ([3, Theorem 1.4]). *Let A be an associative ring and M be a left A -module. Then M has a perfect decomposition if and only if the full subcategory $\text{Add}(M) \subset A\text{-mod}$ has split direct limits.*

Proof: A proof of this theorem can be found in [3], based on preceding results from the books [31, 37] and particularly from the paper [27].

An alternative proof of the (easy) implication “only if” based on contra-module methods is suggested below in Remark 14.2. \square

In this section we prove the following categorical extension of Theorem 10.1.

Theorem 10.2. *Let \mathbf{A} be an idempotent-complete topologically agreeable additive category and $M \in \mathbf{A}$ be an object. Then M has a perfect decomposition if and only if the full subcategory $\text{Add}(M) \subset \mathbf{A}$ has split direct limits.*

Let us explain the plan of our proof of Theorem 10.2. In the second half of Section 9 we have already interpreted the condition of split direct limits in $\text{Add}(M)$ in terms of the topological ring $\mathfrak{R} = \text{Hom}_{\mathbf{A}}(M, M)^{\text{op}}$. In this section we similarly interpret the condition of the existence of a perfect decomposition $M \cong \coprod_{z \in \mathbb{Z}} M_z$ of an object M . Then we translate Theorem 10.1 into the topological ring language, and finally go back to the greater generality of Theorem 10.2.

Let \mathfrak{R} be a complete, separated topological ring with a right linear topology. The notion of the *topological Jacobson radical* $\mathfrak{H}(\mathfrak{R})$ of a topological ring \mathfrak{R} was discussed in the papers [33, Section 3.B] and [45, Section 7]. We recall that, by definition, $\mathfrak{H}(\mathfrak{R})$ is the intersection of all open maximal right ideals in \mathfrak{R} . The topological Jacobson radical $\mathfrak{H}(\mathfrak{R})$ is a closed two-sided ideal in \mathfrak{R} [45, Lemma 7.1].

Lemma 10.3. *Suppose that $\mathfrak{H} \subset \mathfrak{R}$ is a topologically nil closed two-sided ideal in \mathfrak{R} such that the quotient ring $\mathfrak{R}/\mathfrak{H}$ with its quotient topology is (complete and) topologically semisimple. Then the ideal $\mathfrak{H} \subset \mathfrak{R}$ coincides with the topological Jacobson radical of the ring \mathfrak{R} and with the Jacobson radical of the abstract ring \mathfrak{R} (with the topology ignored), that is, $\mathfrak{H} = \mathfrak{H}(\mathfrak{R}) = H(\mathfrak{R})$.*

Proof: This is a straightforward generalization of [45, Lemma 9.1]. One observes that every nonzero element of the quotient ring $\mathfrak{S} = \mathfrak{R}/\mathfrak{H}$ acts nontrivially on a simple discrete right \mathfrak{S} -module (see Remark 6.4), and then argues as in *loc. cit.* \square

The following definition, containing a list of structural conditions on a topological ring \mathfrak{R} , plays the key role. We say that a complete, separated topological ring \mathfrak{R} with right linear topology is *topologically left perfect* if the next three conditions hold:

- (1) the topological Jacobson radical $\mathfrak{H}(\mathfrak{R})$ of the topological ring \mathfrak{R} is topologically left T-nilpotent;
- (2) $\mathfrak{H}(\mathfrak{R})$ is a strongly closed subgroup/ideal in \mathfrak{R} ; and
- (3) the quotient ring $\mathfrak{S} = \mathfrak{R}/\mathfrak{H}(\mathfrak{R})$ with its quotient topology is topologically semisimple.

Theorem 10.4. *Let \mathbf{A} be an idempotent-complete topologically agreeable additive category, $M \in \mathbf{A}$ be an object, and $\mathfrak{R} = \text{Hom}_{\mathbf{A}}(M, M)^{\text{op}}$ be the ring of endomorphisms of M , endowed with its complete, separated, right linear topology. Then the object $M \in \mathbf{A}$ has a perfect decomposition if and only if the topological ring \mathfrak{R} is topologically left perfect.*

Remark 10.5. A relevant concept in the abstract (nontopological) ring theory is that of a semiregular ring. An associative ring R is said to be *semiregular* if its quotient ring by the Jacobson radical $S = R/H(R)$ is von Neumann regular and idempotents can be lifted modulo $H(R)$.

Any topologically left perfect topological ring is semiregular as an abstract ring. Indeed, one has $H(\mathfrak{R}) = \mathfrak{H}(\mathfrak{R})$ by Lemma 10.3; the topologically semisimple quotient ring $\mathfrak{S} = \mathfrak{R}/\mathfrak{H}(\mathfrak{R})$ is von Neumann regular, as was already mentioned in Remark 6.3 (cf. Lemma 11.1 below); and idempotents can be lifted modulo a topologically nil closed two-sided ideal by Lemma 8.1 (see also Corollary 8.6).

Semiregular rings appear in connection with modules with perfect decompositions and in related contexts in [31, Theorem 7.3.15(6)], [2, Propositions 4.1 and 4.2(1)], and [3, Theorem 1.1(4)]. The above theorem provides a more precise description of the class of rings appearing as the endomorphism rings of modules/objects with perfect decompositions in terms of their topological ring structures.

Before proceeding to prove Theorem 10.4, let us formulate and prove three lemmas about right topological/topologically agreeable additive categories in the spirit of Lemmas 3.3-3.4.

Lemma 10.6. *Let \mathbf{A} be a right topological additive category, K, L , and $M \in \mathbf{A}$ be three objects, and $(f_x : K \rightarrow L)_{x \in X}$ be a family of morphisms converging to zero in the group $\text{Hom}_{\mathbf{A}}(K, L)$. Let $\mathfrak{U} \subset \text{Hom}_{\mathbf{A}}(K, M)$ be an open subgroup. Then one has $\text{Hom}_{\mathbf{A}}(L, M)f_x \subset \mathfrak{U}$ for all but a finite subset of indices $x \in X$.*

Proof: Set $N = L \oplus M \in \mathbf{A}$. Let $j : L \rightarrow N$ be the coproduct injection and $q : N \rightarrow M$ be the product projection. By axiom (i), the family of morphisms $(jf_x : K \rightarrow N)_{x \in X}$ converges to zero in the group $\text{Hom}_{\mathbf{A}}(K, N)$. By axiom (ii), it follows that for any open subgroup $\mathfrak{V} \subset \text{Hom}_{\mathbf{A}}(K, N)$ we have $\text{Hom}_{\mathbf{A}}(N, N)jf_x \subset \mathfrak{V}$ for all but a finite subset of indices $x \in X$. By axiom (i), for every open subgroup $\mathfrak{U} \subset \text{Hom}_{\mathbf{A}}(K, M)$ there exists an open subgroup $\mathfrak{V} \subset \text{Hom}_{\mathbf{A}}(K, N)$ such that $q\mathfrak{V} \subset \mathfrak{U}$. Thus we have $q\text{Hom}_{\mathbf{A}}(N, N)jf_x \subset \mathfrak{U}$ for all but a finite subset of indices $x \in X$; and it remains to observe that $q\text{Hom}_{\mathbf{A}}(N, N)j = \text{Hom}_{\mathbf{A}}(L, M)$. □

Lemma 10.7. *Let \mathbf{A} be a topologically agreeable category, $(L_x \in \mathbf{A})_{x \in X}$ be a family of objects in \mathbf{A} , and $M \in \mathbf{A}$ be an object. Then the natural isomorphism $\text{Hom}_{\mathbf{A}}(\coprod_{x \in X} L_x, M) \cong \prod_{x \in X} \text{Hom}_{\mathbf{A}}(L_x, M)$ is an isomorphism of topological abelian groups (where the right-hand side is endowed with the product topology).*

Proof: By axiom (i), in any right topological additive category \mathbf{A} where the coproduct $\coprod_{x \in X} L_x$ exists, the natural map $\text{Hom}_{\mathbf{A}}(\coprod_{x \in X} L_x, M) \rightarrow \text{Hom}_{\mathbf{A}}(L_y, M)$ is continuous for every $y \in X$. Hence so is the map to the product $\text{Hom}_{\mathbf{A}}(\coprod_{x \in X} L_x, M) \rightarrow \prod_{x \in X} \text{Hom}_{\mathbf{A}}(L_x, M)$. It remains to check continuity in the opposite direction, and that is where we will need the assumption that \mathbf{A} is topologically agreeable.

Set $L = \coprod_{x \in X} L_x$. For every $x \in X$, let $\iota_x \in \text{Hom}_{\mathbf{A}}(L_x, L)$ be the coproduct injection and $\pi_x \in \text{Hom}_{\mathbf{A}}(L, L_x)$ be the natural projection. Then the family of morphisms $(\pi_x: L \rightarrow L_x)_{x \in X}$ is summable in the agreeable category \mathbf{A} (as these are the components of the identity morphism $\text{id}_L: L \rightarrow L$). Hence the family of morphisms $(\iota_x \pi_x: L \rightarrow L_x)_{x \in X}$ is summable too.

Since \mathbf{A} is topologically agreeable, it follows that the family of projectors $(\iota_x \pi_x)_{x \in X}$ converges to zero in the topology of the group $\text{Hom}_{\mathbf{A}}(L, L)$. By Lemma 10.6 (applied to the objects $K = L$ and M), every open subgroup \mathfrak{U} in the topological group $\text{Hom}_{\mathbf{A}}(L, M)$ contains the subgroup $\text{Hom}_{\mathbf{A}}(L, M) \iota_x \pi_x = \text{Hom}_{\mathbf{A}}(L_x, M) \pi_x$ for all but a finite subset of indices $x \in X$. In other words, one can say that the whole family of subgroups $\text{Hom}_{\mathbf{A}}(L_x, M) \pi_x$ converges to zero in $\text{Hom}_{\mathbf{A}}(L, M)$. It follows that the map $\prod_{x \in X} \text{Hom}_{\mathbf{A}}(L_x, M) \rightarrow \text{Hom}_{\mathbf{A}}(L, M)$ assigning to a family of morphisms $(f_x: L_x \rightarrow M)_{x \in X}$ the morphism $\sum_{x \in X} f_x \pi_x$ is continuous. \square

Lemma 10.8. *Let \mathbf{A} be a topologically agreeable category, M and $N \in \mathbf{A}$ be two objects, and X be a set. Then the topological group $\text{Hom}_{\mathbf{A}}(M, N^{(X)})$ is isomorphic to the topological group $\text{Hom}_{\mathbf{A}}(M, N)[[X]]$ (where, for any complete, separated topological abelian group \mathfrak{A} , the group $\mathfrak{A}[[X]]$ is endowed with the projective limit topology of $\mathfrak{A}[[X]] = \varprojlim_{\mathfrak{U} \subset \mathfrak{A}} (\mathfrak{A}/\mathfrak{U})[X]$, with \mathfrak{U} ranging over the open subgroups of \mathfrak{A}).*

Proof: For any right topological additive category \mathbf{A} , the abelian group homomorphism $\text{Hom}_{\mathbf{A}}(M, N)[[X]] \rightarrow \text{Hom}_{\mathbf{A}}(M, N^{(X)})$ assigns to every zero-convergent family of morphisms $(f_x: M \rightarrow N)_{x \in X}$ the morphism $\sum_{x \in X} \iota_x f_x: M \rightarrow N^{(X)}$ (where $\iota_x: N \rightarrow N^{(X)}$ are the coproduct injections). The property of this map of being an isomorphism of abelian groups is a restatement of the definition of a topologically agreeable

category \mathbf{A} . The lemma claims that this is a topological isomorphism, so we have to prove continuity in both directions.

Our map decomposes as

$$\mathrm{Hom}_{\mathbf{A}}(M, N)[[X]] \longrightarrow \mathrm{Hom}_{\mathbf{A}}(M, N^{(X)})[[X]] \longrightarrow \mathrm{Hom}_{\mathbf{A}}(M, N^{(X)}).$$

Here the first map assigns to a zero-convergent family of morphisms $(f_x: M \rightarrow N)_{x \in X}$ the family of morphisms $(\iota_x f_x: M \rightarrow N^{(X)})_{x \in X}$, which is zero-convergent by Lemma 3.3. One can see from the proof of Lemma 3.3 that this map is continuous. The second map assigns to a zero-convergent family of elements $(g_x \in \mathfrak{A})_{x \in X}$ of the topological group $\mathfrak{A} = \mathrm{Hom}_{\mathbf{A}}(M, N^{(X)})$ their sum $\sum_{x \in X} g_x$ in \mathfrak{A} . This map is continuous for every complete, separated topological abelian group \mathfrak{A} . Hence the composition is continuous too.

To check that the inverse map $\mathrm{Hom}_{\mathbf{A}}(M, N^{(X)}) \rightarrow \mathrm{Hom}_{\mathbf{A}}(M, N)[[X]]$ is continuous, suppose we are given an open subgroup $\mathfrak{U} \subset \mathrm{Hom}_{\mathbf{A}}(M, N)$. Choose a fixed element $x_0 \in X$, and consider the subgroup $\mathfrak{V}_{\mathfrak{U}, x_0} \subset \mathrm{Hom}_{\mathbf{A}}(M, N^{(X)})$ consisting of all the morphisms $f: M \rightarrow N^{(X)}$ for which the composition $\pi_{x_0} f$ (where $\pi_x: N^{(X)} \rightarrow N$ is the natural projection) belongs to \mathfrak{U} . By the continuity axiom (i), $\mathfrak{V}_{\mathfrak{U}, x_0}$ is an open subgroup in $\mathrm{Hom}_{\mathbf{A}}(M, N^{(X)})$. By axiom (ii), there exists an open $\mathrm{Hom}_{\mathbf{A}}(N^{(X)}, N^{(X)})$ -submodule $\mathfrak{W} \subset \mathrm{Hom}_{\mathbf{A}}(N, N^{(X)})$ such that $\mathfrak{W} \subset \mathfrak{V}_{\mathfrak{U}, x_0}$.

Now for every morphism $f \in \mathfrak{W}$ we have $\sigma_{x_0, x} f \in \mathfrak{W}$ (where $\sigma_{x, y}: N^{(X)} \rightarrow N^{(X)}$ is the automorphism permuting the coordinates x and y). Hence $\pi_x f = \pi_{x_0} \sigma_{x_0, x} f \in \mathfrak{U}$. We have shown that the full preimage of the open subgroup $\mathfrak{U}[[X]] \subset \mathrm{Hom}_{\mathbf{A}}(M, N)[[X]]$ under the map $\mathrm{Hom}_{\mathbf{A}}(M, N^{(X)}) \rightarrow \mathrm{Hom}_{\mathbf{A}}(M, N)[[X]]$ contains an open subgroup $\mathfrak{W} \subset \mathrm{Hom}_{\mathbf{A}}(M, N^{(X)})$, so this map is continuous. \square

Corollary 10.9. *Let \mathbf{A} be a topologically agreeable category, $M \in \mathbf{A}$ be an object, and Y be a set. Let $\mathfrak{R} = \mathrm{Hom}_{\mathbf{A}}(M, M)^{\mathrm{op}}$ be the topological ring of endomorphisms of M . Then the topological ring $\mathrm{Hom}_{\mathbf{A}}(M^{(Y)}, M^{(Y)})^{\mathrm{op}}$ is naturally isomorphic to the ring of row-zero-convergent matrices $\mathfrak{Mat}_Y(\mathfrak{R})$ with the topology defined in Section 5.*

Proof: The ring isomorphism can easily be established, and the description of the topology is provided by Lemmas 10.7-10.8. \square

Proof of the implication “only if” in Theorem 10.4: More generally, for any family of objects $(M_z)_{z \in Z}$ in an agreeable category \mathbf{A} , the ring of endomorphisms $R = \mathrm{Hom}_{\mathbf{A}}(M, M)^{\mathrm{op}}$ of their coproduct $M = \coprod_{z \in Z} M_z$ can be described as the ring of row-summable matrices of morphisms $r = (r_{w, z}: M_w \rightarrow M_z)_{w, z \in Z}$. Here the words “row-summable” mean that for every fixed index $w \in Z$ the family of morphisms $(r_{w, z}: M_w \rightarrow M_z)_{z \in Z}$ must be summable in the agreeable category \mathbf{A} .

Assuming that the family of objects $(M_z)_{z \in Z}$ is locally T-nilpotent, we will denote by X the set of isomorphism classes of the objects M_z . So we have a natural surjective map $Z \rightarrow X$ assigning to every module in the family its isomorphism class. For every element $x \in X$, let $Y_x \subset Z$ denote the full preimage of the element x under this map, so the set Z is the disjoint union of nonempty sets Y_x .

Assume further that the category \mathbf{A} is topologically agreeable. Denote by $\mathfrak{H} \subset \mathfrak{R}$ the subset of all *matrices of nonisomorphisms* in the ring \mathfrak{R} ; that is, an element $h = (h_{w,z})_{w,z \in Z} \in \mathfrak{R}$ belongs to \mathfrak{H} if and only if, for every pair of indices $w, z \in Z$, the morphism $h_{w,z}$ is not an isomorphism. Our next aim is to show that \mathfrak{H} is a closed two-sided ideal in \mathfrak{R} .

Denote by $\mathfrak{H}_{w,z} \subset \mathfrak{R}_{w,z}$ the subset of nonisomorphisms in the topological group of morphisms $\mathfrak{R}_{w,z} = \text{Hom}_{\mathbf{A}}(M_w, M_z)$. Following the discussion at the beginning of this section, $\mathfrak{H}_{z,z}$ is a two-sided ideal in $\mathfrak{R}_{z,z}$. It follows that, for all w and $z \in Z$, the subset $\mathfrak{H}_{w,z} \subset \mathfrak{R}_{w,z}$ is an open subgroup. Indeed, if the elements w and $z \in Z$ do *not* belong to the same equivalence class Y_x , $x \in X$, then we have $\mathfrak{H}_{w,z} = \mathfrak{R}_{w,z}$. If there is $x \in X$ such that both the elements w and z belong to Y_x , then choosing an isomorphism $M_w \cong M_z$ identifies $\mathfrak{R}_{w,z}$ with $\mathfrak{R}_{z,z}$ and $\mathfrak{H}_{w,z}$ with $\mathfrak{H}_{z,z}$. Any proper open right ideal in the topological ring $\mathfrak{R}_{z,z}$ consists of nonisomorphisms. As the topology on $\mathfrak{R}_{z,z}$ is separated, it follows that the ideal of nonisomorphisms $\mathfrak{H}_{z,z} \subset \mathfrak{R}_{z,z}$ is open. Hence $\mathfrak{H}_{w,z} \subset \mathfrak{R}_{w,z}$ is an open subgroup in both cases.

The projection map $\mathfrak{R} \rightarrow \mathfrak{R}_{w,z}$, assigning to a matrix $(a_{w,z})_{w,z \in Z} \in \mathfrak{R}$ its matrix entry at the position $(w, z) \in Z \times Z$, is a continuous group homomorphism (by the continuity axiom (i) of a right topological additive category). The subset $\mathfrak{H} \subset \mathfrak{R}$ is the intersection of the full preimages of the open subgroups $\mathfrak{H}_{w,z} \subset \mathfrak{R}_{w,z}$ over all the pair of indices (w, z) . It follows that \mathfrak{H} is a closed subgroup in \mathfrak{R} .

The multiplication in the ring \mathfrak{R} is computable as follows. Let $a = (a_{v,w})_{v,w \in Z}$ and $b = (b_{w,z})_{w,z \in Z}$ be two matrices representing some elements of \mathfrak{R} . Then, for every fixed v and $z \in Z$, the family of morphisms $(a_{v,w} : M_v \rightarrow M_w)_{w \in Z}$ is summable, and it follows that the family of morphisms $(b_{w,z} a_{v,w} : M_v \rightarrow M_z)_{w \in Z}$ is summable too. In view of Lemma 3.4, the product of two elements $c = ab \in \mathfrak{R}$ is represented by the matrix $(c_{v,z})_{v,z \in Z}$ with the entries

$$c_{v,z} = \sum_{w \in Z} (b_{w,z} a_{v,w})_{v,z},$$

where the infinite summation sign means the limit of finite partial sums in the topology of $\mathfrak{R}_{v,z}$. Furthermore, the composition of any two nonisomorphisms between objects M_z , $z \in Z$, is not an isomorphism, since

these objects are indecomposable, while the composition of an isomorphism and a nonisomorphism (in any order) is obviously a nonisomorphism. Therefore, $\mathfrak{H} \subset \mathfrak{R}$ is a two-sided ideal.

Let us show that \mathfrak{H} is topologically left T-nilpotent. Let $E \subset \mathfrak{R}$ be the set of all matrices with a single nonzero entry which is not an isomorphism. Using Lemmas 10.7 and 10.8, one can see that the subgroup generated by E is dense in \mathfrak{H} . The condition of local T-nilpotency of the family of objects $(M_z)_{z \in Z}$ can be equivalently restated by saying that the subset of elements $E \subset \mathfrak{R}$ is topologically left T-nilpotent. Applying Lemma 7.1, we conclude that the whole ideal $\mathfrak{H} \subset \mathfrak{R}$ is topologically left T-nilpotent.

Our next aim is to describe the quotient ring $\mathfrak{S} = \mathfrak{R}/\mathfrak{H}$. The elements of \mathfrak{S} are represented by matrices with entries in the quotient groups of the groups $\mathfrak{R}_{w,z} = \text{Hom}_A(M_w, M_z)$ by the subgroups of nonisomorphisms $\mathfrak{H}_{w,z} \subset \mathfrak{R}_{w,z}$. These are block matrices supported in the subset $\prod_{x \in X} Y_x \times Y_x \subset Z \times Z$. For every $x \in X$, the related block \mathfrak{S}_x can be described as a similar quotient ring of the topological ring of endomorphisms of the object $\prod_{y \in Y_x} M_y$.

According to the above discussion, the ideal $\mathfrak{H}_{y,y} \subset \mathfrak{R}_{y,y}$ is open and the topology on the quotient ring $D_y = \mathfrak{R}_{y,y}/\mathfrak{H}_{y,y}$ is discrete. All the rings $\mathfrak{R}_{y,y}$, $y \in Y_x$, are isomorphic to each other, since the objects M_y are; hence so are all the rings D_y for $y \in Y_x$. Choosing isomorphisms between the objects M_y in a compatible way, we can put $\mathfrak{R}_x = \mathfrak{R}_{y,y}$ and $D_x = D_y$. So D_x , $x \in X$, are some discrete skew-fields.

By Corollary 10.9, the topological ring of endomorphisms of the object $\prod_{y \in Y_x} M_y$ is isomorphic to the topological ring of row-zero-convergent matrices $\mathfrak{Mat}_{Y_x}(\mathfrak{R}_x)$. It follows that the topological ring \mathfrak{S}_x is isomorphic to the topological ring of row-finite matrices $\mathfrak{Mat}_{Y_x}(D_x)$. By Lemma 10.7, the topology on the ring $\mathfrak{S} = \prod_{x \in X} \mathfrak{S}_x$ is the product topology. It remains to recall Theorem 6.2(iv) and conclude that the topological ring \mathfrak{S} is topologically semisimple.

We also observe that the topological ring \mathfrak{S} is complete, as is clear from the explicit description of its topology that we have obtained. We still have to show that the ideal \mathfrak{H} is strongly closed in \mathfrak{R} .

For this purpose, choose for every $x \in X$ and all $y, z \in Y_x$, some (automatically continuous) section $s_{y,z}: D_{y,z} \rightarrow \mathfrak{R}_{y,z}$ of the natural surjective map $p_{y,z}: \mathfrak{R}_{y,z} \rightarrow \mathfrak{R}_{y,z}/\mathfrak{H}_{y,z}$ onto the discrete group $D_{y,z} = \mathfrak{R}_{y,z}/\mathfrak{H}_{y,z}$, satisfying only the condition that $s_{y,z}(0) = 0$. Define a section $s: \mathfrak{S} \rightarrow \mathfrak{R}$ of the continuous surjective ring homomorphism $p: \mathfrak{S} \rightarrow \mathfrak{R}$ by the rule that every zero matrix entry is lifted to a zero matrix entry, while the maps $s_{y,z}$ are applied to nonzero matrix entries. Then

the map s (does not respect either the matrix addition or the matrix multiplication, of course; but it) is continuous. Applying the map s , one can lift any zero-converging family of elements in \mathfrak{S} to a zero-converging family of elements in \mathfrak{R} .

Finally, we have $\mathfrak{H}(\mathfrak{R}) = \mathfrak{H}$ by Lemma 10.3. This proves the implication “only if”. □

In order to prove the inverse implication, we will need another lemma.

Lemma 10.10. *Let \mathbf{A} be an idempotent-complete agreeable category and $M \in \mathbf{A}$ be an object. Let $(e_z \in \text{Hom}_{\mathbf{A}}(M, M))_{z \in Z}$ be a summable family of orthogonal idempotents such that $\sum_{e \in Z} e_z = \text{id}_M$. Then there exists a unique direct sum decomposition $M \cong \coprod_{z \in Z} M_z$ of the object M such that the projector $\iota_z \pi_z: M \rightarrow M_z \rightarrow M$ on the direct summand M_z in M is equal to e_z for every $z \in Z$.*

Proof: For the uniqueness, one just observes that a direct summand L of an object M in an additive category is determined by the projector $M \rightarrow L \rightarrow M$ onto it. Conversely, since \mathbf{A} is assumed to be idempotent-complete, every idempotent e_z , $z \in Z$, determines a direct summand M_z in M such that e_z is the composition of the projection $\pi_z: M \rightarrow M_z$ and the injection $\iota_z: M_z \rightarrow M$.

It remains to construct an isomorphism $M \cong \coprod_{z \in Z} M_z$. The collection of morphisms $\iota_z: M_z \rightarrow M$ corresponds to a uniquely defined morphism $f: \coprod_{z \in Z} M_z \rightarrow M$. Constructing the desired morphism in the opposite direction $g: M \rightarrow \coprod_{z \in Z} M_z$ is equivalent to showing that the collection of morphisms $\pi_z: M \rightarrow M_z$ is summable.

Now the family of idempotents $(e_z: M \rightarrow M)_{z \in Z}$ is summable by assumption, so we have a morphism $h: M \rightarrow M^{(Z)}$ whose components are the idempotents e_z . Let $\pi: M^{(Z)} \rightarrow \coprod_{z \in Z} M_z$ be the coproduct of the morphisms π_z , that is, $\pi = \coprod_{z \in Z} \pi_z$. Then the desired morphism g can be obtained as $g = \pi \circ h$. This is based on the observation that $\pi_z e_z = \pi_z$ for every $z \in Z$.

The composition $gf: \coprod_{z \in Z} M_z \rightarrow M \rightarrow \coprod_{z \in Z} M_z$ is the identity morphism, since $\pi_z \iota_w = \text{id}_{M_z}$ when $z = w$ and 0 otherwise (the latter property being a consequence of the assumption of orthogonality, $e_z e_w = 0$). Finally, the assertion that the composition $fg: M \rightarrow \coprod_{z \in Z} M_z \rightarrow M$ is the identity endomorphism of M is, by definition, a restatement of the equation $\sum_{z \in Z} e_z = \text{id}_M$. □

Proof of the implication “if” in Theorem 10.4: By Theorem 6.2(iv), we have an isomorphism of topological rings $\mathfrak{S} \cong \prod_{x \in X} \mathfrak{S}_x$, where $\mathfrak{S}_x = \text{Mat}_{\mathfrak{y}_x}(D_x)$ are the rings of row-finite matrices over some discrete skew-fields D_x .

For every $y \in Y_x$, let $\bar{e}_y \in \mathfrak{S}_x$ be the idempotent element represented by the matrix whose only nonzero entry is the element $1 \in D_x$ sitting in the position $(y, y) \in Y_x \times Y_x$. Clearly, $(\bar{e}_y)_{y \in Y_x}$ is a family of orthogonal idempotents in the ring \mathfrak{S}_x , converging to zero in its matrix topology with $\sum_{y \in Y_x} \bar{e}_y = 1$.

Let $Z = \coprod_{x \in X} Y_x$ be the disjoint union of the nonempty sets Y_x . For every $y \in Y_x$, we will consider \bar{e}_y as an element of the ring \mathfrak{S} (which we can do, as \mathfrak{S}_x is naturally a subring in \mathfrak{S} , though with a different unit). Then $(\bar{e}_z)_{z \in Z}$ is a family of orthogonal idempotents in \mathfrak{S} , converging to zero in the product topology of \mathfrak{S} with $\sum_{z \in Z} \bar{e}_z = 1$.

Set $\mathfrak{H} = \mathfrak{H}(\mathfrak{R})$. By Corollary 8.6, there exists a set of orthogonal idempotents $e_z \in \mathfrak{R}$, $z \in Z$, such that $\bar{e}_z = e_z + \mathfrak{H}$, the family of elements $(e_z)_{z \in Z}$ converges to zero in the topology of \mathfrak{R} , and $\sum_{z \in Z} e_z = 1$ in \mathfrak{R} . By Lemma 10.10 (together with Lemma 3.4), there exists a direct sum decomposition $M = \coprod_{z \in Z} M_z$ of the object M such that $e_z = \iota_z \pi_z$ is the composition of the coproduct injection and the product projection $M \rightarrow M_z \rightarrow M$.

The ring \mathfrak{R} can now be viewed as the ring of (row-summable) matrices $r = (r_{w,z})_{w,z \in Z}$ with entries in the topological groups $\mathfrak{R}_{w,z} = \text{Hom}_{\mathbf{A}}(M_w, M_z)$. The topology of $\mathfrak{R}_{w,z}$ can be recovered from that of \mathfrak{R} as the topology on a direct summand with a continuous idempotent projector (see Example 3.6(2)).

The continuous ring homomorphism $p: \mathfrak{R} \rightarrow \mathfrak{S}$ assigns to a matrix $r = (r_{w,z})$ the block matrix $p(r) = (p_{w,z}(r_{w,z}))_{w,z \in Z}$. Here, when there exists $x \in X$ such that both the indices w and z belong to the subset $Y_x \subset Z$, we have a continuous surjective abelian group homomorphism (with a discrete codomain) $p_{w,z}: \mathfrak{R}_{w,z} \rightarrow D_x$. Otherwise, $p_{w,z} = 0$. The ideal $\mathfrak{H} \subset \mathfrak{R}$ consists of all the matrices $h = (h_{w,z})_{w,z \in Z} \in \mathfrak{R}$ such that $h_{w,z} \in \mathfrak{H}_{w,z}$ for all $w, z \in Z$, where $\mathfrak{H}_{w,z} \subset \mathfrak{R}_{w,z}$ is the kernel of the map $p_{w,z}$.

In particular, for every $z \in Z$ the ring $\mathfrak{R}_{z,z} = e_z \mathfrak{R} e_z$ is topologically isomorphic to the topological endomorphism ring $\text{Hom}_{\mathbf{A}}(M_z, M_z)$. The map $p_{z,z}: \mathfrak{R}_{z,z} \rightarrow D_{z,z} = D_x$ is a continuous surjective ring homomorphism with the kernel $\mathfrak{H}_{z,z} \subset \mathfrak{H}$. Since the ideal $\mathfrak{H} \subset \mathfrak{R}$ is topologically left T-nilpotent by assumption, so is the open ideal $\mathfrak{H}_{z,z} \subset \mathfrak{R}_{z,z}$. It follows that $\mathfrak{R}_{z,z}$ is a local ring and $\mathfrak{H}_{z,z}$ is its maximal ideal.

Let $f: M_w \rightarrow M_z$ be a morphism that does not belong to the subgroup $\mathfrak{H}_{w,z} \subset \mathfrak{R}_{w,z}$. Our next aim is to show that f is an isomorphism. Indeed, there exists $x \in X$ such that both w and z belong to Y_x , for otherwise $\mathfrak{H}_{w,z} = \mathfrak{R}_{w,z}$. Consider the element $d = p_{w,z}(f) \in D_x$. Since $d \neq 0$, there exists an inverse element $d^{-1} \in D_x$.

The map $p_{z,w}: \mathfrak{R}_{z,w} \rightarrow D_x$ is surjective, so we can choose a preimage $g: M_z \rightarrow M_w$ of the element d^{-1} . Consider the compositions $fg \in \mathfrak{R}_{z,z}$ and $gf \in \mathfrak{R}_{w,w}$. We have $p_{z,z}(fg) = p_{w,z}(f)p_{z,w}(g) = 1 \in D_x$ and $p_{w,w}(gf) = p_{z,w}(g)p_{w,z}(f) = 1 \in D_x$. Since the rings $R_{z,z}$ and $R_{w,w}$ are local with the residue skew-fields D_x , it follows that the elements fg and gf are invertible in these rings. Hence our morphism f is an isomorphism.

Finally, let $E \subset \mathfrak{R}$ denote the subset of all elements represented by matrices with a single nonzero entry which is not an isomorphism. We have shown that $E \subset \mathfrak{H}$. Since the ideal \mathfrak{H} is topologically left T-nilpotent in \mathfrak{R} by assumption, it follows that so is the set E . The latter observation is equivalent to saying that the family of objects $(M_z \in \mathbf{A})_{z \in Z}$ is locally T-nilpotent. □

Having proved Theorem 10.4, we can now deduce the results promised at the beginning of this section.

Corollary 10.11. *Let \mathfrak{R} be a complete, separated topological ring with a right linear topology. Then the full subcategory of projective \mathfrak{R} -contramodules $\mathfrak{R}\text{-contra}_{\text{proj}}$ is closed under direct limits in $\mathfrak{R}\text{-contra}$ if and only if the topological ring \mathfrak{R} is topologically left perfect.*

Proof: By Corollary 4.4, there exists an associative ring A and a left A -module M such that the topological ring \mathfrak{R} is isomorphic to the topological ring of endomorphisms of the A -module M endowed with the finite topology, that is, $\mathfrak{R} \cong \text{Hom}_A(M, M)^{\text{op}}$.

Assume that the full subcategory $\mathfrak{R}\text{-contra}_{\text{proj}}$ is closed under direct limits in $\mathfrak{R}\text{-contra}$. By Corollary 9.9, this means that the full subcategory $\text{Add}(M) \subset A\text{-mod}$ has split direct limits. According to Theorem 10.1, it follows that the left A -module M has a perfect decomposition. Using Theorem 10.4, we can conclude that the topological ring \mathfrak{R} is topologically left perfect.

Conversely, assume that \mathfrak{R} is topologically left perfect. By Theorem 10.4, this means that the left A -module M has a perfect decomposition. According to Theorem 10.1, it follows that the additive category $\text{Add}(M)$ has split direct limits. From Corollary 9.9 we conclude that the class of all projective left \mathfrak{R} -contramodules is closed under direct limits in $\mathfrak{R}\text{-contra}$. □

Proof of Theorem 10.2: Let $\mathfrak{R} = \text{Hom}_A(M, M)^{\text{op}}$ be the topological ring of endomorphisms of an object M in a topologically agreeable category \mathbf{A} .

Assume that the object $M \in \mathbf{A}$ has a perfect decomposition. By Theorem 10.4, it follows that the topological ring \mathfrak{R} is topologically left perfect. According to Corollary 10.11, this means that the full subcategory $\mathfrak{R}\text{-contra}_{\text{proj}}$ is closed under direct limits in $\mathfrak{R}\text{-contra}$. From Corollary 9.9 we can conclude that the full subcategory $\text{Add}(M) \subset \mathbf{A}$ has split direct limits.

Conversely, assume that the category $\text{Add}(M)$ has split direct limits. By Corollary 9.9, this means that the class of all projective left \mathfrak{R} -contramodules is closed under direct limits in $\mathfrak{R}\text{-contra}$. According to Corollary 10.11, it follows that the topological ring \mathfrak{R} is topologically left perfect. By Theorem 10.4, we can conclude that the object $M \in \mathbf{A}$ has a perfect decomposition. \square

11. Split contramodule categories are semisimple

In this section we prove Theorems 3.15 and 6.6. We also give a negative answer (present a counterexample) to a question posed in [47, Section 1.2].

The following lemma is a straightforward generalization of [54, Proposition V.6.1].

Lemma 11.1. *Let \mathbf{A} be a split abelian category and $M \in \mathbf{A}$ be an object. Then the endomorphism ring $R = \text{Hom}_{\mathbf{A}}(M, M)^{\text{op}}$ is von Neumann regular.* \square

Corollary 11.2. *Let \mathfrak{R} be a complete, separated topological ring with a right linear topology. Assume that the abelian category $\mathfrak{R}\text{-contra}$ is split. Then any closed topologically left T -nilpotent two-sided ideal in \mathfrak{R} is zero.*

First proof: We will even prove that any topologically nil two-sided ideal $J \subset \mathfrak{R}$ is zero. Indeed, by [45, Lemma 7.6(a)], J is contained in the Jacobson radical $H(\mathfrak{R})$ of the ring \mathfrak{R} . On the other hand, the ring \mathfrak{R} (viewed as an abstract ring with the topology ignored) is the opposite ring to the ring of endomorphisms $\text{Hom}^{\mathfrak{R}}(\mathfrak{R}, \mathfrak{R})$ of the free left \mathfrak{R} -contramodule \mathfrak{R} with one generator. By Lemma 11.1, the ring \mathfrak{R} is von Neumann regular; hence $H(\mathfrak{R}) = 0$ [29, Corollary 1.2(c)]. \square

Second proof: Let $\mathfrak{J} \subset \mathfrak{R}$ be a closed topologically left T -nilpotent two-sided ideal. For any left \mathfrak{R} -contramodule \mathfrak{D} , consider the subcontramodule $\mathfrak{C} = \mathfrak{J} \ltimes \mathfrak{D} \subset \mathfrak{D}$ (see [45, Section 2.10]). Since the category $\mathfrak{R}\text{-contra}$ is split by assumption, \mathfrak{C} is a direct summand in \mathfrak{D} . Hence $\mathfrak{J} \ltimes \mathfrak{C} = \mathfrak{C}$. By the contramodule Nakayama lemma [45, Lemma 6.2], it follows that $\mathfrak{C} = 0$. In particular, taking $\mathfrak{D} = \mathfrak{R}$ to be the free left \mathfrak{R} -contramodule with one generator, one obtains $0 = \mathfrak{C} = \mathfrak{J} \ltimes \mathfrak{R} = \mathfrak{J} \subset \mathfrak{R}$. \square

Proof of Theorem 6.6: Let \mathfrak{R} be a complete, separated topological ring with a right linear topology for which the abelian category $\mathfrak{R}\text{-contra}$ is split. Then all left \mathfrak{R} -contramodules are projective, so the class of all projective left \mathfrak{R} -contramodules is closed under direct limits in $\mathfrak{R}\text{-contra}$. By Corollary 10.11, it follows that the topological ring \mathfrak{R} is topologically left perfect.

So the topological Jacobson radical $\mathfrak{H}(\mathfrak{R})$ is a closed topologically left T-nilpotent two-sided ideal in \mathfrak{R} . According to Corollary 11.2, we can conclude that $\mathfrak{H}(\mathfrak{R}) = 0$. Thus $\mathfrak{R} = \mathfrak{S}$ is a topologically semisimple topological ring, and the abelian category $\mathfrak{R}\text{-contra}$ is Ab5 and semisimple by Theorem 6.2(i). \square

Proof of Theorem 3.15: Let \mathbf{A} be a topologically agreeable split abelian category. Given an object $G \in \mathbf{A}$, consider the full subcategory $\text{Add}(G) \subset \mathbf{A}$. Since \mathbf{A} is split abelian, the category $\text{Add}(G)$ is split abelian too. Furthermore, the full subcategory $\text{Add}(G) \subset \mathbf{A}$ is closed under kernels and colimits. Since \mathbf{A} is topologically agreeable, Theorem 3.14(i)(iii) tells us that $\text{Add}(G) \cong \mathfrak{R}\text{-contra}_{\text{proj}}$, where $\mathfrak{R} = \text{Hom}_{\mathbf{A}}(G, G)^{\text{op}}$ is the topological ring of endomorphisms of the object G . So the additive category of projective left \mathfrak{R} -contramodules is split abelian. It follows that all left \mathfrak{R} -contramodules are projective, that is, $\mathfrak{R}\text{-contra} = \mathfrak{R}\text{-contra}_{\text{proj}}$ (cf. the discussion in the proof of Theorem 6.2(iii) \implies (i)). Hence the abelian category $\mathfrak{R}\text{-contra}$ is split. By Theorem 6.6, the category $\mathfrak{R}\text{-contra}$ is Grothendieck and semisimple. Thus so is the category $\text{Add}(G) \cong \mathfrak{R}\text{-contra}_{\text{proj}} = \mathfrak{R}\text{-contra}$. As this holds for every object $G \in \mathbf{A}$, it follows easily that the category \mathbf{A} is Ab5 and semisimple. \square

Remark 11.3. Conversely, it is easy to deduce Theorem 6.6 back from Theorem 3.15. Indeed, for any complete, separated topological ring \mathfrak{R} with right linear topology, the additive category $\mathfrak{R}\text{-contra}_{\text{proj}}$ is topologizable by Remark 3.12 (or by Corollary 4.4 and Theorem 3.14(i)(iii)). If $\mathfrak{R}\text{-contra} = \mathfrak{R}\text{-contra}_{\text{proj}}$ is split abelian and all topologizable split abelian categories are Ab5 and semisimple, then $\mathfrak{R}\text{-contra}$ is Ab5 and semisimple, and so a Grothendieck category.

Example 11.4. It was observed in [47, Section 1.2] that the category $\mathfrak{R}\text{-contra}_{\text{proj}}$ is always agreeable, and the question was asked whether the converse holds, in the following sense. Let \mathbf{B} be a locally presentable abelian category with a projective generator such that the full subcategory of projective objects $\mathbf{B}_{\text{proj}} \subset \mathbf{B}$ is agreeable (cf. Lemma 3.13). Does there exist a complete, separated topological ring \mathfrak{R} with a right linear topology such that \mathbf{B} is equivalent to the category of left \mathfrak{R} -contramodules?

Now we can show that the answer is negative. Let \mathbf{B} be a spectral category. By definition, \mathbf{B} is a Grothendieck abelian category; in particular, it is locally presentable and has a generator G . Furthermore, all objects of \mathbf{B} are projective, so $G \in \mathbf{B}_{\text{proj}} = \mathbf{B}$. Assume that there exists a topological ring \mathfrak{A} such that $\mathbf{B} = \mathfrak{A}\text{-contra}$. By Theorem 6.6, it would then follow that $\mathfrak{A}\text{-contra}$ is a semisimple Grothendieck category, i.e., it is discrete spectral. Thus any nonzero continuous spectral category \mathbf{B} (such as, e.g., the one described in Example 2.9) is a counterexample.

The above argument is quite involved and indirect, as it is based on the results of the theory of direct sum decompositions of modules (the “if” assertion of Theorem 10.1). A somewhat simpler and more direct alternative argument, producing a narrower class of counterexamples, is discussed below in Remark 14.11.

On the other hand, if \mathbf{B} is a cocomplete abelian category with a projective generator P such that the category \mathbf{B}_{proj} is *topologically* agreeable, and $\mathfrak{A} = \text{Hom}_{\mathbf{B}_{\text{proj}}}(P, P)^{\text{op}}$ is the topological ring of endomorphisms of the object P , then the category \mathbf{B} is equivalent to $\mathfrak{A}\text{-contra}$. This equivalence can be constructed by applying the result of Theorem 3.14(iii) to the additive category $\mathbf{A} = \mathbf{B}_{\text{proj}}$ and the object $M = P$. Indeed, one has $\text{Add}(P) = \mathbf{B}_{\text{proj}} \subset \mathbf{B}$, so it remains to take into account the uniqueness assertion in Theorem 3.14(i).

12. Countable topologies and countably generated modules

Let R be an associative ring. A right R -module N is said to be *coperfect* if every descending chain of cyclic R -submodules in N terminates, or equivalently if every descending chain of finitely generated R -submodules in N terminates [12, Theorem 2]. Clearly, any submodule and any quotient module of a coperfect module is coperfect. The class of coperfect right R -modules is also closed under direct limits.

A right R -module N is said to be *Σ -coperfect* if the right R -module $N^{(\omega)} = \bigoplus_{n=0}^{\infty} N$ is coperfect, or equivalently, the right R -module N^n is coperfect for every $n \geq 1$. Clearly, if an R -module N is Σ -coperfect, then the R -module $N^{(X)}$ is Σ -coperfect for any set X .

Let A be an associative ring, let M be a left A -module, and let $R = \text{Hom}_A(M, M)^{\text{op}}$ be its ring of endomorphisms. An A -module M is said to be *endocoperfect* if M is a coperfect right R -module. We will say that an A -module M is *endo- Σ -coperfect* if M is a Σ -coperfect module over its endomorphism ring, that is, a Σ -coperfect R -module.

The following theorem was proved in the paper [3].

Theorem 12.1 ([3, Corollary 2.3]). *Let A be an associative ring and M be a left A -module. Assume that, for every sequence of left A -module endomorphisms $M \rightarrow M \rightarrow M \rightarrow \dots$, the induced morphism of left A -modules $\bigoplus_{n=0}^{\infty} M \rightarrow \varinjlim_{n \geq 0} M$ is split. Then the left A -module M is endo- Σ -coperfect. In particular, any left A -module M with a perfect decomposition is endo- Σ -coperfect. \square*

The aim of this section is to prove the following theorem providing a partial converse assertion to Theorem 12.1.

Theorem 12.2. *Let A be an associative ring and M be a countably generated endo- Σ -coperfect left A -module. Then the left A -module M has a perfect decomposition.*

We start with the following simple lemma interpreting endo- Σ -coperfectness as a property of the topological ring of endomorphisms.

Lemma 12.3 (cf. [3, Proposition 2.2(1) \iff (2)]). *Let A be an associative ring, M be a left A -module, and $\mathfrak{R} = \text{Hom}_A(M, M)^{\text{op}}$ be the topological ring of endomorphisms of M (with the finite topology). Then the right \mathfrak{R} -module M is Σ -coperfect if and only if all discrete right \mathfrak{R} -modules are coperfect.*

Proof: “If”: It is clear from the definition of the finite topology in Example 3.7(1) that M is a discrete right \mathfrak{R} -module. Hence so is $M^{(\omega)}$.

“Only if”: assuming that $M^{(\omega)}$ is a coperfect right \mathfrak{R} -module, we have to show that the cyclic right \mathfrak{R} -module $\mathfrak{R}/\mathfrak{J}$ is coperfect for every open right ideal $\mathfrak{J} \subset \mathfrak{R}$. Indeed, by the definition of the finite topology, there exists a finite set of elements $m_1, \dots, m_n \in M$ such that \mathfrak{J} contains the intersection of the annihilators of the elements m_j in \mathfrak{R} . Let $m = (m_1, \dots, m_n, 0, 0, \dots) \in M^n \subset M^{(\omega)}$ be the related element. Then the cyclic right \mathfrak{R} -module $\mathfrak{R}/\mathfrak{J}$ is a quotient module of the cyclic right \mathfrak{R} -module $m\mathfrak{R} \subset M^{(\omega)}$. It remains to recall that the class of all coperfect right \mathfrak{R} -modules is closed under the passages to submodules and quotients. \square

Let \mathfrak{R} be a complete, separated topological ring with a right linear topology. We say that \mathfrak{R} is *topologically right coperfect* if all discrete right \mathfrak{R} -modules are coperfect. We will see below in Section 14 that all topologically left perfect rings (in the sense of Section 10) are topologically right coperfect.

The proof of Theorem 12.2 is based on the following theorem, which is the key technical result of this section.

Theorem 12.4. *Let \mathfrak{R} be a complete, separated topological ring with a countable base of neighborhoods of zero consisting of open right ideals. Assume that the topological ring \mathfrak{R} is topologically right coperfect. Then \mathfrak{R} is topologically left perfect.*

The proof of Theorem 12.4 follows below in the form of a sequence of lemmas. Given an associative ring R and a right R -module M , the radical of M , denoted by $\text{rad}(M) \subset M$, is defined as the intersection of all the maximal (proper) R -submodules in M .

Lemma 12.5. *For any R -module M , one has $\text{rad}(M/\text{rad}(M)) = 0$.*

Proof: All the maximal submodules of M contain $\text{rad}(M)$, so maximal submodules of M correspond bijectively to maximal submodules of $M/\text{rad}(M)$. □

Lemma 12.6. *Let M be an R -module with $\text{rad}(M) = 0$. Then any simple submodule of M is a direct summand in M .*

Proof: Suppose $S \subset M$ is a simple submodule. Since $\text{rad}(M) = 0$, there exists a maximal R -submodule $N \subset M$ with $S \not\subset N$. Then we have $S \cap N = 0$ since S is simple, and $S + N = M$ since N is maximal; hence $M = S \oplus N$. □

Lemma 12.7. *Any nonzero coperfect R -module has a simple submodule.*

Proof: Any nonzero R -module is either simple or it contains a nonzero proper cyclic submodule. □

Lemma 12.8. *Let M be a coperfect R -module with $\text{rad}(M) = 0$. Then M is a direct sum of simple R -modules.*

Proof: It suffices to show that all cyclic R -submodules of M are semisimple (since a sum of semisimple modules is always semisimple). Let $N \subset M$ be a cyclic submodule. By Lemma 12.7, if $N \neq 0$, then N has a simple R -submodule S_1 . By Lemma 12.6, any simple submodule of N is a direct summand in M , hence also in N . So we have $N = S_1 \oplus N_1$ for some R -submodule $N_1 \subset N$.

The \mathfrak{R} -module N_1 is also cyclic, as a direct summand of a cyclic \mathfrak{R} -module N . If $N_1 \neq 0$, then N_1 has a simple R -submodule S_2 , and the same argument shows that $N_1 = S_2 \oplus N_2$. Proceeding in this way, we get a descending chain of cyclic submodules $N \supseteq N_1 \supseteq N_2 \supseteq \dots$ in the R -module M . By the assumption of coperfectness of M , this chain must terminate; so there exists $k \geq 1$ such that $N_k = 0$. Hence $N = \bigoplus_{i=1}^k S_k$, and we are done. □

The following assertion (as well as some other properties of coperfect modules) can be found in the book [56, Section 31.8].

Lemma 12.9. *Let M be a coperfect R -module. Then $M/\text{rad}(M)$ is a semisimple R -module.*

Proof: Follows from Lemmas 12.5 and 12.8. □

Given an R -module M , we denote by $\text{top}(M)$ the quotient R -module $M/\text{rad}(M)$.

Lemma 12.10. *Assume that a topological ring \mathfrak{R} is topologically right coperfect, and let $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$ be a short exact sequence of discrete right \mathfrak{R} -modules. Then*

- (i) *the short sequence $\text{top}(K) \rightarrow \text{top}(L) \rightarrow \text{top}(M) \rightarrow 0$ is right exact;*
- (ii) *the map $\text{rad}(L) \rightarrow \text{rad}(M)$ is surjective.*

Proof: First of all, one needs to notice that both rad and top are functors $\text{discr-}\mathfrak{R} \rightarrow \text{discr-}\mathfrak{R}$, and there is a short exact sequence of functors $0 \rightarrow \text{rad} \rightarrow \text{Id}_{\text{discr-}\mathfrak{R}} \rightarrow \text{top} \rightarrow 0$, where $\text{Id}_{\text{discr-}\mathfrak{R}}$ denotes the identity functor. These observations do not depend on the assumption of topological coperfectness of \mathfrak{R} yet.

(i) Denote by $(\text{discr-}\mathfrak{R})^{\text{ss}}$ the full subcategory of semisimple discrete right \mathfrak{R} -modules in $\text{discr-}\mathfrak{R}$. Clearly, $(\text{discr-}\mathfrak{R})^{\text{ss}}$ is an abelian category and the inclusion functor $(\text{discr-}\mathfrak{R})^{\text{ss}} \rightarrow \text{discr-}\mathfrak{R}$ is exact.

By Lemma 12.9, one has $\text{top}(M) \in (\text{discr-}\mathfrak{R})^{\text{ss}}$ for any $M \in \text{discr-}\mathfrak{R}$. On the other hand, for any module $N \in (\text{discr-}\mathfrak{R})^{\text{ss}}$, any \mathfrak{R} -module morphism $f: M \rightarrow N$ annihilates the submodule $\text{rad}(M) \subset M$, since $\text{rad}(N) = 0$. So f factorizes (uniquely) as $M \rightarrow \text{top}(M) \rightarrow N$. In other words, this means that $\text{top}: \text{discr-}\mathfrak{R} \rightarrow (\text{discr-}\mathfrak{R})^{\text{ss}}$ is a left adjoint functor to the inclusion $(\text{discr-}\mathfrak{R})^{\text{ss}} \rightarrow \text{discr-}\mathfrak{R}$. It follows that top is right exact as a functor $\text{discr-}\mathfrak{R} \rightarrow (\text{discr-}\mathfrak{R})^{\text{ss}}$, and consequently also as a functor $\text{discr-}\mathfrak{R} \rightarrow \text{discr-}\mathfrak{R}$.

In part (ii), the short exact sequence of functors $0 \rightarrow \text{rad} \rightarrow \text{Id}_{\text{discr-}\mathfrak{R}} \rightarrow \text{top} \rightarrow 0$ applied to the short sequence $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$ produces a short exact sequence of 3-term complexes. A simple diagram chase (or an application of the snake lemma) shows that (i) implies (ii). □

Lemma 12.11. *Assume that a topological ring \mathfrak{R} is topologically right coperfect and has a countable base of neighborhoods of zero. Then, for any open right ideal $\mathfrak{J} \subset \mathfrak{R}$, one has $\text{top}(\mathfrak{R}/\mathfrak{J}) = \mathfrak{R}/(\mathfrak{J} + \mathfrak{H})$, where $\mathfrak{H} \subset \mathfrak{R}$ is the topological Jacobson radical.*

Proof: Let us show that $\text{rad}(\mathfrak{R}/\mathfrak{J}) = (\mathfrak{J} + \mathfrak{H})/\mathfrak{J}$. Consider the projective system of discrete right \mathfrak{R} -modules $\mathfrak{R}/\mathfrak{J}$, where \mathfrak{J} ranges over the open right ideals in \mathfrak{R} , and the projective subsystem formed by the submodules $\text{rad}(\mathfrak{R}/\mathfrak{J}) \subset \mathfrak{R}/\mathfrak{J}$. By Lemma 12.10(ii), for any open right ideals $\mathfrak{J} \subset \mathfrak{I} \subset \mathfrak{R}$, the map $\text{rad}(\mathfrak{R}/\mathfrak{J}) \rightarrow \text{rad}(\mathfrak{R}/\mathfrak{I})$ is surjective. Since the poset of all open right ideals $\mathfrak{J} \subset \mathfrak{R}$ has a countable cofinal subposet, it follows that the projection map

$$\varprojlim_{\mathfrak{J} \subset \mathfrak{R}} \text{rad}(\mathfrak{R}/\mathfrak{J}) \longrightarrow \text{rad}(\mathfrak{R}/\mathfrak{J})$$

is surjective.

The projective limit $\varprojlim_{\mathfrak{J} \subset \mathfrak{R}} \text{rad}(\mathfrak{R}/\mathfrak{J})$ is a subgroup in $\varprojlim_{\mathfrak{J} \subset \mathfrak{R}} \mathfrak{R}/\mathfrak{J} = \mathfrak{R}$. Let us compute this subgroup. We have

$$\text{rad}(\mathfrak{R}/\mathfrak{J}) = \frac{\bigcap_{\mathfrak{M} \supset \mathfrak{J}} \mathfrak{M}}{\mathfrak{J}},$$

where the intersection is taken over all the (necessarily open) maximal right ideals $\mathfrak{M} \subset \mathfrak{R}$ containing \mathfrak{J} . Consequently,

$$\varprojlim_{\mathfrak{J} \subset \mathfrak{R}} \text{rad}(\mathfrak{R}/\mathfrak{J}) = \bigcap_{\mathfrak{J} \subset \mathfrak{R}} \bigcap_{\mathfrak{M} \supset \mathfrak{J}} \mathfrak{M} = \bigcap_{\mathfrak{M}} \mathfrak{M} = \mathfrak{H}.$$

We have shown that the natural surjective map $\mathfrak{R} \rightarrow \mathfrak{R}/\mathfrak{J}$ restricts to a surjective map $\mathfrak{H} \rightarrow \text{rad}(\mathfrak{R}/\mathfrak{J})$. Therefore, $\text{rad}(\mathfrak{R}/\mathfrak{J}) = (\mathfrak{J} + \mathfrak{H})/\mathfrak{J}$, as desired. \square

Now at last we can prove the promised theorems.

Proof of Theorem 12.4: By [45, Lemma 2.3], the ideal \mathfrak{H} is strongly closed in \mathfrak{R} . In particular, the quotient ring $\mathfrak{S} = \mathfrak{R}/\mathfrak{H}$ is complete in its quotient topology.

By [45, Corollary 7.7], the topological Jacobson radical \mathfrak{H} of the ring \mathfrak{R} is topologically left T-nilpotent. It remains to prove that the topological ring \mathfrak{S} is topologically semisimple.

For any topological ring \mathfrak{R} with a closed two-sided ideal $\mathfrak{H} \subset \mathfrak{R}$, the category of discrete modules over the topological quotient ring $\mathfrak{R}/\mathfrak{H}$ is equivalent to the full subcategory in $\text{discr-}\mathfrak{R}$ consisting of all the discrete right \mathfrak{R} -modules annihilated by \mathfrak{H} . In the situation at hand, in view of Theorem 6.2(ii), it remains to show that every discrete right \mathfrak{R} -module N annihilated by \mathfrak{H} is semisimple.

Since a sum of semisimple modules is semisimple, it suffices to consider the case when N is a cyclic discrete right \mathfrak{R} -module annihilated by \mathfrak{H} . So we have $N \cong \mathfrak{R}/\mathfrak{J}$, where $\mathfrak{J} \subset \mathfrak{R}$ is an open right ideal containing \mathfrak{H} . Now by Lemma 12.11 we have $\text{rad}(\mathfrak{R}/\mathfrak{J}) = 0$, and it remains to invoke Lemma 12.8. \square

Proof of Theorem 12.2: Given a set of generators $\{m_y \in M \mid y \in Y\}$ of a left A -module M , the annihilators of finite subsets of $\{m_y\}$ form a base of neighborhoods of zero in the topological ring $\mathfrak{R} = \text{Hom}_A(M, M)^{\text{op}}$. Hence \mathfrak{R} has a countable base of neighborhoods of zero whenever M is countably generated.

Now let M be a countably generated endo- Σ -coperfect left A -module. Then all discrete right \mathfrak{R} -modules are coperfect by Lemma 12.3. According to Theorem 12.4, it follows that the topological ring \mathfrak{R} is topologically left perfect. Applying Theorem 10.4 to the object $M \in \mathbf{A} = A\text{-mod}$, we conclude that the left A -module M has a perfect decomposition. \square

13. Topological coherence and coperfectness

In this section we discuss some results of the paper [52], which imply that a topologically right coperfect topological ring \mathfrak{R} is topologically left perfect under the additional assumption of *topological right coherence* of \mathfrak{R} .

We refer to the book [1, Sections 1.A and 2.A] for the definitions of *finitely accessible* and *locally finitely presentable* categories. In this section, we are interested in additive categories. Let us point out that the terminology in the literature is not consistent: what we, following [1], call finitely accessible categories are called “locally finitely presented categories” in the papers [17, 34].

Let \mathbf{A} be a finitely accessible additive category and $\mathbf{A}_{\text{fp}} \subset \mathbf{A}$ be the full subcategory of finitely presentable objects. Then \mathbf{A}_{fp} is an essentially small idempotent-complete additive category. Conversely, any small idempotent-complete additive category \mathbf{D} can be realized as the category \mathbf{A}_{fp} for a finitely accessible additive category \mathbf{A} . The category \mathbf{A} can be recovered as the category of ind-objects in \mathbf{D} [1, Theorem 2.26]. Alternatively, \mathbf{A} is the category of flat functors $\mathbf{D}^{\text{op}} \rightarrow \mathbf{Ab}$ (where, given a small preadditive category \mathbf{D} , a contravariant additive functor $F: \mathbf{D}^{\text{op}} \rightarrow \mathbf{Ab}$ is called *flat* if the tensor product $G \mapsto F \otimes_{\mathbf{D}} G$ is an exact functor on the abelian category of covariant additive functors $G: \mathbf{D} \rightarrow \mathbf{Ab}$) [17, Theorem 1.4], [34, Proposition 5.1].

A finitely accessible additive category \mathbf{A} is locally finitely presentable if and only if \mathbf{A} has cokernels. In this case, the full subcategory \mathbf{A}_{fp} is closed under cokernels in \mathbf{A} . Conversely, if the category \mathbf{A}_{fp} has cokernels, then so does \mathbf{A} [17, Section 2.2], [34, Lemma 5.7]. In this case, an additive functor $(\mathbf{A}_{\text{fp}})^{\text{op}} \rightarrow \mathbf{Ab}$ is flat if and only if it is left exact, i.e., takes cokernels in \mathbf{A}_{fp} to kernels in \mathbf{Ab} . So the category \mathbf{A} can be recovered as the category of left exact functors $(\mathbf{A}_{\text{fp}})^{\text{op}} \rightarrow \mathbf{Ab}$.

The definition of a *locally finitely generated* category can be found in [1, Section 1.E]. Any locally finitely generated abelian category (hence, in particular, any locally finitely presentable abelian category) is Grothendieck. Moreover, any locally finitely presentable abelian category is locally finitely generated and an object of it is finitely generated if and only if it is a quotient object of a finitely presentable object [1, Proposition 1.69].

Let \mathcal{C} be a locally finitely generated abelian category. An object $C \in \mathcal{C}$ is said to be *coherent* if C is finitely generated and, for every finitely generated object $D \in \mathcal{C}$, the kernel of any morphism $D \rightarrow C$ in \mathcal{C} is also finitely generated. In any locally finitely generated abelian category, the class of all coherent objects is closed under kernels, cokernels, and extensions, so coherent objects form an abelian subcategory. A locally finitely generated abelian category \mathcal{C} is called *locally coherent* if has a set of coherent generators [52, Section 2].

An abelian category \mathcal{C} is locally coherent if and only if it is locally finitely presentable and the full subcategory \mathcal{C}_{fp} is closed under kernels in \mathcal{C} . Equivalently, a locally finitely presentable abelian category \mathcal{C} is locally coherent if and only if any finitely generated subobject of a finitely presentable object in \mathcal{C} is finitely presentable, and if and only if the category \mathcal{C}_{fp} , viewed as an abstract category, is abelian (in this case, the inclusion functor $\mathcal{C}_{\text{fp}} \rightarrow \mathcal{C}$ is exact). In a locally coherent abelian category, an object is coherent if and only if it is finitely presentable. Conversely, for any small abelian category \mathcal{D} there exists a unique locally coherent abelian category \mathcal{C} such that the category \mathcal{D} is equivalent to \mathcal{C}_{fp} . The category \mathcal{C} can be recovered as the category of ind-objects in \mathcal{D} , or which is the same, the category of flat (equivalently, left exact) functors $\mathcal{D}^{\text{op}} \rightarrow \text{Ab}$.

A locally finitely generated abelian category \mathcal{C} is said to be *locally coperfect* [52, Section 3] if it has a set of (finitely generated) generators (C_α) with the property that any descending chain of finitely generated subobjects in any one of the objects C_α terminates. Equivalently, this means that any descending chain of finitely generated subobjects in any object of \mathcal{C} terminates. A locally coherent abelian category \mathcal{C} is locally coperfect if and only if the abelian category \mathcal{C}_{fp} is Artinian, that is, any descending chain of (sub)objects in the category \mathcal{C}_{fp} terminates.

Let \mathfrak{R} be a complete, separated topological ring with right linear topology. Then the abelian category of discrete right \mathfrak{R} -modules $\text{discr-}\mathfrak{R}$ is locally finitely generated. Moreover, an object $N \in \text{discr-}\mathfrak{R}$ is finitely generated in the sense of the definition in [1, Section 1.E] if and only if it is a finitely generated right \mathfrak{R} -module (so our terminology is consistent).

The category $\text{discr-}\mathfrak{R}$ is locally coperfect if and only if all discrete right \mathfrak{R} -modules are coperfect (in the sense of the discussion in Section 12). We recall that in this case the topological ring \mathfrak{R} is said to be *topologically right coperfect*.

Following [52, Definition 4.3], we say that the topological ring \mathfrak{R} is *topologically right coherent* if the category $\text{discr-}\mathfrak{R}$ is locally coherent.

Lemma 13.1 ([52, Remark 2 in Section 4]). *The topological ring \mathfrak{R} is topologically right coherent if and only if there exists a set of open right ideals B forming a base of neighborhoods of zero in \mathfrak{R} such that, for every pair of open right ideals \mathfrak{I} and $\mathfrak{J} \in B$ and any integer $n \geq 1$, the kernel of any right \mathfrak{R} -module morphism $(\mathfrak{R}/\mathfrak{I})^n \rightarrow \mathfrak{R}/\mathfrak{J}$ is a finitely generated right \mathfrak{R} -module.*

Proof: Assume that the category $\text{discr-}\mathfrak{R}$ is locally coherent. Then let B denote the set of all open right ideals $\mathfrak{J} \subset \mathfrak{R}$ such that the right \mathfrak{R} -module $\mathfrak{R}/\mathfrak{J}$ is a coherent object of $\text{discr-}\mathfrak{R}$. Let us show that B is a base of neighborhoods of zero in \mathfrak{R} . Indeed, let $\mathfrak{I} \subset \mathfrak{R}$ be an open right ideal, and let M be a coherent object of $\text{discr-}\mathfrak{R}$ admitting an epimorphism $M \rightarrow \mathfrak{R}/\mathfrak{I}$. Let $m \in M$ be any preimage of the element $1 + \mathfrak{I} \in \mathfrak{R}/\mathfrak{I}$, and let \mathfrak{J} be the annihilator of m in \mathfrak{R} . Then $\mathfrak{R}/\mathfrak{J}$ is a finitely generated subobject of a coherent object M , hence the object $\mathfrak{R}/\mathfrak{J} \in \text{discr-}\mathfrak{R}$ is coherent and $\mathfrak{J} \in B$. By construction, we have $\mathfrak{J} \subset \mathfrak{I}$.

To show that B is closed under finite intersections, one observes that, for any open right ideals $\mathfrak{J}', \mathfrak{J}'' \in B$, the object $\mathfrak{R}/(\mathfrak{J}' \cap \mathfrak{J}'') \in \text{discr-}\mathfrak{R}$ is a finitely generated subobject of the coherent object $\mathfrak{R}/\mathfrak{J}' \oplus \mathfrak{R}/\mathfrak{J}''$. Finally, for any open right ideal $\mathfrak{I} \subset \mathfrak{R}$ and any $\mathfrak{J} \in B$, the kernel of a morphism from the finitely generated object $(\mathfrak{R}/\mathfrak{I})^n$ to the coherent object $\mathfrak{R}/\mathfrak{J}$ is finitely generated.

Conversely, let B be a base of neighborhoods of zero in \mathfrak{R} satisfying the condition of the lemma. Then the object $\mathfrak{R}/\mathfrak{J} \in \text{discr-}\mathfrak{R}$ is coherent for all $\mathfrak{J} \in B$. Indeed, let M be a finitely generated discrete right \mathfrak{R} -module and $M \rightarrow \mathfrak{R}/\mathfrak{J}$ be a morphism. Then there exists an open right ideal $\mathfrak{I} \in B$ and an integer $n \geq 1$ such that there is a surjective \mathfrak{R} -module morphism $(\mathfrak{R}/\mathfrak{I})^n \rightarrow M$. The kernel of the composition $(\mathfrak{R}/\mathfrak{I})^n \rightarrow M \rightarrow \mathfrak{R}/\mathfrak{J}$ is finitely generated by assumption, hence the kernel of the morphism $M \rightarrow \mathfrak{R}/\mathfrak{J}$ is finitely generated too. Now the discrete right \mathfrak{R} -modules $\mathfrak{R}/\mathfrak{J}, \mathfrak{J} \in B$ form a set of coherent generators of the locally finitely generated abelian category $\text{discr-}\mathfrak{R}$. \square

Proposition 13.2 ([52, Theorem 6]). *Any locally coperfect locally coherent abelian category \mathcal{C} can be realized as the category of discrete right modules over a (topologically right coperfect and right coherent) topological ring, $\mathcal{C} \cong \text{discr-}\mathfrak{R}$.*

Proof: For any locally coherent abelian category \mathcal{C} , one considers the category \mathbf{A} of all left exact, direct limit-preserving covariant functors $\mathcal{C} \rightarrow \mathbf{Ab}$. Then \mathbf{A} is also a locally coherent abelian category. In the terminology of [52], \mathbf{A} is called the *conjugate* locally coherent abelian category to \mathcal{C} . Then \mathcal{C} is also conjugate to \mathbf{A} . In fact, \mathcal{C} is the category of left exact functors $(\mathcal{C}_{\text{fp}})^{\text{op}} \rightarrow \mathbf{Ab}$ and \mathbf{A} is the category of left exact functors $\mathcal{C}_{\text{fp}} \rightarrow \mathbf{Ab}$, so the small abelian categories \mathbf{A}_{fp} and $(\mathcal{C}_{\text{fp}})^{\text{op}}$ are naturally equivalent.

Hence the category \mathcal{C}_{fp} is Artinian if and only if the category \mathbf{A}_{fp} is Noetherian. It follows that the category \mathcal{C} is locally coperfect if and only if the category \mathbf{A} is locally Noetherian. Assuming that this is the case, the paper [52] suggests choosing a *big* injective object $J \in \mathbf{A}$, which means an object such that the full subcategory of injective objects $\mathbf{A}_{\text{inj}} \subset \mathbf{A}$ coincides with $\text{Add}(J)$. Equivalently, an injective object $J \in \mathbf{A}$ is big if and only if it contains a representative of every isomorphism class of indecomposable injectives in \mathbf{A} . Any big injective object is an injective cogenerator, but the converse is not true, in general.

Now let $F: \mathcal{C} \rightarrow \mathbf{Ab}$ be the functor corresponding to the chosen big injective object $J \in \mathbf{A}$. Then there is a complete, separated topological ring \mathfrak{R} with a right linear topology and a category equivalence $\mathcal{C} \cong \text{discr-}\mathfrak{R}$ transforming the functor $F: \mathcal{C} \rightarrow \mathbf{Ab}$ into the forgetful functor $\text{discr-}\mathfrak{R} \rightarrow \mathbf{Ab}$. This is one of the results of [52, Theorem 6].

Essentially, $\mathfrak{R} = \text{Hom}_{\mathbf{A}}(J, J)^{\text{op}}$ is the opposite ring to the ring of endomorphisms of the object J , endowed with the finite topology, as in Example 3.7(2). This means that annihilators of finitely generated (= Noetherian) subobjects $E \subset J$ form a base of neighborhoods of zero in \mathfrak{R} . Notice that the functor $F|_{\mathcal{C}_{\text{fp}}}: \mathcal{C}_{\text{fp}} \rightarrow \mathbf{Ab}$ can be computed as the functor $\text{Hom}_{\mathbf{A}}(-, J)|_{\mathbf{A}_{\text{fp}}}: (\mathbf{A}_{\text{fp}})^{\text{op}} \rightarrow \mathbf{Ab}$. Now, for any object $N \in \mathbf{A}$, the abelian group $\text{Hom}_{\mathbf{A}}(N, J)$ has a natural structure of right \mathfrak{R} -module, and this right \mathfrak{R} -module is discrete for $N \in \mathbf{A}_{\text{fp}}$. This defines an exact functor $\mathcal{C}_{\text{fp}} = (\mathbf{A}_{\text{fp}})^{\text{op}} \rightarrow \text{discr-}\mathfrak{R}$, which can be uniquely extended to a direct limit-preserving exact functor $G: \mathcal{C} \rightarrow \text{discr-}\mathfrak{R}$.

It is explained in [52, first part of proof of Theorem 4 in Section 4] that the functor G takes the objects of \mathcal{C}_{fp} to coherent objects of $\text{discr-}\mathfrak{R}$. In particular, since $G(E^{\text{op}}) = \text{Hom}_{\mathbf{A}}(E, J) \cong \mathfrak{R}/\mathfrak{I}$ for any Noetherian subobject $E \subset J$ and its annihilator ideal $\mathfrak{I} \subset \mathfrak{R}$, the discrete right \mathfrak{R} -module $\mathfrak{R}/\mathfrak{I}$ is coherent. So the category $\text{discr-}\mathfrak{R}$ has a set of coherent generators and the topological ring \mathfrak{R} is topologically right coherent. Furthermore, by [52, Lemma 4.1], the functor $G|_{\mathcal{C}_{\text{fp}}}: \mathcal{C}_{\text{fp}} \rightarrow \text{discr-}\mathfrak{R}$ is fully faithful.

Since coherent objects in a locally coherent category (here the category $\text{discr-}\mathfrak{R}$) are finitely presentable, it follows that the functor $G: \mathbf{C} \rightarrow \text{discr-}\mathfrak{R}$ is fully faithful too. Hence the essential image of G is a full subcategory closed under kernels, cokernels, and coproducts in $\text{discr-}\mathfrak{R}$. Since the essential image of G contains a set of generators of $\text{discr-}\mathfrak{R}$, as we have seen, we can conclude that G is an equivalence of categories. \square

Theorem 13.3. *Let \mathfrak{R} be a topologically right coherent, topologically right coperfect, complete and separated topological ring with a right linear topology. Then the topological ring \mathfrak{R} is topologically left perfect.*

Proof: The category of discrete right \mathfrak{R} -modules $\mathbf{C} = \text{discr-}\mathfrak{R}$ is a locally coperfect, locally coherent abelian category. Left exact, direct limit-preserving functors $\mathbf{C} \rightarrow \mathbf{Ab}$ form a locally Noetherian abelian category \mathbf{A} conjugate to \mathbf{C} (see the discussion in the previous proof). In particular, the forgetful functor $F: \text{discr-}\mathfrak{R} \rightarrow \mathbf{Ab}$ corresponds to an object $J \in \mathbf{A}$. The functor $F|_{\mathbf{C}_{\text{fp}}}$ can be computed as the functor $\text{Hom}_{\mathbf{A}}(-, J)|_{\mathbf{A}_{\text{fp}}}$. Since the functor $F|_{\mathbf{C}_{\text{fp}}}$ is exact, it follows that the functor $\text{Hom}_{\mathbf{A}}(-, J)$ is exact in restriction to \mathbf{A}_{fp} ; thus the object $J \in \mathbf{A}$ is injective. Following [52, Corollary 5.1], J is actually a big injective object in \mathbf{A} (but we do not need to use this fact).

The topological ring \mathfrak{R} can be uniquely recovered as the opposite ring to the ring of endomorphisms of the functor F , endowed with the finite topology, as explained in Section 4 above. This is the same thing as the ring of endomorphisms of the object $J \in \mathbf{A}$, endowed with the finite topology, as in the proof of Proposition 13.2.

According to Example 9.4, the category $\text{Add}(J) = \mathbf{A}_{\text{inj}} \subset \mathbf{A}$ has split direct limits. By Corollary 9.9, it follows that the full subcategory $\mathfrak{R}\text{-contra}_{\text{proj}} \subset \mathfrak{R}\text{-contra}$ is closed under direct limits. It remains to apply Corollary 10.11 in order to conclude that \mathfrak{R} is topologically left perfect.

Alternatively, one can prove directly from the definition that, for any injective object J in a locally Noetherian category \mathbf{A} , the decomposition of J into a direct sum of indecomposable injectives is a perfect decomposition. By Theorem 10.4, it follows that the topological ring $\mathfrak{R} = \text{Hom}_{\mathbf{A}}(J, J)^{\text{op}}$ is topologically left perfect. \square

14. Topologically perfect topological rings

Recall the definition given in Section 10: a complete, separated topological ring \mathfrak{R} with right linear topology is *topologically left perfect* if its topological Jacobson radical $\mathfrak{H} = \mathfrak{H}(\mathfrak{R})$ is topologically left T-nilpotent and strongly closed in \mathfrak{R} , and the quotient ring $\mathfrak{S} = \mathfrak{R}/\mathfrak{H}$ in its quotient topology is topologically semisimple. In this section we discuss equivalent conditions characterizing topologically perfect topological rings.

We consider a number of such conditions. Some of them are indeed equivalent to topological perfectness, as we prove. The equivalence of the rest of the conditions is a conjecture. This conjecture is equivalent to a positive answer to Question 0.1, as we explain. In addition to the results obtained above in this paper, we make use of some results from the papers [45] and [9].

Given a full subcategory \mathcal{C} in a category \mathbf{A} , a morphism $c: C \rightarrow A$ in \mathbf{A} is said to be a \mathcal{C} -precover (of the object A) if $C \in \mathcal{C}$ and for every morphism $c': C' \rightarrow A$ in \mathbf{A} with $C' \in \mathcal{C}$ there exists a morphism $f: C' \rightarrow C$ such that $c' = cf$. A \mathcal{C} -precover $c: C \rightarrow A$ is called a \mathcal{C} -cover (of A) if the equation $cf = c$ for a morphism $f: C \rightarrow C$ implies that f is an isomorphism. We refer to the papers [45, Sections 4 and 10] and [9, Sections 3 and 7] for a discussion of projective covers in contramodule categories.

Let \mathfrak{R} be a complete, separated topological ring with a right linear topology. A left \mathfrak{R} -contramodule \mathfrak{F} is said to be flat if the functor of contratensor product $-\odot_{\mathfrak{R}}\mathfrak{F}: \text{discr}\mathfrak{R} \rightarrow \text{Ab}$ is exact. All projective left \mathfrak{R} -contramodules are flat, and the class of all flat left \mathfrak{R} -contramodules is closed under direct limits in $\mathfrak{R}\text{-contra}$, so all the direct limits of projective left \mathfrak{R} -contramodules are flat [45, Section 3].

It is not known whether the converse is true (i.e., whether an analogue of the Govorov–Lazard description of flat modules holds for contramodules)². Nevertheless, we can prove the following theorem, which is the main result of this section.

Theorem 14.1. *Let \mathfrak{R} be a complete, separated topological ring with a right linear topology. Then the following conditions are equivalent:*

- (i) *all flat left \mathfrak{R} -contramodules have projective covers;*
- (i') *all the direct limits of projective left \mathfrak{R} -contramodules have projective covers in $\mathfrak{R}\text{-contra}$;*
- (ii) *all left \mathfrak{R} -contramodules have projective covers;*
- (iii) *all flat left \mathfrak{R} -contramodules are projective;*
- (iii') *the class of all projective left \mathfrak{R} -contramodules $\mathfrak{R}\text{-contra}_{\text{proj}}$ is closed under direct limits in $\mathfrak{R}\text{-contra}$;*
- (iv) *the topological ring \mathfrak{R} is topologically left perfect.*

Proof: (ii) \implies (i), (iii) \implies (i), and (iii') \implies (i') Obvious.

(i) \implies (i') and (iii) \implies (iii') These implications hold because all the direct limits of projective contramodules are flat (see the above discussion).

²A counterexample is available now: Example 10.2 in [46].

(iv) \implies (iii) The argument uses the construction of the reduction functor $\mathfrak{R}\text{-contra} \longrightarrow \mathfrak{S}\text{-contra}$ taking a left \mathfrak{R} -contramodule \mathfrak{C} to the left \mathfrak{S} -contramodule $\mathfrak{C}/\mathfrak{H} \ltimes \mathfrak{C}$, where $\mathfrak{H} = \mathfrak{H}(\mathfrak{R})$ (see [45, Sections 2.10 and 2.12]). By [45, Theorem 9.3], a flat left \mathfrak{R} -contramodule \mathfrak{F} is projective if and only if the left \mathfrak{S} -contramodule $\mathfrak{F}/\mathfrak{H} \ltimes \mathfrak{F}$ is projective. By Theorem 6.2(i), all left \mathfrak{S} -contramodules are projective.

(iii') \iff (iv) This is Corollary 10.11.

(i') \implies (iii') This is [9, Corollary 7.5].

(iii) or (iii') \implies (ii) This is [45, Theorem 10.1 or Corollary 10.2].

(iv) \implies (ii) Provable along the lines of [45, second proof of Theorem 10.4]. \square

Remark 14.2. The above proofs of the implications (iv) \implies (iii) \implies (iii') in Theorem 14.1 allow us to obtain a contramodule-based proof of the implication “only if” in Theorem 10.1. Indeed, let A be an associative ring and M be a left A -module with a perfect decomposition. Consider the ring of endomorphisms $\mathfrak{R} = \text{Hom}_A(M, M)^{\text{op}}$, and endow it with the finite topology (see Example 3.7(1)).

By Theorem 10.4, the topological ring \mathfrak{R} is topologically left perfect. So we conclude from the above argument based on [45, Theorem 9.3] that the class of all projective left \mathfrak{R} -contramodules $\mathfrak{R}\text{-contra}_{\text{proj}}$ is closed under direct limits in $\mathfrak{R}\text{-contra}$. By Corollary 9.9, it follows that the full subcategory $\text{Add}(M) \subset A\text{-mod}$ has split direct limits.

Let \mathfrak{R} be a complete, separated topological ring with a right linear topology. A *Bass flat left \mathfrak{R} -contramodule* [45, Section 5] is, by definition, the direct limit in the category of left \mathfrak{R} -contramodules $\mathfrak{R}\text{-contra}$ of a sequence of free left \mathfrak{R} -contramodules \mathfrak{R} with one generator and left \mathfrak{R} -contramodule morphisms between them,

$$\mathfrak{B} = \varinjlim (\mathfrak{R} \longrightarrow \mathfrak{R} \longrightarrow \mathfrak{R} \longrightarrow \dots).$$

Now we can formulate our conjecture.

Conjecture 14.3. *Let \mathfrak{R} be a complete, separated topological associative ring with a right linear topology. Then the following conditions are equivalent to each other and to the conditions listed in Theorem 14.1:*

- (i^b) *all Bass flat left \mathfrak{R} -contramodules have projective covers;*
- (iii^b) *all Bass flat left \mathfrak{R} -contramodules are projective;*
- (v) *all discrete right \mathfrak{R} -modules are coperfect.*

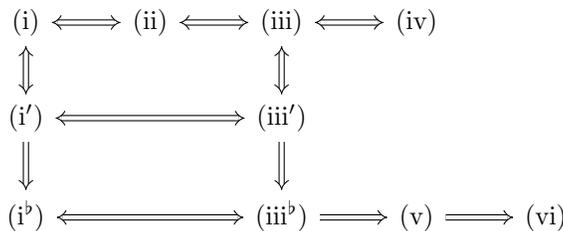
A partial generalization of Conjecture 14.3 to locally presentable abelian categories with a projective generator is formulated in [8, Main Conjecture 4.6].

The following additional property will be useful in the discussion below. We do not expect it to be equivalent to the other conditions in Conjecture 14.3 in general, but sometimes it is, as we will see:

- (vi) all the discrete quotient rings of \mathfrak{R} (i.e., the quotient rings of \mathfrak{R} by its open two-sided ideals) are left perfect.

The next theorem lists those implications in the above conjecture that we can prove unconditionally.

Theorem 14.4. *The following implications between the properties in Theorem 14.1, in Conjecture 14.3, and the additional property (vi) hold true:*



Proof: The equivalence of all the conditions in the first two lines is the result of Theorem 14.1. The equivalence $(i^b) \iff (iii^b)$ is provided by [9, Corollary 3.10].

The rest is essentially explained in [45, proofs of Theorems 11.1 and 13.4] (though the generality level of the exposition in [45] is somewhat more restricted than in the present paper; cf. Remark 14.6 below). Specifically, the implication $(iii^b) \implies (v)$ is [45, Proposition 5.3 and Lemma 7.3]. The implication $(i^b) \implies (vi)$ is [45, Corollary 5.7]. The implication $(v) \implies (vi)$ is explained in [45, proof of Theorem 11.1]. \square

In the rest of this section we discuss some special cases when one can prove Conjecture 14.3 and/or some partial/related results in its direction.

Theorem 14.5. *The implication $(vi) \implies (iv)$ holds under any one of the following three additional assumptions of [45, Section 11]: either*

- (a) *the ring \mathfrak{R} is commutative, or*
- (b) *the topological ring \mathfrak{R} has a countable base of neighborhoods of zero consisting of two-sided ideals, or*

(c) *the topological ring \mathfrak{R} has a base of neighborhoods of zero consisting of two-sided ideals and only a finite number of semisimple Artinian discrete quotient rings,*

as well as under the further assumption (d) formulated in [45, Section 13].

Proof: This is the result of [45, Propositions 11.2 and 13.5] together with [45, Lemma 9.1] (cf. our Lemma 10.3). \square

Remark 14.6. The main difference between the generality levels of the expositions in [45] and in the present paper is that the topological quotient ring $\mathfrak{S} = \mathfrak{R}/\mathfrak{H}(\mathfrak{R})$ is assumed to be the topological product of some discrete simple Artinian rings in the paper [45] (see [45, Section 9]). This, of course, means rings of *finite-sized* matrices over skew-fields. In the present paper, the same role is played by *topologically* semisimple topological rings, which are topological products of the topological rings of *infinite-sized*, row-finite matrices over skew-fields (see Theorem 6.2).

The explanation is that the idea of the proof of the main results in [45] is to deduce the condition (iv) from (vi), as in Theorem 14.5. This is doable under one of the assumptions (a), (b), (c), or (d), which are designed to make this approach work. Any one of these four assumptions, under which the main results of the paper [45] are obtained, implies that \mathfrak{S} is the topological product of discrete simple Artinian rings.

It is instructive to consider the particular case when $\mathfrak{R} = \mathfrak{Mat}_Y(D)$ is the topological ring of row-finite matrices of some infinite size Y over a skew-field D . Then \mathfrak{R} has *no* nonzero proper closed two-sided ideals. As \mathfrak{R} itself is not discrete, it follows that \mathfrak{R} has no nonzero discrete quotient rings. Though not a counterexample to a possible implication (vi) \implies (iv), this simple example seems to suggest that topological rings with right linear topologies may have too few discrete quotient rings for the condition (vi) to be interesting in the general case. That is why we doubt the general validity of (vi) \implies (iv), and therefore do not include the condition (vi) in the list of conditions in Conjecture 14.3. (Cf. Proposition 14.15 below, which concerns the case of a topological ring \mathfrak{R} with a topology base of two-sided ideals.)

Corollary 14.7. *Conjecture 14.3 holds true under any one of the additional assumptions (a), (b), (c), or (d) of the paper [45]. In particular, the conjecture is true for commutative rings \mathfrak{R} .*

Proof: Follows from Theorems 14.4 and 14.5. \square

Theorem 14.8. *Conjecture 14.3 holds true for any complete, separated topological ring \mathfrak{R} with a countable base of neighborhoods of zero consisting of open right ideals.*

Proof: In view of Theorem 14.4, it suffices to prove the implication (v) \implies (iv). For topological rings \mathfrak{R} with a countable base of neighborhoods of zero, it is provided by Theorem 12.4. \square

Remark 14.9. The implication (v) \implies (iv) in Conjecture 14.3 is true *if and only if* the answer to Question 0.1 is positive.

Indeed, given a left A -module M , one considers its ring of endomorphisms $\mathfrak{R} = \text{Hom}_A(M, M)^{\text{op}}$, endowed with the finite topology. Conversely, given a complete, separated topological ring \mathfrak{R} with right linear topology, by Corollary 4.4 one can find an associative ring A and a left A -module M such that \mathfrak{R} is isomorphic to $\text{Hom}_A(M, M)^{\text{op}}$ as a topological ring.

Now, according to Lemma 12.3, the right \mathfrak{R} -module M is Σ -coperfect if and only if the topological ring \mathfrak{R} satisfies condition (v). On the other hand, by Theorem 10.4, the left A -module M has a perfect decomposition if and only if the topological ring \mathfrak{R} satisfies condition (iv). (Cf. the proof of Theorem 12.2.)

It follows that the whole Conjecture 14.3 is equivalent to a positive answer to Question 0.1.

Corollary 14.10. *Any endo- Σ -coperfect left module M over an associative ring A with a commutative endomorphism ring $\mathfrak{R} = \text{Hom}_A(M, M)^{\text{op}}$ has a perfect decomposition.*

Proof: Follows from Corollary 14.7, Lemma 12.3, and Theorem 10.4, as explained in the preceding remark. \square

Remark 14.11. Now we can present an alternative argument for the negative answer to the question posed in [47, Section 1.2], which allows us to avoid using the theory of direct sum decompositions of modules, as discussed above in Example 11.4. Let \mathbb{B} be a nondiscrete spectral category with a generator G such that the ring of endomorphisms $R = \text{Hom}_{\mathbb{B}}(G, G)^{\text{op}}$ is commutative. It is explained in Example 2.9 how such categories \mathbb{B} can be produced. We would like to show that there does *not* exist a complete, separated topological ring \mathfrak{T} with right linear topology such that the abelian category \mathbb{B} is equivalent to \mathfrak{T} -contra.

Indeed, suppose that such a topological ring \mathfrak{T} exists. Then the category \mathbb{B}_{proj} is topologically agreeable by Remark 3.12. Hence the ring $\mathfrak{R} = R = \text{Hom}_{\mathbb{B}}(G, G)^{\text{op}}$ also acquires a topology such that there is an equivalence of categories $\mathbb{B} \cong \mathfrak{R}$ -contra taking the projective generator $G \in \mathbb{B}$

to the free \mathfrak{R} -contramodule with one generator $\mathfrak{R} \in \mathfrak{R}\text{-contra}$ (see the last paragraph of Example 11.4).

Now if the category $\mathfrak{R}\text{-contra}$ is split abelian, then all left \mathfrak{R} -contramodules are projective, and in particular, all flat left \mathfrak{R} -contramodules are projective, so condition (iii) is satisfied. As \mathfrak{R} is commutative, Theorem 14.5 applies, providing the implication (vi) \implies (iv). The implication (iii) \implies (i^b) is obvious, and the argument from Theorem 14.4 for the implication (i^b) \implies (vi) can be used. All of this is covered by [45, Theorem 11.1]. Hence we can conclude that the topological ring \mathfrak{R} satisfies (iv), i.e., \mathfrak{R} is topologically left perfect.

Taking into account Corollary 11.2, it follows that the topological ring \mathfrak{R} is topologically semisimple, so it is a topological product of discrete fields. In particular, R is isomorphic to a product of fields as an abstract ring, which implies that the category \mathfrak{B} is semisimple. The contradiction proves our claim.

Theorem 14.12. *Conjecture 14.3 holds true for any topologically right coherent topological ring \mathfrak{R} .*

Proof: In view of Theorem 14.4, it suffices to prove either one of the implications (v) \implies (iii') or (v) \implies (iv). For topologically right coherent topological rings \mathfrak{R} , these are provided by Theorem 13.3. \square

Corollary 14.13. *Let M be an endo- Σ -coperfect left module over an associative ring A , and let $\mathfrak{R} = \text{Hom}_A(M, M)^{\text{op}}$ be the (opposite ring to) the endomorphism ring of M , viewed as a topological ring in the finite topology. Assume that the topological ring \mathfrak{R} is topologically right coherent. Then the A -module M has a perfect decomposition.*

Proof: Follows from Theorem 14.12, Lemma 12.3, and Theorem 10.4, as explained in Remark 14.9. Alternatively, one can apply directly Theorem 13.3, Corollary 9.9, and Theorem 10.1. \square

Before we finish this section, let us list two partial or conditional results in the direction of the main conjecture.

Proposition 14.14. *Let \mathfrak{R} be a complete, separated topological ring with a right linear topology. Suppose that \mathfrak{R} satisfies condition (v), i.e., all discrete right \mathfrak{R} -modules are coperfect. Then the topological Jacobson radical $\mathfrak{J}(\mathfrak{R})$ of the topological ring \mathfrak{R} is topologically left T -nilpotent (which is a part of condition (iv)).*

Proof: This is [45, Corollary 7.7]. \square

Proposition 14.15. *Let \mathfrak{R} be a complete, separated topological ring with a base of neighborhoods of zero consisting of open two-sided ideals. Then the equivalence (v) \iff (vi) in Conjecture 14.3 holds true.*

Proof: This is explained in [45, proof of Theorem 11.1]. □

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