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TWISTED L^2-TORSION ON THE CHARACTER VARIETY

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Abstract: We define a twisted L^2-torsion on the character variety of a 3-manifold M and study some of its properties. In the case where M is hyperbolic of finite volume, we prove that the L^2-torsion is a real-analytic function in a neighbourhood of any lift of the holonomy representation.

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1. Introduction

Let M be a compact oriented 3-manifold with μ toroidal boundary components, $\pi = \pi_1(M)$ its fundamental group, and assume that its interior admits a complete hyperbolic metric. Then the Riemannian volume $\mathrm{Vol}(M)$ given by this metric is finite and it is a topological invariant of M by the Mostow–Prasad rigidity theorem. The complete hyperbolic structure also gives rise to a *holonomy representation* from π to the isometry group $\mathrm{PSL}_2(\mathbb{C})$ of hyperbolic 3-space $\mathbb{H}^3$ (see for example [**20**, p. 139–152]). The latter lifts to a finite set of representations $\rho_0\colon \pi \to \mathrm{SL}_2(\mathbb{C})$; in this paper we will usually pick one of these arbitrarily and call it the holonomy representation. In general the lifts are not isomorphic to each other as representations of π but this does not matter for our purposes here.

Another topological invariant of M is the *combinatorial* L^2-torsion $\tau^{(2)}(M)$. It is defined in a manner similar to the classical combinatorial torsions but using a chain complex for the universal cover with the action of the fundamental group, and the Fuglede–Kadison determinants of equivariant operators. For hyperbolic manifolds it can be computed using analytic means, by a result of Wolfgang Lück and Thomas Schick [**15**]. It follows that

$$\tau^{(2)}(M) = e^{-\frac{\mathrm{Vol}(M)}{6\pi}}.$$

Given a linear representation of the fundamental group of M we can use it to twist the chain complex, and for the holonomy representation of a hyperbolic 3-manifold the result of Lück–Schick has been extended in the PhD thesis of Benjamin Wasserman (see [**22**] where the corresponding result is announced). In this paper we want to check that the L^2-torsion $\tau^{(2)}(M,\rho)$ twisted by an $\mathrm{SL}_2(\mathbb{C})$-representation ρ is well defined, at least near the holonomy. We study the properties of the function $\rho \mapsto \tau^{(2)}(M,\rho)$ defined on the $\mathrm{SL}_2(\mathbb{C})$-character variety. We will discuss the extension of the formula in terms of volume below.

1.1. Well-definedness and regularity of the L^2-torsion function. For hyperbolic knot complements, such a twisted L^2-torsion was defined in Weiping Li and Weiping Zhang's paper [**10**]. However, the nontriviality of this invariant was never addressed in this reference, as it is not established there that the relevant complexes are of determinant class. We remedy this and observe that in fact the argument used to check well-definedness also implies the following result.

Theorem A. *The twisted L^2-torsion function*

$$\tau^{(2)}(M,\rho)\colon X(M) \longrightarrow \mathbb{R}_{\geq 0}$$

is real-analytic in an open neighbourhood U of a holonomy character $[\rho_0]$ of M in its character variety $X(M)$.

This result fits within a broader programme. In general, the question of defining L^2-torsions for twisted L^2-chain complexes, and studying their continuity in the twisting representation, is asked by Lück in [**14**] (see e.g. Problem 10.11 there). It has been answered positively by Yi Liu [**11**] in the "abelian" case: for coefficients coming from 1-dimensional representations. Our result gives a partial answer to this problem in a nonabelian setting.

Theorem A is local in character, that is, we do not have an explicit description of the locus U. Our argument is not well suited for this purpose, as we use a spectral gap condition to check the determinant class condition (see Lemma 5.3) and we establish the latter through a continuity argument. It would be interesting to prove this spectral gap directly for all holonomies of cone-manifold structures; ideally the theorem would be proved to hold on the whole *Dehn surgery space* (see [**7**]). On the rest of the character variety we see no reason why there should be such a spectral gap and we see no other way to check the determinant class condition, outside of the dense subset of characters with values in $\overline{\mathbb{Q}}$, where the determinant class holds by well-known arguments (see [**13**, Chapter 13]).

Let us say a few more words on the proof of Theorem A. As we said above, the main ingredient is to establish a spectral gap for the relevant operators. By a standard continuity argument we immediately deduce this from the result for the holonomy representation, which was essentially established by Nicolas Bergeron and Akshay Venkatesh in [**1**] (under the name of "strong acyclicity"). The analyticity of $\tau^{(2)}(M,\rho)$ in this neighbourhood then follows from a general result, Lemma 3.6, which seems standard but we could not find in the literature, and from the regularity of the twisted Laplacians on the character variety, as operator-valued functions.

1.2. The L^2-torsion function and geometric invariants. In view of the results of Lück–Schick and their extension it is natural to ask whether there is a relationship between this kind of twisted L^2-torsion and the volume function Vol defined on $X(M)$ (see also [**12**, p. 486] and [**10**, p. 248]).

Using the relationship between the twisted L^2-torsion and the volume for the family of hyperbolic manifolds obtained by Dehn surgery on M and a surgery formula, one might expect an expression of the L^2-torsion function as the volume function plus a sum of lengths of the length functions L_i, $i = 1, \dots, \mu$, appearing in [**16**], which for our purposes can be defined as follows: if ρ is the holonomy of a cone-manifold structure on a Dehn filling of M, then $L_i(\rho)$ equals the length of the core curve of the i-th cusp.

The surgery formula only computes the L^2-torsion of some intermediate cover of M, and we do not see how to extract enough information on the L^2-torsion of the universal cover from the latter. However, the Lück–Schick result and the continuity of $\tau^{(2)}(M,\rho)$ imply that $\log \tau^{(2)}(M,\rho) = -\frac{11}{12\pi}\operatorname{Vol}(\rho) + o(1)$ near the holonomy representation ρ_0. A corollary of the analyticity of the L^2-torsion proved in Theorem A and of [**16**, equation (3)] is the following first-order approximation:

$$\log \tau^{(2)}(M,\rho) = -\frac{11}{12\pi}\operatorname{Vol}(\rho) + \frac{11}{24}\sum_{i=1}^{\mu} L_i(\rho) + O(\max L_i(\rho)^2). \tag{1}$$

This can be deduced as follows. First, $\tau^{(2)}(M,\rho_0) = \exp\bigl(-\frac{11}{12\pi}\operatorname{Vol}(M)\bigr)$, as follows from the Cheeger–Müller theorem of Wasserman [**22**] and the computation of analytic L^2-torsion by Bergeron–Venkatesh [**1**]. Now the L_i^2 are local analytic parameters near the holonomy, the volume function admits a Puiseux development of valuation $1/2$, and the expansion in (1) is the only one that is differentiable at ρ_0 (this is the content of [**16**, equation (3)]). Since the L^2-torsion is analytic by Theorem A, it proves equation (1).

We do not know a closed formula for the twisted L^2-torsion outside of the holonomy representation and so we are not even able to determine whether it is non-constant in a neighbourhood of the holonomy representation. Computing higher-order terms in (1) would be an interesting related question. Note that it follows from our Proposition 3.4(i) that $\tau^{(2)}(\rho) = e^{-\frac{1}{3\pi}\operatorname{Vol}(M)}$ for unitary representations ρ, hence the torsion is not constant on the whole character variety in general.

1.3. L^2-torsion function for nonhyperbolic manifolds. Our definition of the twisted L^2-torsion is rather general, and it can be considered on the character variety of any 3-manifold. The main point is to ensure that this invariant is not trivial (i.e. zero). To this purpose one needs to check whether some delicate conditions hold, namely that the complex under consideration is *weakly acyclic* and of *determinant class*. This is achieved in several cases in this paper. Aside from hyperbolic manifolds, we consider the case where N is a Seifert fibred manifold. We prove:

Proposition 1.1. *Let N be a compact Seifert manifold. Let ρ be an irreducible representation of $\pi_1(N)$. Then the complex $C_*(N,\rho)$ is weakly L^2-acyclic and of determinant class, and $\tau^{(2)}(N;\rho) = 1$.*

From Proposition 1.1 we deduce a JSJ formula for the twisted L^2-torsion:

Theorem B. *Let M be a compact aspherical 3-manifold and $N_1, \dots, N_r$ be the hyperbolic components in the JSJ decomposition of M. Let $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ be a representation such that, for any i, the complex $C_*^{(2)}(N_i, \rho|_{\pi_1(N_i)})$ is L^2-acyclic and of determinant class and that, for any Seifert piece $N \subset M$, the restriction $\rho|_{\pi_1(N)}$ is irreducible. We have*

$$\tau^{(2)}(M;\rho) = \prod_{i=1}^{r} \tau^{(2)}(N_i;\rho).$$

In particular, if M is a graph manifold, then $\tau^{(2)}(M;\rho) = 1$ for any representation ρ that restricts to an irreducible representation on each Seifert fibred piece.

The hypotheses in the theorem are somewhat unnecessary but give a cleaner statement:

(i) We do not know whether it can happen that a hyperbolic piece is not of determinant class; on the other hand we will see that Seifert pieces always are.
(ii) If the representation is reducible on a Seifert piece, we can extract a formula from the proof which involves further factors (see Remark 4.4 below).

(iii) In principle it is possible to give a similar formula for the twisted L^2-torsion of any 3-manifold by decomposing it into prime manifolds, which are either elliptical (for which the torsion is a classical Franz–Reidemeister torsion), $\mathbb{S}^2 \times \mathbb{S}^1$ (for which the computation is easy) or aspherical (for which the above result applies).

We also relate our invariant with its abelian cousin studied in [**4**] by Jérôme Dubois, Stefan Friedl, and Wolfgang Lück, and treat the case of unitary representations in Proposition 3.4.

We note that in general when M has a nontrivial JSJ decomposition it is not clear whether there are representations to $\mathrm{SL}_2(\mathbb{C})$ which satisfy the hypotheses of the theorem, or indeed any representations with infinite image.

Organization of the paper. Section 2 contains preliminary facts on character varieties of hyperbolic manifolds. In Section 3 we recall the general theory of L^2-invariants, define the twisted L^2-torsion, and state various technical results. In Section 4 we prove Theorem B and Proposition 1.1. Finally, in Section 5 we prove Theorem A.

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2. Preliminaries on character varieties and the volume function

In this section we collect facts about the $\mathrm{SL}_2(\mathbb{C})$-character variety of a hyperbolic 3-manifold. In Subsection 2.1 we define the character variety in general, while in Subsection 2.2 we specialize to the case of hyperbolic manifolds.

2.1. Character varieties. We refer to [**19**] for a survey of character varieties in general. If Γ is a finitely generated group and G a complex affine algebraic group, the representation variety $R(\Gamma, G)$ is the set $\mathrm{Hom}(\Gamma, G)$, which has the structure of an affine variety, whose defining equations are given by the relations defining Γ and by the defining equations of G. The group G acts by conjugation on $R(\Gamma, G)$; if G is reductive, then the categorical quotient $R(\Gamma, G)/\!\!/G$ is a well-defined affine algebraic variety. It is called the *G-character variety* of Γ and is denoted by $X(\Gamma, G)$. For a representation $\rho \in R(\Gamma, G)$ we will denote its image in $X(\Gamma, G)$ by $[\rho]$.

In the sequel we will only consider the case $G = \mathrm{SL}_2(\mathbb{C})$. The character variety $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is not homeomorphic in general to the topological quotient $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$; however, there is an $\mathrm{SL}_2(\mathbb{C})$-invariant open subset for which the two quotients coincide as we now describe. Recall that a representation is irreducible if it does not admit a proper nonzero invariant subspace. Let $R^*(\Gamma, \mathrm{SL}(\mathbb{C}))$ be the subset of irreducible representations, which is a Zariski-open subset of the space $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$. Orbits of $\mathrm{SL}_2(\mathbb{C})$ on $R^*(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ are closed and so its image $X^*(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ in $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is homeomorphic to $R^*(\Gamma, \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$.

2.2. Character varieties of hyperbolic 3-manifolds. Now we restrict to the case of interest to us of studying $X(\pi, \mathrm{SL}_2(\mathbb{C}))$, where $\pi = \pi_1(M)$ is the fundamental group of an orientable hyperbolic manifold with μ cusps. We will use the notations

$$R(M) = R(\pi, \mathrm{SL}_2(\mathbb{C})), \quad X(M) = X(\pi, \mathrm{SL}_2(\mathbb{C})).$$

The complete hyperbolic structure of M corresponds to a *holonomy representation* $\overline{\rho}_0 \colon \pi \to \mathrm{PSL}_2(\mathbb{C})$ such that $M = \overline{\rho}_0(\pi)\backslash\mathbb{H}^3$. It is known (see [**2**]) that this representation lifts to a representation $\rho_0 \colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$, which we will also call the holonomy representation. There is in general no unique choice for ρ_0; we assume that an arbitrary one has been made. It is an irreducible representation and we call its image $[\rho_0]$ in $X(M)$ the holonomy character.

We will need a way to work directly with representations rather than characters, for which the following well-known lemma (see [**17**, Proposition 3.2]) will be useful.

Lemma 2.1. *There exists a section of the projection $R(M) \to X(M)$ which is real-analytic in a neighbourhood of the holonomy character $[\rho_0]$.*

3. The L^2-torsion

We introduce the twisted L^2-torsion $\tau^{(2)}(M, \rho)$ for unimodular representations of 3-manifold groups. Twisted L^2-torsions are considered in a general context in [**14**] and we use this and the book [**13**] as a reference. In Subsection 3.1 we recall without proof all the results needed to make the paper self-contained. In Subsection 3.2 we use this to define $\tau^{(2)}(M, \rho)$. In Subsection 3.3 we give some useful properties and some simple examples. Finally, in Subsection 3.4 we prove a regularity lemma for the determinant which we will use in Section 5.

3.1. L^2-invariants. Let π be a discrete group and $\mathbb{C}\pi$ its group ring. The reduced von Neumann algebra $\mathcal{N}\pi$ is the commutant of the left action of $\mathbb{C}\pi$ on $\ell^2(\pi)$ in the algebra of bounded operators on the Hilbert space $\ell^2(\pi)$ (see [**13**, Definition 1.1]). A finitely generated Hilbert $\mathcal{N}\pi$-module is a quotient of some $\ell^2(\pi)^n$ by a closed $\mathcal{N}\pi$-invariant subspace ([**13**, Definition 1.5]). There is a functor Λ from the category of based free $\mathbb{C}\pi$-modules to the category of Hilbert $\mathcal{N}\pi$-modules ([**14**, p. 727]); it can be defined as the L^2-completion of M with respect to the basis $g \cdot b$, where b is an element of the $\mathbb{C}\pi$-basis of M and $g \in \pi$. In the sequel we will often make no distinction between M and ΛM or f and Λf.

For V a Hilbert $\mathcal{N}\pi$-module we denote by $\mathcal{B}_{\mathcal{N}\pi}(V)$ the set of bounded $\mathcal{N}\pi$-equivariant operators on V. The subset of $\mathcal{B}_{\mathcal{N}\pi}(V)$ consisting of bounded operators with bounded inverse is denoted by $\mathrm{GL}_{\mathcal{N}\pi}(V)$. We also denote by $\mathrm{tr}_{\mathcal{N}\pi}$ the usual trace on $\mathcal{B}_{\mathcal{N}\pi}(V)$ if V is of finite type (see [**13**, Definitions 1.2, 1.8]). For the definition of the L^2-torsion we need the definition of the Fuglede–Kadison determinant $\det_{\mathcal{N}\pi}(A)$ of a morphism A between two Hilbert $\mathcal{N}\pi$-modules. If A is a positive self-adjoint bounded operator in $\mathcal{B}_{\mathcal{N}\pi}(V)$ with trivial kernel, then the operator $\log(A)$ is well-defined – but not bounded in general. If $\log(A)$ is in $\mathcal{B}_{\mathcal{N}\pi}(V)$, the Fuglede–Kadison determinant of A can be computed by the formula:

$$\det_{\mathcal{N}\pi}(A) = \exp(\mathrm{tr}_{\mathcal{N}\pi} \log(A)). \tag{2}$$

The operator $\log(A)$ is in $\mathcal{B}_{\mathcal{N}\pi}(V)$ if and only if $A \in \mathrm{GL}_{\mathcal{N}\pi}(V)$, which is the main case we will consider in the sequel.

The Fuglede–Kadison determinant is defined in general, and the equality above holds formally (see [**13**, Definition 3.11]). When the trace is $-\infty$ the determinant is zero. If $\det_{\mathcal{N}\pi}(A) > 0$, then the operator A is said to be *of determinant class*. Being of determinant class depends only on the behaviour at 0 of the spectral density function of A; the simplest case is when its Novikov–Shubin invariants (see [**13**, Definitions 2.8, 2.16]) are positive.

Remark 3.1. It is difficult to know whether a morphism is of determinant class. A notable exception occurs when π is virtually abelian; in this case all morphisms between finitely generated $\mathbb{C}\pi$-modules are of determinant class, as follows from [**13**, Example 3.13] and [**6**, Lemma 1.8].

For many groups π, including all residually finite groups, so in particular all 3-manifold groups, if f is given by a matrix over the integral group ring $\mathbb{Z}\pi$, then it is of determinant class (see [**13**, Chapter 13]). So if ρ is defined over $\mathbb{Q}$ or even $\overline{\mathbb{Q}}$, we have $\tau^{(2)}(M, \rho) \neq 0$. However, when π is nonabelian and the representation has transcendental coefficients it is essentially completely open to give general criteria.

A complex of Hilbert $\mathcal{N}\pi$-modules (C_*, d_*) is L^2*-acyclic* if its reduced L^2-homology ([**13**, Definition 1.17]) vanishes, that is, if $\operatorname{Im}(d_p)$ is dense in $\ker(d_{p-1})$. In this situation the combinatorial Laplace operators

$$\Delta_p = d_p d_{p+1}^* + d_{p+1} d_p^* \tag{3}$$

have zero kernel. They are always positive, and the complex (C_*, d_*) is said to be *of determinant class* if all Δ_p are of determinant class. Then the L^2-torsion $\tau^{(2)}(C_*) \in \,]0, +\infty[$ is defined by

$$\tau^{(2)}(C_*)^2 = \prod_{p=0}^{n} \det_{\mathcal{N}\pi}(\Delta_p)^{(-1)^p p}. \tag{4}$$

We extend the definition of L^2-torsion to all complexes by taking the convention that $\tau^{(2)}(C_*) = 0$ if C_* is not of determinant class.

3.2. Twisted L^2-torsion for unimodular representations. Let M be a finite CW-complex, π its fundamental group, and $\rho\colon \pi \to \mathrm{GL}(E)$ a finite-dimensional unimodular representation. There is a twisted chain complex of $\mathbb{C}(\pi)$-modules

$$C_*(\widetilde{M}, \rho) = C_*(\widetilde{M}) \otimes_{\mathbb{C}} E,$$

where any element γ of the group π acts by $\gamma \otimes \rho(\gamma)$ on $\mathbb{C}(\pi) \otimes_{\mathbb{C}} E$ by the right regular representation on the first factor times ρ acting on the left on the second one; this is denoted by $\eta_E C_*(\widetilde{M})$ in [**14**].

We choose a $\mathbb{C}\pi$-basis for $C_*(\widetilde{M})$, that is, an orientation and a lift to the universal cover for each cell. We choose a basis for E as well, and we get a complex of free based $\mathbb{C}\pi$-modules. If its completion $\Lambda C_*(\widetilde{M}, \rho)$, which we will rather denote by $C_*^{(2)}(M, \mathcal{N}\pi \otimes \rho)$ in the rest of this paper, is L^2-acyclic, then we define the twisted L^2-torsion of (M, ρ) by:

$$\tau^{(2)}(M, \rho) = \tau^{(2)}(C_*^{(2)}(M, \mathcal{N}\pi \otimes \rho)). \tag{5}$$

Since the representation ρ is unimodular, it follows immediately from [**14**, Theorem 6.7(1), Lemma 3.2(1)] that (5) does not depend on the choices of bases for $C_*(\widetilde{M})$ and E.

We are interested in the twisted L^2-torsion as defined above for 3-manifolds; however, in some occurences we will need the L^2-torsion for more general covers. The twisted L^2-torsion of a cover $\widehat{M} \to M$ associated to a surjective morphism $\pi \to \Lambda$ is defined exactly as above, using the chain complexes $C_*(\widehat{M})$ instead of $C_*(\widetilde{M})$ and replacing the group ring and von Neumann algebra $\mathbb{C}\pi$, $\mathcal{N}\pi$ by $\mathbb{C}\Lambda$, $\mathcal{N}\Lambda$ respectively. In this case we will denote the complex by $C_*^{(2)}(M, \mathcal{N}\Lambda \otimes \rho)$, and the L^2-torsion by $\tau^{(2)}(M, \mathcal{N}\Lambda \otimes \rho)$.

3.3. Properties and first examples. The following basic fact follows immediately from [**14**, Lemma 3.2(1)].

Lemma 3.2. *Let $\rho, \rho' : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ be two conjugate representations. Then $\tau^{(2)}(M, \rho) = \tau^{(2)}(M, \rho')$.*

The next well-known lemma (see for example [**4**, Lemma 3.1]) will be useful to compute the L^2-torsion of Seifert manifolds.

Lemma 3.3. *Let*

$$C_* = 0 \longrightarrow \mathbb{C}\pi^k \xrightarrow{A} \mathbb{C}\pi^{k+l} \xrightarrow{B} \mathbb{C}\pi^l \longrightarrow 0.$$

Let $A' \in M_{k,k}(\mathbb{C}\pi)$ and $B' \in M_{l,l}(\mathbb{C}\pi)$ be the square matrices obtained from the k first lines of A and l last columns of B respectively. If A', B' are acyclic and of determinant class, then so is C_* and moreover

$$\tau^{(2)}(C_*) = \det_{\mathcal{N}\pi}(A')^{-1} \det_{\mathcal{N}\pi}(B').$$

Proof: Consider the diagram

$$\begin{array}{ccccccccccc}
E_* & & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{C}\pi^l & \xrightarrow{B'} & \mathbb{C}\pi^l & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \| & & \\
C_* & & 0 & \longrightarrow & \mathbb{C}\pi^k & \xrightarrow{A} & \mathbb{C}\pi^{k+l} & \xrightarrow{B} & \mathbb{C}\pi^l & \longrightarrow & 0 \\
\downarrow & & & & \| & & \downarrow & & \downarrow & & \\
D_* & & 0 & \longrightarrow & \mathbb{C}\pi^k & \xrightarrow{A'} & \mathbb{C}\pi^k & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

where the vertical arrows in the middle are given by natural inclusions and projections. This gives a short exact sequence of complexes $0 \to E_* \to C_* \to D_* \to 0$ and it follows from the additivity of the L^2-torsion [**13**, Theorem 3.35(1)] (the terms on the right in this equation vanish because of acyclicity and the fact that the determinants of the inclusions and projections are 1) that C_* is acyclic, of determinant class and

$$\tau^{(2)}(C_*) = \tau^{(2)}(D_*) \cdot \tau^{(2)}(E_*).$$

Since the torsion of a complex with a single nonzero differential d is equal to $\det_{\mathcal{N}\pi}(d)^{(-1)^p}$, where p is the degree of d, the result follows. ☐

In some very special cases the computation of the L^2-torsion is immediate.

Proposition 3.4.

(i) *If M is hyperbolic and the representation $\rho\colon \pi_1(M) \to \mathrm{SU}(2) \subset \mathrm{SL}_2(\mathbb{C})$ is unitary, then*

$$\tau^{(2)}(M,\rho) = \tau^{(2)}(M)^2 = \exp(-\operatorname{Vol}(M)/3\pi).$$

(ii) *Assume that $H_1(M) = \mathbb{Z}$, and let $\phi\colon \pi_1(M) \to \mathbb{Z}$ be a choice of an abelianization map. If $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ is reducible, conjugated to*

$$\rho(\gamma) = \begin{pmatrix} \lambda^{\phi(\gamma)} & * \\ 0 & \lambda^{-\phi(\gamma)} \end{pmatrix}$$

for some $\lambda \in \mathbb{C}^*$, then

$$\tau^{(2)}(M,\rho) = \tau^{(2)}(M,\phi)(|\lambda|)\tau^{(2)}(M,\phi)(|\lambda|^{-1}),$$

where the abelian-twisted L^2-torsion $\tau^{(2)}(M,\phi)$ is defined in [**4**].

Proof: Point (i) is a consequence of [**14**, Theorem 4.1(4)].

For (ii), observe that using Lemma 3.2 one can assume that ρ has the triangular form above. It induces an exact sequence of π-modules

$$0 \longrightarrow \ell^2(\pi) \longrightarrow \ell^2(\pi)^2 \longrightarrow \ell^2(\pi) \longrightarrow 0\,,$$

where the π-actions are given by the representations λ^{ϕ}, ρ, $\lambda^{-\phi}$ respectively. Now we can apply [**14**, Lemma 3.3] and conclude since we have the equality $\tau^{(2)}(M,-\phi)(|\lambda|) = \tau^{(2)}(M,\phi)(|\lambda|^{-1})$. □

3.4. Regularity under a spectral gap. Let V be a Hilbert space and $\mathcal{B}(V)$ the space of bounded linear operators on V. On $\mathcal{B}(V)$ we will use the operator norm defined by

$$|||T||| = \sup_{\|v\|=1} \|Tv\|.$$

Let $\mathcal{S}(V) \subset \mathcal{B}(V)$ be the subspace of self-adjoint operators. By the spectral theorem any self-adjoint operator T in $\mathcal{S}(V)$ has a spectrum $\sigma(T)$ which is a closed subset of $\mathbb{R}$. We will use the following lemma.

Lemma 3.5. *The function $\mathcal{S}(V) \to \mathbb{R}$ defined by $T \mapsto \inf \sigma(T)$ is continuous.*

Proof: It also follows from the spectral theorem that

$$\inf \sigma(T) = \inf_{v \in V\setminus\{0\}} \frac{\langle Tv, v\rangle}{\|v\|^2} = \inf_{\|v\|=1} \langle Tv, v\rangle. \tag{6}$$

Since $T \in \mathcal{S}(V)$ is bounded we have that $\inf \sigma(T) = \lambda > -\infty$. Let $\varepsilon > 0$. Then for any $v \in V$ with $\|v\| = 1$ we have $\langle Tv, v\rangle \geq \lambda$. If $|||S - T||| < \varepsilon$, then for all such v we have $\langle Sv, v\rangle \geq \langle Tv, v\rangle - \|Sv - Tv\|$ by Cauchy–Schwarz and the right-hand side is then at least $\lambda - \varepsilon$. Together with (6) this proves that σ is lower-continuous. On the other hand there exists a $w \in V$, $\|w\| = 1$, such that $\langle Tw, w\rangle \leq \lambda + \varepsilon$. For S as above we have $\langle Sw, w\rangle \leq \langle Tw, w\rangle + \|Sw - Tw\| \leq \lambda + 2\varepsilon$ so $\sigma(S) \leq \lambda + 2\varepsilon$. With (6) this proves that σ is upper-continuous. $\square$

Let $\mathcal{S}^{>0}(V)$ be the set of bounded self-adjoint operators $A \in \mathcal{S}(V)$ such that $\inf \sigma(A) > 0$, and let $\mathcal{S}^{>0}_{\mathcal{N}\pi}(V) = \mathcal{S}^{>0}(V) \cap \mathcal{B}_{\mathcal{N}\pi}(V)$. By Lemma 3.5, the set $\mathcal{S}^{>0}_{\mathcal{N}\pi}(V)$ is an open subset in $\mathcal{S}_{\mathcal{N}\pi}(V)$.

It is an open question to determine exactly the domain of continuity in $\mathcal{B}_{\mathcal{N}\pi}(V)$ of the Fuglede–Kadison determinant. However, it is a general principle that it is "as regular as possible" on the open subset $\mathrm{GL}_{\mathcal{N}\pi}(V)$; we will use an instance of this that is valid in the real-analytic category.

The space $\mathcal{B}_{\mathcal{N}\pi}(V)$ is a Banach space for the norm operator, as a norm-closed subspace of $\mathcal{B}(V)$. If $U \subset \mathbb{R}^n$ is an open subset and E a Banach space, we say that a function $A\colon U \to E$ is *real-analytic* if it admits an expression as a convergent power series in the neighbourhood of every $x \in U$.

We will use the usual notations: for an n-tuple of integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $x \in \mathbb{R}^n$, let $|\alpha| = \sum_i \alpha_i$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

The former real-analyticity condition can be conveniently formulated as follows: for every $x \in U$ there exists a sequence $(a_\alpha) \in E^{\mathbb{N}^n}$ and $r_0 > 0$ such that $\sum_\alpha \|a_\alpha\|_E r_0^{|\alpha|} < +\infty$ and for any $v \in \mathbb{R}^n$ such that $|v_i| < r_0$ for $1 \leq i \leq n$ (that is, $\|v\|_\infty < r_0$) we have $A(x + v) = \sum_{\alpha \in \mathbb{N}^m} a_\alpha v^\alpha$.

Lemma 3.6. *Let V be a Hilbert-$\mathcal{N}\pi$-module of finite type, U an open subset of $\mathbb{R}^n$, and $A\colon U \to \mathcal{S}^{>0}_{\mathcal{N}\pi}(V)$ a real-analytic map. Then the map $x \mapsto \det_{\mathcal{N}\pi} A(x)$ is a real-analytic function on U.*

Proof: We see from (2) that we need to prove that the function $x \mapsto \mathrm{tr}_{\mathcal{N}\pi} \log(A(x))$ is real-analytic. This will follow immediately from the two following points:

(a) If ℓ is a continuous linear form on a Banach space $\mathcal{B}$ and $x \mapsto T(x)$ a real-analytic map $U \to \mathcal{B}$, then $x \mapsto \ell(T(x))$ is real-analytic.
(b) If $I \subset \mathbb{R}$, f is a real-analytic function with a power series expansion which converges on I, and $\sigma(A(x)) \subset I$ for all $x \in U$, then $x \mapsto f(A(x))$ is real-analytic. Note that $f = \log$ satisfies this assumption for any $I \subset]0, +\infty[$.

Let us prove (a): write $\sum_{\alpha \in \mathbb{N}^n} T_\alpha v^\alpha$ a convergent power series with a positive radius of convergence for T, then by continuity of ℓ the series $\sum_\alpha \ell(T_\alpha) v^\alpha$ converges as well, and it equals the image of the former by the linear map ℓ on the domain of convergence.

Now we prove (b): let $f(t) = \sum_k c_k t^k$ be a power series expression for $f\colon I \subset \mathbb{R} \to \mathbb{R}$, let $x \in U$, and let $\sum_{\alpha\in\mathbb{N}^n} a_\alpha v^\alpha$ be the power series expansion for $A(x+v)$, which converges for $\|v\|_\infty \le r_1$ for some $r_1 > 0$. We denote $M = \sum_{\alpha\in\mathbb{N}^n} \|a_\alpha\|_{\mathcal{B}} r_1^{|\alpha|}$.

Assuming $a_0 = 0$ for convenience, we want to show that the formal series $\sum_{k\ge 0} c_k \big(\sum_{\alpha\in\mathbb{N}^n} a_\alpha v^\alpha\big)^k$ is convergent in a neighbourhood of $v = 0$. Let r_0 be the radius of convergence for $\sum_k c_k t^k$; then for $\|v\|_\infty < r_1 r_0/M$ we have that $\sum_{\alpha\in\mathbb{N}^n} \|a_\alpha\|_{\mathcal{B}}\, |v^\alpha| < r_0$ by convexity.

For those v the series $\sum_k |a_k| \big(\sum_{\alpha\in\mathbb{N}^n} \|a_\alpha\|_{\mathcal{B}} |v^\alpha|\big)^k$ converges; in other words the power series $\sum_{k\ge 0} a_k \big(\sum_{\alpha\in\mathbb{N}^n} a_\alpha v^\alpha\big)^k$ is absolutely convergent in a neighbourhood of 0. $\square$

4. Seifert manifolds and JSJ decomposition

In this section we prove Theorem B from the introduction, whose statement we recall here. Note that a similar statement, with a much simpler proof, also holds for torsions with unitary coefficients.

Theorem 4.1. *Let M be a compact aspherical 3-manifold and $N_1, \dots, N_r$ be the hyperbolic components in the JSJ decomposition of M. Let $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ be a representation such that for any Seifert piece $N \subset M$ the restriction $\rho|_{\pi_1(N)}$ is irreducible, and for every hyperbolic piece $C_*(N_i, \mathcal{N}\pi_1(N_i) \otimes \rho|_{\pi_1(N_i)})$ is of determinant class. Then we have*

$$\tau^{(2)}(M;\rho) = \prod_{i=1}^{r} \tau^{(2)}(N_i;\rho).$$

In particular, if M is Seifert or more generally a graph manifold, then $\tau^{(2)}(M;\rho) = 1$.

Proof: Since M is aspherical it is in particular irreducible and we can perform a JSJ decomposition of M. Its JSJ components are either hyperbolic or Seifert fibred manifolds. The theorem then follows immediately from the following two claims:

(a) If N_1, N_2 are compact with toric boundary and N is a gluing of N_1, N_2 along a collection of incompressible boundary components, then
$$\tau^{(2)}(N,\rho) = \tau^{(2)}(N_1, \rho|_{\pi_1(N_1)})\tau^{(2)}(N_2, \rho|_{\pi_1(N_2)}).$$

(b) For any $\rho\colon \pi_1(N) \to \mathrm{SL}_2(\mathbb{C})$, if N is Seifert (compact with toric boundary components), then the complex $C_*^{(2)}(N, \mathcal{N}\pi_1(N) \otimes \rho|_{\pi_1(N)})$ is L^2-acyclic and of determinant class. Moreover, if ρ is an irreducible representation of $\pi_1(N)$, then $\tau^{(2)}(N;\rho) = 1$.

Let $N = N_1 \cup N_2$ as in the statement of claim (a) and $T_1, \dots, T_r$ the boundary tori along which N_1 and N_2 are glued to each other. Let $\pi = \pi_1(N)$, for $X = N_1, N_2, T_j$ we denote by $C_*^{(2)}(X, \rho)$ the twisted complex of Hilbert $\mathcal{N}\pi$-modules associated to the preimage of X in the universal cover $\widetilde{N}$. Since the T_j are incompressible, this preimage is a disjoint union of copies of $\widetilde{X}$. By the restriction law [**14**, Theorem 6.7(6)] we have that

$$\tau^{(2)}(C_*^{(2)}(X, \rho)) = \tau^{(2)}(X, \rho|_{\pi_1(X)})$$

for those X. As we have an exact sequence

$$0 \longrightarrow \bigoplus_{j=1}^r C_*^{(2)}(T_j, \rho) \longrightarrow C_*^{(2)}(N_1, \rho) \oplus C_*^{(2)}(N_2, \rho) \\ \longrightarrow C_*^{(2)}(N, \rho) \longrightarrow 0$$

it follows by using the gluing formula [**14**, Theorem 6.7(3)] that

$$\tau^{(2)}(N, \rho) = \frac{\tau^{(2)}(N_1, \rho|_{\pi_1(N_1)}) \cdot \tau^{(2)}(N_2, \rho|_{\pi_1(N_2)})}{\prod_{j=1}^r \tau^{(2)}(T_j, \rho|_{\pi_1(T_j)})}.$$

In Lemma 4.2 below we show that the L^2-torsion of a 2-torus with coefficients in any local system is 1 so $\tau^{(2)}(T_j, \rho|_{\pi_1(T_j)}) = 1$ for all j and we finally obtain the formula in claim (a).

Claim (b) is a subcase of the more general Proposition 4.3 that we prove below. The arguments are essentially lifted from [**9**] and adapted to the L^2 setting. □

We now prove a lemma on tori used in the proof above and which we will also need to deal with Seifert manifolds. It could be deduced from general Poincaré duality but we prefer to give a more explicit proof.

Lemma 4.2. *Let $\rho\colon \mathbb{Z}^2 \to \mathrm{SL}(V)$ be any unimodular representation, Λ an infinite group, and $\mathbb{Z}^2 \to \Lambda$ a surjective morphism. Then the complex $C_*^{(2)}(\mathbb{T}^2, \mathcal{N}\Lambda \otimes \rho)$ is L^2-acyclic and of determinant class, and its torsion $\tau^{(2)}(\mathbb{T}^2, \mathcal{N}\Lambda \otimes \rho) = 1$.*

Proof: The determinant class condition is always satisfied (see Remark 3.1). So we need to compute the torsion and show it equals 1.

First we reduce to the case where ρ is semisimple. Let m, ℓ be generators for $\pi_1(\mathbb{T}^2)$. As $\rho(m)$ and $\rho(\ell)$ commute they are triagonalizable with Jordan blocks of the same shape. It follows that there exists a sequence $g_n \in \mathrm{SL}(V)$ such that $\rho_n(\ell) := g_n \rho(\ell) g_n^{-1}$ and $\rho_n(m) := g_n \rho(m) g_n^{-1}$ converge to a pair of commuting semisimple elements $\rho_\infty(\ell)$, $\rho_\infty(m) \in \mathrm{SL}(V)$. By Lemma 3.2 we have $\tau^{(2)}(\mathbb{T}^2, \rho) = \tau^{(2)}(\mathbb{T}^2, \rho_n)$. It follows from the formula for the Fuglede–Kadison determinant over abelian groups that $\rho \mapsto \tau^{(2)}(\mathbb{T}^2, \rho)$ is continuous. So $\tau^{(2)}(\mathbb{T}^2, \rho) = \tau^{(2)}(\mathbb{T}^2, \rho_\infty)$.

We can now assume ρ to be semisimple, so we can decompose $\rho \cong \bigoplus_{j=1}^{\dim(V)} \chi_j$, where χ_j are 1-dimensional. It follows that $\tau^{(2)}(\mathbb{T}^2, \rho) = \prod_j \tau^{(2)}(\mathbb{T}^2, \chi_j)$ so we can just assume that ρ is 1-dimensional.

Now the complex whose L^2-torsion we have to compute is just

$$0 \longrightarrow \ell^2(\mathbb{Z}^2) \xrightarrow{d_1} \ell^2(\mathbb{Z}^2)^2 \xrightarrow{d_0} \ell^2(\mathbb{Z}^2) \longrightarrow 0\,,$$

with the boundary operators given by the following matrices over the group ring:

$$\begin{aligned} d_1 &= \begin{pmatrix} 1 - \ell \otimes \rho(\ell) \\ 1 - m \otimes \rho(m) \end{pmatrix}, \\ d_0 &= \begin{pmatrix} m \otimes \rho(m) - 1 & \ell \otimes \rho(\ell) - 1 \end{pmatrix}. \end{aligned}$$

It is immediate that the complex is L^2-acyclic, and it follows from Lemma 3.3 that the torsion is equal to $\det_{\mathcal{N}\Lambda}(1 - \ell \otimes \rho(\ell))^{-1} \det_{\mathcal{N}\Lambda}(1 - \ell \otimes \rho(\ell)) = 1$. $\square$

Proposition 4.3. *Let N be a compact Seifert manifold. Let ρ be an irreducible representation of $\pi_1(N)$. Then the complex $C_*(N, \rho)$ is weakly L^2-acyclic and of determinant class, and $\tau^{(2)}(N; \rho) = 1$.*

Before starting the proof, let us recall some facts about oriented Seifert fibred spaces (see [**8**]). In a Seifert fibred space N with boundary a union of k tori, there is a finite number of exceptional fibres $F_1, \dots, F_r$, each of which is given with a pair of integers (p_i, q_i). We denote by T_i a tubular neighbourhood of a singular fibre F_i, which is homeomorphic to a solid torus. The manifold $N \setminus (T_1 \cup \cdots \cup T_r)$ is an S^1-bundle on a surface with $k + r$ boundary components. The obstruction for this bundle to be trivial can be concentrated in the neighbourhood of a single regular fibre F_0 with neighbourhood T_0, so that $N \setminus (T_0 \cup T_1 \cup \cdots \cup T_r)$ is homeomorphic to the product $N_0 = S_{g,k+r+1} \times S^1$ of a surface of genus g with $k + r + 1$ boundary components, and the fibration twists b times around T_0.

If the surface S is orientable, we have the following presentation for the fundamental group of N:

$$(7) \quad \pi_1(N) = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_k, f_0, \dots, f_r, h \\ \mid hx = xh\, \forall x \in \pi_1(N),\ f_i^{p_i} h^{q_i} = 1, [a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_k f_0 \cdots f_r = 1 \rangle,$$

where the a_i, b_i are standard generators of the surface group, the c_i represent the boundary curves of the surface which do not bound a fibre, and f_i represents a curve on the surface such that $f_i \times S^1 \simeq \partial T_i$. Additionally $(p_0, q_0) = (1, -b)$, and finally h denotes the class of the generic fibre.

Proof: Weak acyclicity and the determinant class property follow from the explicit computation we will make for the complexes involved. Lemma 4.2 yields

$$\tau^{(2)}(\partial T_i, \mathcal{N}\pi_1(T_i) \otimes \rho|_{\pi_1(\partial T_i)}) = 1.$$

Applying the multiplicativity formula for the L^2-torsion we get

$$\tau^{(2)}(N, \rho) = \tau^{(2)}(N_0, \mathcal{N}\pi_1(N_0) \otimes \rho|_{\pi_1(N_0)}) \prod_{i=0}^{r} \tau^{(2)}(T_i, \rho|_{\pi_1(T_i)}).$$

First we will prove that $\tau^{(2)}(T_i, \rho|_{\pi_1(T_i)}) = 1$ for any index $i = 0, \ldots, r$, and then that $\tau^{(2)}(N_0, \mathcal{N}\pi_1(N) \otimes \rho|_{\pi_1(N_0)}) = 1$.

Since ρ is irreducible and h is central in $\pi_1(N)$, necessarily $\rho(h) = \pm\,\mathrm{Id}$, and it follows that for each i, the generator ℓ_i of $\pi_1(T_i)$ has finite order through ρ. In particular its eigenvalues λ_i, λ_i^{-1} are unitary complex numbers, and the operator $1 - \ell_i \otimes \rho(\ell_i)\colon \ell^2(\pi_1(T_i))^2 \to \ell^2(\pi_1(T_i))^2$ has Fuglede–Kadison determinant equal to 1. Since T_i retracts onto the circle ℓ_i, its L^2-torsion is $\tau^{(2)}(T_i, \rho_{\pi_1(T_i)}) = \det_{\mathcal{N}\pi_1(T_i)}(1 - \ell_i \otimes \rho(\ell_i))^{-1} = 1$, and the first assertion is proved.

Now N_0 retracts onto a 2-complex Y, which is a product of a circle with a bouquet of $2g+k+r$ circles indexed by $a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_k$, $f_1, \ldots, f_r$ (the generator f_0 does not appear thanks to the last relation in $\pi_1(N)$).

By the same computation as in [**9**, Proof of Proposition 4.2] the differentials in the $\pi_1(N)$-complex $C_*(Y, \mathcal{N}\pi_1(N) \otimes \rho|_{\pi_1(Y)})$ are given by

$$A = (a_1 \otimes \rho(a_1) - 1 \cdots b_g \otimes \rho(b_g) - 1 \; c_1 \otimes \rho(c_1) - 1 \cdots f_r \otimes \rho(f_r) - 1 \; h \otimes \rho(h) - 1)$$

and

$$B = \begin{pmatrix} 1 - h \otimes \rho(h) & 0 & \cdots & \\ 0 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & & 1 - h \otimes \rho(h) \\ 1 - a_1 \otimes \rho(a_1) & 1 - b_1 \otimes \rho(b_1) & \cdots & 1 - f_r \otimes \rho(f_r) \end{pmatrix}.$$

By Lemma 3.3 the L^2-torsion of the complex

$$0 \longrightarrow L^2(\pi_1(N))^{2g+k+r} \xrightarrow{B} L^2(\pi_1(N))^{2g+k+r+1} \xrightarrow{A} L^2(\pi_1(N)) \longrightarrow 0$$

is equal to $\det_{\mathcal{N}\pi_1(N)}(B')\det_{\mathcal{N}\pi_1(N)}(A')^{-1}$, where

$$B' = \begin{pmatrix} 1-h\otimes\rho(h) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1-h\otimes\rho(h) \end{pmatrix}$$

is the $(2g+k+r)\times(2g+k+r)$ matrix obtained by keeping all but the last line of B and

$$A' = (1-h\otimes\rho(h))$$

is the matrix obtained by deleting all but the last column of A. Both of these Fuglede–Kadison determinants equal 1. So $\tau^{(2)}(Y, N\pi_1(N)\otimes\rho|_{\pi_1(Y)}) = 1$, and by homotopy invariance [**14**, Theorem 6.7(2)] it follows that $\tau^{(2)}(N_0, \mathcal{N}\pi_1(N)\otimes\rho|_{\pi_1(N_0)}) = 1$ as well, and it proves the second assertion and finishes the proof of Proposition 4.3 in this case.

It remains to deal with the case where S is not orientable; this can be done following a similar scheme as above (see [**9**] for the classical case). A simpler argument in our setting is to apply the orientable case to the double cover associated to $\pi_1(N)\to\pi_1(S)\to\mathbb{Z}/2$ and use the multiplicativity of the twisted L^2-torsion in covers [**14**, Theorem 6.7(5)]. □

Remark 4.4. If ρ is not irreducible, then the arguments above are still valid except that the image $\rho(h)$ of the generic fibre can have any complex number as an eigenvalue, hence $\det_{\mathcal{N}\pi_1(N)}(1-h\otimes\rho(h))$ is not necessarily equal to 1. In general, denoting by λ_ρ an eigenvalue of $\rho(h)$ of modulus ≥ 1, we have

$$\tau^{(2)}(N,\rho) = |\lambda_\rho|^{2g+k+r-2-\sum_{i=1}^{r}\frac{1}{p_i}}. \tag{8}$$

For example, if $M(p,q)$ is the complement of the (p,q)-torus knot, we have

$$\tau^{(2)}(M(p,q),\rho) = |\lambda_\rho|^{1-\frac{1}{p}-\frac{1}{q}},$$

since in this case $M(p,q)$ is a fibration on a disk with two singular fibres with singularities $(p,1)$ and $(q,1)$. Note that this can also be computed directly by a simpler method analogous to [**5**], using the following presentation of the knot group $\pi_1(M(p,q)) = \langle a,b \mid a^p = b^q\rangle$.

Proof of Remark 4.4: With the same matrices as in the proof of Proposition 4.3, we just have to compute the determinants involved. First we carefully compute the core curves of the solid tori T_i. For given index $i = 0, \ldots, r$, a meridian of the solid torus T_i is given by the null-homotopic curve $f_i^{p_i} h^{q_i}$; see (7). Hence a longitude ℓ_i is given by $\ell_i = f_i^{n_i} h^{m_i}$ for $m_i\, p_i - n_i\, q_i = 1$. Now the eigenvalues of $\rho(\ell_i)$ are $\lambda_\rho^{\pm 1/p_i}$, and we obtain

$$\begin{aligned} \det_{\mathcal{N}\pi_1(N)}(1 - \ell_i \otimes \rho(\ell_i)) &= |\lambda_\rho|^{1/p_i}, \\ \det_{\mathcal{N}\pi_1(N)}(B') &= |\lambda_\rho|^{2g+k+r}, \\ \det_{\mathcal{N}\pi_1(N)}(A') &= |\lambda_\rho|, \end{aligned}$$

and the result (8) follows from

$$\tau^{2)}(N,\rho) = \frac{\det_{\mathcal{N}\pi_1(N)}(B')}{\det_{\mathcal{N}\pi_1(N)}(A') \prod_{i=0}^{r} \det_{\mathcal{N}\pi_1(N)}(1 - \ell_i \otimes \rho(\ell_i))}. \qquad \square$$

Finally, we note that for reducible representation ρ the element $\rho\big(\prod_i [a_i, b_i]\big)$ has trace ± 2. In the case that N has empty boundary ($k = 0$), if $b \neq 0$ in (7), then the eigenvalues of $\rho(h)$ must lie on the unit circle.

5. Twisted L^2-torsion for hyperbolic manifolds

In this section we conclude the proof of Theorem A, whose statement we recall here:

Theorem 5.1. *The twisted L^2-torsion function*

$$\tau^{(2)}(M,\rho)\colon X(M) \longrightarrow \mathbb{R}$$

is real-analytic in an analytic neighbourhood U of any lift $[\rho_0]$ of the holonomy representation of M in the character variety $X(M)$.

In Subsections 5.1 and 5.2 we give alternative definitions for the L^2-torsion $\tau^{(2)}(M,\rho)$, which allow us to work with operators on a fixed Hilbert space (Lemma 5.2) and for which we have a spectral gap (Lemma 5.3) that allows us to apply Lemma 3.6 to deduce Theorem A. We give the proof of Lemma 5.3 in Subsection 5.3, using comparison with the analytic L^2-invariants and the spectral gap property of the holonomy representation established in [**1**, Lemma 4.1].

5.1. Combinatorial Laplacians.

Lemma 5.2. *There exists a graded Hilbert $\mathcal{N}\pi$-module $V = V_0 \oplus \cdots \oplus V_3$ and functions $D_p\colon R(M) \to \operatorname{Hom}_{\mathcal{N}\pi}(V_p, V_{p-1})$, $p = 1, \ldots, 3$, which are regular in a Zariski neighbourhood of ρ_0 in $R(M)$ and such that for every ρ the complex $(V_*, D_*(\rho))$ is isomorphic to $C_*^{(2)}(M, \mathcal{N}\pi \otimes \rho)$.*

In particular, if we set $A_p = D_p^* D_p + D_{p+1} D_{p+1}^*$, then $\rho \mapsto A_p(\rho)$ are analytic in a neighbourhood of ρ_0 and we have

$$\tau^{(2)}(M, \rho) = \prod_{p=1}^{3} \det{}_{\mathcal{N}\pi}(A_p(\rho))^{p(-1)^p}. \tag{9}$$

Proof: We set

$$L_p = C_p(\widetilde{M}) \otimes_{\mathbb{C}} \mathbb{C}^2,$$

where π acts by $\gamma \cdot (e \otimes v) = (\gamma e) \otimes v$. We choose an arbitrary $\mathbb{C}\pi$-base B_p for $C_p(\widetilde{M})$; then L_p is isomorphic (as a $\mathbb{C}\pi$-module) to $C_p(\widetilde{M}) \otimes \mathbb{C}^2$ with diagonal action (as introduced in Subsection 3.2) via the map $I_p(\rho)\colon \gamma e \otimes v \mapsto \gamma e \otimes \rho(\gamma) v$ for $e \in B$, $\gamma \in \pi$. Choosing any base of $\mathbb{C}^2$ we get a basis of L_p, and the completion $V_p = \Lambda L_p$ is isomorphic to $C_p^{(2)}(M, \mathcal{N}\pi \otimes \rho)$ (see also [**14**, Lemma 1.1]).

Let ∂_p be the differentials of $C_*(M, \mathcal{N}\pi \otimes \rho)$, which are given by $\partial_p(e \otimes v) = \partial_p e \otimes v$. The corresponding boundary maps of V_* are given by $D_p(\rho) = I_{p-1}^{-1}(\rho) \partial p I_p(\rho)$. The coefficients of $I_p(\rho)$ in the $\mathbb{C}\pi$-bases of $L_*, C_*(\widetilde{M}) \otimes \mathbb{C}^2$ are rational functions of ρ because they depend only on the coefficients of a finite number of $\rho(\gamma)$, so the first part of the lemma is proved.

Obviously $A_p = I_p^{-1} \Delta_p I_p$ and (9) follows. □

5.2. L^2-torsion.

5.2.1. L^2-cochain complexes. We define in the same way as in Subsection 3.2 the L^2-cochain complexes $C^*_{(2)}(M, \mathcal{N}\pi \otimes \rho)$ as the completion of the complex $C^*(\widetilde{M}) \otimes \mathbb{C}^2$ with diagonal action and similarly for $C^*_{(2)}(\partial M, \mathcal{N}\pi \otimes \rho)$. The relative L^2-cochain complex $C^*_{(2)}(M, \partial M, \mathcal{N}\pi \otimes \rho)$ is then defined by the exact sequence

$$0 \longrightarrow C^*_{(2)}(M, \partial M, \mathcal{N}\pi \otimes \rho) \longrightarrow C^*_{(2)}(M, \mathcal{N}\pi \otimes \rho) \xrightarrow{i^*} C^*_{(2)}(\partial M, \mathcal{N}\pi \otimes \rho|_{\partial M}) \longrightarrow 0.$$

We denote by $\Delta^p_{\mathrm{rel}}\colon C^p_{(2)}(M, \partial M, \mathcal{N}\pi \otimes \rho) \to C^p_{(2)}(M, \partial M, \mathcal{N}\pi \otimes \rho)$ the combinatorial Laplacians of the complex $C^*_{(2)}(M, \partial M, \mathcal{N}\pi \otimes \rho)$. The crucial point for us is then the following, which we prove in the next subsection.

Lemma 5.3. *There exists $\lambda_0 > 0$ and a neighbourhood U of the holonomy such that for all $[\rho] \in U$, for all $p = 0, \dots, 3$, we have $\sigma(\Delta^p_{\mathrm{rel}}(\rho)) \subset]\lambda_0, +\infty[$.*

5.2.2. Poincaré duality. We follow the references [**13**] (see the proofs of Theorems 1.36(3) and 3.93(3)) and [**14**, Theorem 6.7(7)]. There is a homotopy equivalence $P_*\colon C_*(\widetilde{M}) \to C^{3-*}(\widetilde{M}, \partial\widetilde{M})$ between the untwisted chain and relative cochain complexes; a construction is given in the proof of Theorem 2.1 in [**21**]. We can extend it by the identity to a homotopy equivalence

$$P_*\colon C^{(2)}_*(M, \mathcal{N}\pi \otimes \rho) \longrightarrow C^{3-*}_{(2)}(M, \partial M, \mathcal{N}\pi \otimes \rho)$$

between the completed twisted complexes. By [**13**, Theorem 2.19] it follows that they have the same Novikov–Shubin invariants. In particular, $\Delta_p(\rho)$ has a spectral gap for any $\rho \in U$ (where U is the neighbourhood of ρ_0 given by Lemma 5.3).

5.2.3. Proof of Theorem A. We have that the relative cochain complex $C^*_{(2)}(M, \partial M, \mathcal{N}\pi \otimes \rho)$ is of determinant class for ρ in the neighbourhood U given by Lemma 5.3. By Poincaré duality (see preceding paragraph) we get that $C^{(2)}_*(\widetilde{M}, \rho)$ is of determinant class and that for $\rho \in U$, $\inf \sigma(\Delta_p(\rho)) > 0$, and by Lemma 5.2 we get that $(V_*, D_*(\rho))$ are of determinant class and the $A_p(\rho)$ have a spectral gap for $\rho \in U$.

The hypotheses of Lemma 3.6 are thus satisfied by $[\rho] \mapsto A_p(\rho)$, where ρ belongs to the section of the map $R(M) \to X(M)$ given by Lemma 2.1. It follows that the functions $[\rho] \mapsto \det_{\mathcal{N}\pi} A_p(\rho)$ are real-analytic in U and so is the L^2-torsion by (9). $\square$

5.3. Proof of Lemma 5.3.

5.3.1. Preliminaries on analytic L^2-invariants. In this subsection we define the twisted analytic Laplacian operators and derive some of their properties. For our purposes here we need only consider twisting by the holonomy representation ρ_0 but the discussion can be adapted to deal with any ρ.

We consider the trivial rank 2 bundle $\mathbb{H}^3 \times \mathbb{C}^2$ with the $\pi_1(M)$-action

$$\gamma \cdot (\widetilde{x}, v) = (\gamma \cdot \widetilde{x}, \rho_0(\gamma)\, v)$$

for any $\gamma \in \pi_1(M)$, $\widetilde{x} \in \mathbb{H}^3$, $v \in \mathbb{C}^2$, and the associated complex $\Omega^*_c(\mathbb{H}^3, \mathbb{C}^2)$ of compactly supported, $\mathbb{C}^2$-valued forms on $\mathbb{H}^3$ with the natural Γ-equivariant differential $d_p\colon \Omega^p(\mathbb{H}^3, \mathbb{C}^2) \to \Omega^{p+1}(\mathbb{H}^3, \mathbb{C}^2)$. On the bundle $\mathbb{H}^3 \times \mathbb{C}^2$ we choose an arbitrary $\mathrm{SL}_2(\mathbb{C})$-invariant norm, which amounts to choosing a norm on $\mathbb{C}^2$ above a base point.

The spaces $\Omega_c^p(\mathbb{H}^3, \mathbb{C}^2)$ admit a Hermitian product associated with this metric on $\mathbb{H}^3 \times \mathbb{C}^2$. We will use a uniform notation for various spaces of differential forms associated with this inner product: if X is a manifold, a prefix R before $\Omega^p(X, \rho)$ indicates that we consider the completion associated to ρ of the p-forms with coefficients in the trivial bundle and regularity R. For example, $L^2\Omega^p$ stands for square-integrable forms and $H^l\Omega^p$ for Sobolev spaces; note that these Hilbert spaces and their $\mathcal{N}\pi$-module structure depend on ρ. The norm on any $R\Omega^p(X, \rho)$ will be denoted by $\|\cdot\|_R$. For the Sobolev spaces from here on, an integer $l > \dim M = 3$ is fixed.

The Hodge Laplacian Δ_p on the space $L^2\Omega^p(\mathbb{H}^3, \rho)$ is defined as follows: if d_p^* is the formal adjoint of d_p, then the operator defined by $\Delta_p = d_p^* d_p + d_{p+1} d_{p+1}^*$ on the space $\Omega_c^p(\mathbb{H}^3, \mathbb{C}^2)$ of compactly supported smooth forms extends to a unique essentially self-adjoint operator on the completion $L^2\Omega^p(\mathbb{H}^3, \rho)$. This operator has a well-defined spectrum $\sigma(\Delta_p)$ and it follows from (6) that we have

(10)
$$\inf \sigma(\Delta_p) = \min\left(\inf_{\omega \in \Omega_c^p(\mathbb{H}^3, \mathbb{C}^2)\setminus\{0\}} \frac{\|d_p\omega\|_{L^2}^2}{\|\omega\|_{L^2}^2}, \inf_{\omega \in \Omega_c^{p+1}(\mathbb{H}^3, \mathbb{C}^2)\setminus\{0\}} \frac{\|d_{p+1}\omega\|_{L^2}^2}{\|\omega\|_{L^2}^2} \right).$$

The following result essentially follows from [**1**, Lemma 4.1]; we give a short justification as this reference nominally covers only compact locally symmetric spaces. To be more specific: the statement of [**1**, Lemma 4.1] is given only for the discrete spectrum but the argument also applies to the continuous spectrum. This is justified by the fact that the representations in the packets contributing to the spectrum on p-forms also satisfy the condition $\mathrm{Hom}_K(\wedge^p \mathfrak{p}^* \otimes \mathbb{C}^2, \cdot) \neq 0$ which is the only point used to give a lower bound on the Casimir eigenvalue in loc. cit. As rigorously justifying this would take too much time and preliminaries we give a suboptimal bootstrap argument below.

Lemma 5.4. *There exists $\delta > 0$ such that for the holonomy ρ_0 and $0 \le p \le 3$ we have $\sigma(\Delta^p(\rho_0)) \subset [\delta, +\infty[$.*

Proof: The standard representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$ satisfies the hypotheses of [**1**, Lemma 4.1]. Hence there exists a $\delta > 0$ such that $\sigma(\Delta_M^p(\rho_M)) \ge \delta$ for any closed hyperbolic 3-manifold M, where ρ_M is the holonomy representation of M and $\Delta_M^p(\rho_M)$ the Laplacian on p-forms with coefficients of the associated $\mathbb{C}^2$-bundle over M with the Hermitian metric descended from that described above on $\mathbb{H}^3$.

Now let M_n be a tower of coverings of a fixed closed hyperbolic manifold such that the injectivity radius $\mathrm{inj}(M_n)$ goes to infinity. Let μ_n

be the spectral density measure of $\Delta^p_M(\rho_M)$; it is an atomic measure by the spectral theorem in Riemannian geometry and by the preceding paragraph it is supported on the interval $[\delta,+\infty[$ for all n. Now it follows from the lemma in the proof of [**1**, Theorem 4.5] that the measures $\mu_n/\operatorname{Vol}(M_n)$ weakly converge to the spectral measure μ of $\Delta^p(\rho_0)$.

More precisely, the lemma states (in our notation) that for fixed $t>0$ we have that $\int_{M_n}(\operatorname{tr}(e^{-t\Delta^p_{M_n}(\rho_{M_n})})-\operatorname{tr}(e^{-t\Delta^p(\rho_0)}))\,d\operatorname{Vol}$ goes to 0. The integrals of the heat kernels can be rewritten using the spectral measures of $\Delta^p_{M_n}(\rho_{M_n})$ and $\Delta^p(\rho_0)$ and we get that for any $t>0$ we have that $\int_0^{+\infty} e^{-t\lambda}\,d\mu_n(\lambda)/\operatorname{Vol}(M_n)$ converges to $\int_0^{+\infty} e^{-t\lambda}\,d\mu$. The latter implies weak convergence of the measures since the functions $\lambda\mapsto e^{-t\lambda}$ are dense in uniform convergence on compact subsets. See also [**3**, Theorem 1.1], whose proof extends to our setting using [**1**, Lemma 3.8].

In particular, if φ is any continuous function supported in $]-\delta,\delta[$, we have that $0=\int\varphi\,d\mu_n/\operatorname{Vol}(M_n)\to\int\varphi\,d\mu$ and so the latter also vanishes, which means that $\Delta^p(\rho_0)$ has no spectrum in $]-\delta,\delta[$. □

5.3.2. Comparison of analytic and combinatorial invariants for the holonomy. The main step in the proof of Lemma 5.3 is now to prove the spectral gap for the representation $\rho=\rho_0$. We compare the spectra of Laplace operators on the complex $C^*_{(2)}(\widetilde{M},\partial\widetilde{M},\mathcal{N}\pi\otimes\rho)$ and on the Sobolev complexes $H^{l-*}\Omega^*(\mathbb{H}^3,\rho)$ using Whitney maps. Our arguments are essentially those leading to [**13**, Lemma 2.71], which we adapt to the twisted setting. We will use the notation $\|\cdot\|_{\ell^2}$ for the norm on the cochain complex $C^{(2)}_*(\widetilde{M},\partial\widetilde{M},\mathcal{N}\pi\otimes\rho)$, and $\|\cdot\|_{\widetilde{x}}$ for the norm of the fibre above a point $\widetilde{x}$ in $\mathbb{H}^3$.

First we recall the definition of Whitney maps. Using the analogous of Lemma 5.2 for cohomology, we use a different – although isomorphic – model for the cochain complex $C^*(M,\mathcal{N}\pi\otimes\rho)$. We see it as the completion of the complex $C^*(\widetilde{M})\otimes\mathbb{C}^2$ but with action $\gamma\cdot(f\otimes v)=(\gamma f)\otimes(\rho(\gamma)v)$ and differentials $d(f\otimes v)=(df)\otimes v$. If we fix a $\mathbb{C}\pi$-basis of $C_p(\widetilde{M})$, the map $\gamma f\otimes v\mapsto\gamma f\otimes\rho(\gamma)v$ gives an isomorphism of $\mathbb{C}\pi$-complexes from our former model to this one. We choose a smooth partition of unity e_c on M, indexed by vertices c of the triangulation, and we lift it to $\widetilde{M}$. The important property that we require is that e_c has its support contained in the open simplices adjacent to c. Then, if f_σ is the cochain with value 1 on σ and 0 on other simplices, we define

$$W^p(f_\sigma\otimes v)=p!\sum_{c\in\sigma}\omega_{\sigma,c}\otimes v,$$

where $\omega_{\sigma,c}$ is defined by the usual formula: if $c_0, \dots, c_p$ are the vertices of σ ordered according to orientation, $\omega_{\sigma,c_i} = (-1)^i e_{c_i} \cdot \bigwedge_{j\neq i} de_{c_j}$. The form $W^p f_\sigma \otimes v$ is supported in the open star of σ; in particular it is compactly supported in $\widetilde{M} \setminus \partial\widetilde{M}$ if σ is not contained in $\widetilde{\partial M}$.

Thus restricting W defines a map $C^*(\widetilde{M}, \partial\widetilde{M}) \otimes \mathbb{C}^2 \to C_c^\infty \Omega^*(\mathbb{H}^3, \mathbb{C}^2)$, and a direct computation (see [**23**, p. 140]) shows that it is a chain map – note that in the model we use here, on both sides the differentials satisfy $d(f \otimes v) = df \otimes v$, $d(\omega \otimes v) = d\omega \otimes v$, so we can use this computation as it is for our twisted case. This map from a $\mathbb{C}\pi$-module of finite rank extends to a bounded chain map

$$W\colon C^*_{(2)}(M, \partial M, \mathcal{N}\pi \otimes \rho) \longrightarrow H^{l-*}\Omega^*(\mathbb{H}^3, \rho).$$

To prove our claim about the spectral gap of $C^*_{(2)}(\widetilde{M}, \partial\widetilde{M}, \rho)$ it suffices to prove the following inequalities: there exists $C, c > 0$ such that for all p and any cochain $\phi \in C^p_{(2)}(\widetilde{M}, \partial\widetilde{M}, \mathcal{N}\pi \otimes \rho)$ we have

$$c\|\phi\|_{\ell^2} \leq \|W^p\phi\|_{H^{l-p}} \leq C\|\phi\|_{\ell^2}. \tag{11}$$

Indeed, if this holds, then for any ϕ the Rayleigh quotient of $W^p\phi$ satisfies

$$\frac{\|d(W^p\phi)\|_{H^{l-p-1}}}{\|W^p\phi\|_{H^{l-p}}} = \frac{\|W^p(d\phi)\|_{H^{l-p-1}}}{\|W^p\phi\|_{H^{l-p}}} \leq (c^{-1}C)\frac{\|d\phi\|_{\ell^2}}{\|\phi\|_{\ell^2}}$$

and by Lemma 5.4 and (10) it follows that we must have $\frac{\|d\phi\|_{\ell^2}}{\|\phi\|_{\ell^2}} \geq \frac{c}{C}\delta$. As (10) also applies to the combinatorial Laplacian we get that $\inf \sigma(\Delta^p_{\mathrm{rel}}) \geq \frac{c}{C}\delta$. This will prove Lemma 5.3 for the holonomy representation ρ_0, after we show inequality (11).

Since W is bounded, the content of (11) is the lower bound $c\|\phi\|_{\ell^2} \leq \|W^p\phi\|_{H^{l-p}}$, which we will now prove. First we note (see [**18**, Section 7]) that on the image of W^p the Sobolev and L^2-norms are equivalent, so it suffices to show that $c\|\phi\|_{L^2} \leq \|W^p\phi\|_{L^2}$. To do so, for a p-simplex σ in $\widetilde{M} = \mathbb{H}^3$ let $\widetilde{U}_\sigma$ be the maximal open subset of $\mathbb{H}^3$ on which $\sum_{c\in\sigma} e_c = 1$ and no e_c vanishes for $c \in \sigma$. This is nonempty, and we have $\widetilde{U}_\sigma \cap \widetilde{U}_\tau = \emptyset$ if $\tau \neq \sigma$ is another p-simplex. In addition, if $\sigma = [c_0, \dots, c_p]$, replacing every instance of e_{c_0} in $W^p(f_\sigma \otimes v)$ with $\left(1 - \sum_{i=1}^p e_{c_i}\right) \otimes v$ (which is valid on $\widetilde{U}_\sigma$ by definition), we get that

$$W^p(f_\sigma \otimes v) = (de_{c_1} \wedge \cdots \wedge de_{c_p}) \otimes v \text{ on } \widetilde{U}_\sigma;$$

in particular it does not vanish there and so the integral of its norm on $\widetilde{U}_\sigma$ is strictly positive. As there are only finitely many p-simplexes modulo π

and the maps $e_{\gamma c}$ are π-equivariant we get that there exists $a > 0$ such that

$$\|W^p(f_\sigma \otimes v)\|_{L^2(U_\sigma)} \geq a \inf_{\widetilde{x} \in \widetilde{U}_\sigma} \|v\|_{\widetilde{x}} \tag{12}$$

for all σ.

Let $\|\cdot\|$ be the norm on $\mathbb{C}^2$, and B the $\mathbb{C}\pi$-basis of $C^*(\widetilde{M})$, that were used to define the norm $\|\cdot\|_{\ell^2}$ on $C^*_{(2)}(\widetilde{M}, \partial\widetilde{M}, \mathcal{N}\pi \otimes \rho)$. For any σ let γ_σ be the element of π such that $\gamma_\sigma^{-1}\sigma$ belongs to B. Then by definition $\|f_\sigma \otimes v\|_{\ell^2} = \|\rho(\gamma_\sigma)^{-1}v\|$. As the $\widetilde{U}_\sigma$ are relatively compact and B is finite, there is some a' (independent of σ) such that $\inf_{\widetilde{x} \in \widetilde{U}_\sigma} \|v\|_{\widetilde{x}} > a'\|v\|$ for all $\sigma \in B$. As $\widetilde{x} \mapsto \|v\|_{\widetilde{x}}$ is π-invariant, it follows that for any σ we have

$$\inf_{\widetilde{x} \in \widetilde{U}_\sigma} \|v\|_{\widetilde{x}} > a'\|\rho(\gamma^{-1})v\| = a'\|f_\sigma \otimes v\|_{\ell^2}.$$

With (12) we finally get that

$$\|W^p(f_\sigma \otimes v)\|_{L^2(U_\sigma)} \geq a''\|f_\sigma \otimes v\|_{\ell^2}$$

for $a'' = aa' > 0$ and all σ. Now if ϕ is an arbitrary relative cochain, we can write it as

$$\phi = \sum_\sigma b_\sigma f_\sigma \otimes v_\sigma$$

so that we have

$$\|\phi\|^2_{\ell^2} = \sum_\sigma b_\sigma^2 \|f_\sigma \otimes v_\sigma\|^2_{\ell^2}.$$

As the $\widetilde{U}_\sigma$ are disjoint we get that

$$\begin{aligned}
\|W^p(\phi)\|^2_{L^2} &\geq \sum_\sigma \|W^p(\phi)\|^2_{L^2(U_\sigma)} \\
&= \sum_\sigma \|W^p(f_\sigma \otimes v)\|^2_{L^2(U_\sigma)} \\
&\geq (a'')^2 \sum_\sigma b_\sigma^2 \|f_\sigma \otimes v\|^2_{\ell^2} = (a'')^2 \|\phi\|^2_{\ell^2},
\end{aligned}$$

where the inequality on the second line follows from the fact that $W^p(f_\tau \otimes v)$ vanishes identically on $\widetilde{U}_\sigma$ whenever $\tau \neq \sigma$, and the one on the third follows from the previous inequality. The last line is the inequality we were after.

5.3.3. Conclusion of the proof of Lemma 5.3. By Lemma 5.2 and the isomorphisms $V_* \to C_*^{(2)}(\widetilde{M}, \mathcal{N}\pi \otimes \rho) \to C^*_{(2)}(\widetilde{M}, \partial\widetilde{M}, \mathcal{N}\pi \otimes \rho)$ the matrices of the operators $\Delta_p(\rho)$ have coefficients which vary continuously for $[\rho]$ in their domain of definition D. In particular, each map $\Delta^p_{\mathrm{rel}} \colon D \to \mathcal{B}(C^p_{(2)}(\widetilde{M}, \partial\widetilde{M}, \mathcal{N}\pi \otimes \rho))$ is continuous for the operator norm. Lemma 5.3 then follows from Lemma 3.5 since it holds at $[\rho_0]$ by Subsubsection 5.3.2 and Lemma 5.4. $\square$

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